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M. KHORASHADIZADEH – A. H. REZAEI ROKNABADI
G. R. MOHTASHAMI BORZADARAN

Variance residual life function in discrete random ageing

Summary - The random variable $X_t = X - t|X \geq t$, which is called residual life random variable, has gathered the attention of most researchers in reliability. The mean and the variance of this variable in continuous distribution have been studied by several authors. But, in discrete case, only in recent years, some studies have been done for the mean of this variable. In this paper, we define and study the properties of variance of $T_k = T - k|T \geq k$ where T is a discrete random variable. Besides similar results for discrete and continuous lifetime distributions, relationships with its mean, monotonicity and the associated ageing classes of distributions are obtained for discrete cases. Furthermore, some characterization results about the class of increasing (decreasing) variance residual life distributions based on mean residual life and residual coefficient of variation, are presented and the lower and upper bound for them are achieved.

Key Words - Hazard rate; Mean residual life; Variance residual life; Residual coefficient of variation.

1. Introduction

In reliability literature, the additional life time given that the component has survived up to time t, is called the residual life function (RLF) of the component. More specifically, if the random variable X denotes the lifetime of a unit, then the random variable $X_t = X - t|X \geq t$ is called the residual life time.

The mean and the variance of the random variable X_t has received considerable attention in reliability. The concepts of residual or remaining life based on the current age is effectively used to infer properties of the underlying life distributions. The mean residual life function (MRL) has many applications, for example, in life insurance, maintenance and product quality control, economics and social studies. The mean residual life function has been studied

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by various authors such as Gupta (1975), Hall and wellner (1981), Ebrahimi (1986), Bradley and Gupta (2003) and Navarro and Hernandez (2008). It has been shown that by Gupta (1981) that MRL determines the distributions uniquely.

Another quantity of interest which has generated interest in recent years is the variance residual life (VRL). Karlin (1982) has studied the monotonic behavior of VRL when the density is log-convex or log-concave. Gupta (1987, 2006) studied the VRL, its monotonicity and the associated aging classes of life distributions. Launer (1984) and Gupta et al. (1987) discussed the class of life distributions having decreasing (increasing) variance residual life. Lynn and Singpurwalla (1997) viewed the burn-in concept as a process of reduction of uncertainty of the lifetime of a component. One approach to this is to minimize VRL. Combining this with maximizing the MRL leads Block et al. (2002) to consider balancing mean residual life and residual variance through minimizing the residual coefficient of variation (CV).

The role and properties of the variance residual life and the residual coefficient of variation in reliability have been discussed considerably for continuous lifetime random variables by various authors such as Gupta and Kirmani (1998, 2000, 2004), El-Arishi (2005), Al-Zahrani and Stoyanov (2008) and Abu-Youssef (2004, 2007, 2009). It remains an open problem to find similar characterization and properties for discrete distributions.

In this paper, we study the variance residual life in discrete lifetime distributions which the results are different from continuous case. Its relationship with mean residual life and residual coefficient of variation are obtained. Also, its monotonicity and the associated ageing classes of distributions are discussed. Some characterization results of the class of increasing (decreasing) variance residual life, which is denoted by D-IVRL(D-DVRL), are presented and the lower and upper bounds for variance residual life under some conditions are obtained.

2. Preliminaries

Let X be a non-negative continuous random variable with cumulative distribution function (cdf), $F(x)$ and probability density function (pdf), $f(x)$.

The MRL, VRL and CV functions for random variable X are,

$$\mu(t) = E(X_t) = \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(u)du,$$
$$\sigma^2(t) = \mathrm{Var}(X_t) = \frac{2}{\overline{F}(t)} \int_t^\infty \int_y^\infty \overline{F}(u)dudy - \mu^2(t),$$

and

$$CV(t) = \frac{\sigma(t)}{\mu(t)},$$

respectively.

It has been shown by many authors that,

$$\begin{aligned}\frac{d}{dt}\sigma^2(t) &= h(t)[\sigma^2(t) - \mu^2(t)],\\ &= \mu(t)\left(1 + \frac{d}{dt}\mu(t)\right)[CV^2(t) - 1],\end{aligned}$$

where $h(t) = \frac{f(t)}{\overline{F}(t)}$ is the hazard rate function. Gupta and Kirmani (2000) showed that if X has increasing mean residual life (IMRL) property, then $\sigma^2(t) \geq \mu^2(t)$ or equivalently $CV^2(t) \geq 1$, thus X has the IVRL property. Also, they characterized the distribution by $\mu(t)$ and under additional information $\sigma^2(t)$ with the following equations,

$$\overline{F}(t) = \frac{\mu(0)}{\mu(t)} \exp\left[-\int_0^t \frac{dx}{\mu(x)}\right],$$

and

$$\overline{F}(t) = \exp\left\{-\int_0^t \frac{\frac{d}{dy}\sigma^2(y)}{\sigma^2(y) - \mu^2(y)} dy\right\},$$

respectively.

Now, let T be a non-negative discrete random variable with cdf, $F(k)$ and probability mass function (pmf), $p(k)$, the hazard rate of T is defined as,

$$h(k) = \frac{p(k)}{\overline{F}(k-1)}. \tag{1}$$

The MRL function in discrete lifetime distributions is defined by,

$$\alpha(k) = E(T - k | T \geq k) = \frac{1}{\overline{F}(k-1)} \sum_{i=k}^{\infty} \overline{F}(i), \tag{2}$$

where its relation with hazard rate is as follows,

$$h(k) = 1 - \frac{\alpha(k)}{1 + \alpha(k+1)}. \tag{3}$$

Barlow and Proschan (1981) have been shown that $h(k)$ and $\alpha(k)$ characterized the distribution by,

$$\overline{F}(k) = \prod_{i=0}^{k}[1 - h(i)] = \prod_{i=0}^{k}\left(\frac{\alpha(i)}{1 + \alpha(i + 1)}\right); \quad k \in \{0, 1, 2, ...\}.$$

Note that $\alpha^*(k) = E(T - k|T > k)$ is also another definition that is applied in some cases.

3. Main Results

For finding new results that are mentioned in this section as variance residual life function in discrete random ageing, we first consider the following definitions and arguments.

If $E(T^2) < \infty$, we define the discrete variance residual life function (D-VRL) and denote it by $\beta(k)$ as,

$$\beta(k) = \text{Var}(T - k|T \geq k).$$

Definition 3.1. F is said to have increasing (decreasing) variance residual life property and denote it by D-IVRL(D-DVRL), if $\beta(k)$ is increasing (decreasing) with respect to k.

For determining the relationship between $\alpha(k)$ and $\beta(k)$, we need the following lemma.

Lemma 3.2. *If $p(k)$ is the pmf of discrete non-negative random variable T and $\overline{F}(k) = 1 - F(k)$ be its survival function, then we have the following relations,*

$$E(T|T \geq k) = k + \frac{\sum_{i=k}^{\infty} \overline{F}(i)}{\overline{F}(k-1)}, \tag{4}$$

$$E(T^2|T \geq k) = k^2 + \frac{\sum_{i=k}^{\infty}(2i + 1)\overline{F}(i)}{\overline{F}(k-1)}. \tag{5}$$

Proof. According to $p(i) = \overline{F}(i-1) - \overline{F}(i)$, the relations are easily proved. □

In the next theorem, the relation between $\beta(k)$ and $\alpha(k)$ is obtained.

Theorem 3.3. *Let T_k be the discrete residual life random variable, then the variance residual life function ($\beta(k)$) and mean residual life function ($\alpha(k)$) are related as follows,*

$$\beta(k) = \frac{2\sum_{i=k}^{\infty} i\overline{F}(i)}{\overline{F}(k-1)} - (2k-1)\alpha(k) - \alpha^2(k). \tag{6}$$

Proof. We have,

$$\begin{aligned}\beta(k) &= E\left[(T-k)^2|T \geq k\right] - E^2(T-k|T \geq k)\\ &= E(T^2|T \geq k) - kE(T|T \geq k) - k\alpha(k) - \alpha^2(k),\end{aligned}$$

Using Lemma 3.2, we get the required result. □

Remark 3.4. It can be easily seen that constancy of $\beta(k)$ characterizes the geometric distribution.

In the next theorem, we obtain a lower (upper) bound for $\beta(k)$, when $\alpha(k)$ is increasing (decreasing) in k.

Theorem 3.5. *If the non-negative discrete random variable T, has D-IMRL property, then,*

$$\beta(k) > \alpha(k)(1 + \alpha(k)); \quad k > 0, \tag{7}$$

and for $k = 0$ the above inequality becomes equality.

Proof. According to (2), we have,

$$\begin{aligned}\frac{2}{\overline{F}(k-1)}\sum_{i=k}^{\infty}\overline{F}(i-1)\alpha(i) &= \frac{2}{\overline{F}(k-1)}\sum_{i=k}^{\infty}\sum_{j=i}^{\infty}\overline{F}(j) = \frac{2}{\overline{F}(k-1)}\sum_{j=k}^{\infty}\sum_{i=k}^{j}\overline{F}(j)\\ &= \frac{2}{\overline{F}(k-1)}\sum_{j=k}^{\infty}(j-k+1)\overline{F}(j)\\ &= \frac{2}{\overline{F}(k-1)}\sum_{j=k}^{\infty} j\overline{F}(j) - (2k-2)\alpha(k)\\ &= \alpha(k) + \alpha^2(k) + \beta(k).\end{aligned}$$

Thus, it implies,

$$\begin{aligned}\beta(k) - \alpha^2(k) &= \frac{2}{\overline{F}(k-1)}\sum_{i=k}^{\infty}\overline{F}(i-1)\alpha(i) - \alpha(k) - 2\alpha^2(k)\\ &= \frac{2}{\overline{F}(k-1)}\sum_{i=k}^{\infty}\overline{F}(i-1)(\alpha(i) - \alpha(k)) + \alpha(k) > \alpha(k),\end{aligned}$$

since $\alpha(k)$ is increasing, $\alpha(i) - \alpha(k) > 0$ for $i \geq k$. Hence the required result is followed. □

Remark 3.6. When the non-negative discrete random variable T, has D-DMRL property, we can get an upper bound for $\beta(k)$ by reversing the inequality in (7).

Now, we investigate the connection between D-DVRL(D-IVRL) and other classes of distributions.

Theorem 3.7. *If $\alpha(k)$ is increasing (decreasing) in k, then $\beta(k)$ is increasing (decreasing) in k, i.e., the D-IMRL(D-DMRL) property is stronger than the D-IVRL(D-DVRL) property.*

Proof. Using (6) we have,

$$\begin{aligned}\beta(k+1)-\beta(k) = {} & \frac{2\sum_{i=k+1}^{\infty} i\overline{F}(i)}{\overline{F}(k)} - (2k+1)\alpha(k+1) - \alpha^2(k+1) \\ & - \frac{2\sum_{i=k}^{\infty} i\overline{F}(i)}{\overline{F}(k-1)} + (2k-1)\alpha(k) + \alpha^2(k).\end{aligned} \tag{8}$$

On the other hand, one can show that,

$$\frac{2\sum_{i=k+1}^{\infty} i\overline{F}(i)}{\overline{F}(k)} - \frac{2\sum_{i=k}^{\infty} i\overline{F}(i)}{\overline{F}(k-1)} = \frac{2\sum_{i=k+1}^{\infty} i\overline{F}(i)}{\overline{F}(k)}h(k) - 2k\frac{\overline{F}(k)}{\overline{F}(k-1)}, \tag{9}$$

$$\alpha(k+1)-\alpha(k) = \alpha(k+1)h(k) - \frac{\overline{F}(k)}{\overline{F}(k-1)}, \tag{10}$$

so, by replacing the equations (9) and (10) into (8), we have,

$$\beta(k+1)-\beta(k) = h(k)\left[\beta(k+1) - \alpha(k)\alpha(k+1) - \alpha(k)\right]. \tag{11}$$

Since, $\alpha(k)$ is increasing (decreasing),

$$\beta(k+1)-\beta(k) \geq (\leq) h(k)\left[\beta(k+1) - \alpha^2(k+1) - \alpha(k+1)\right],$$

so on using Theorem 3.5, we get the required results. □

Remark 3.8. If F has increasing failure rate (IFR) property, then it has D-DVRL property. Since, it is well-known in discrete distributions that IFR property implies D-DMRL property.

According to (11), we can have the following definition.

Definition 3.9. A discrete non-negative random variable, T, is said to have D-DVRL(D-IVRL) property if and only if,

$$\beta(k+1) \leq (\geq) \alpha(k)\alpha(k+1) + \alpha(k).$$

Although, $\alpha(.)$ characterizes the distribution, but in the following theorem, we investigate that if it is possible to characterize the distributions by the knowledge of $\beta(.)$.

Theorem 3.10. *If $\alpha(k)$ be the MRL function and $\beta(k)$ be the D-VRL function of the discrete random variable, then,*

$$\overline{F}(k)=\prod_{i=0}^{k}\left(1-\frac{\beta(i+1)-\beta(i)}{\beta(i+1)-\alpha(i)\alpha(i+1)-\alpha(i)}\right);\quad k\in\{0,1,2,...\}. \tag{12}$$

Proof. By using (11) and knowing $h(k)=1-\frac{\overline{F}(k)}{\overline{F}(k-1)}$, the Theorem would be proved by induction. □

We define the discrete residual coefficient of variation by,

$$\gamma(k)=\frac{\sqrt{\beta(k)}}{\alpha(k)}.$$

Here, another characterizations of the D-IVRL(D-DVRL) and D-IMRL(D-DMRL) classes of distributions based on $\gamma(k)$ are obtained.

Theorem 3.11. *For non-negative discrete random variable T we have,*

i) *T has D-IVRL property if and only if,*

$$\gamma(k)\geq\frac{\alpha(k-1)}{\alpha(k)},$$

ii) *T has D-DVRL property if and only if,*

$$\gamma(k)\leq\frac{\alpha(k-1)}{\alpha(k)}[1-h(k-1)]^{-1/2},$$

iii) *If T has D-IMRL property then,*

$$\gamma(k)\geq 1,$$

iv) *If T has D-DMRL property then,*

$$\gamma(k)\leq[1+1/\alpha(k)]^{1/2}.$$

Proof. On using (11) and (3) and knowing that $0\leq h(k)\leq 1$, the proof is completed. □

In the next theorem, we present a characterization about $\beta(k)$ which is not quit similar in continuous case.

Theorem 3.12. *$\beta(k)$ is increasing (decreasing) with respect to k if and only if,*

$$\varphi(k) > (<)1,$$

where $\varphi(k) = \frac{\beta(k)}{\alpha(k-1)\alpha^{+}(k-1)}$ *and* $\alpha^{+}(k) = E(T - k|T > k)$.

Proof. Using (11) and $\alpha(k) = \alpha^{+}(k-1) - 1$ the required result is obvious. □

4. Conclusion

In this paper, variance residual life function in discrete lifetime distributions is introduced and its relation with discrete mean residual life, discrete coefficient of variation and some discrete ageing classes of distributions are discussed. The following partial chain in discrete lifetime, is also obtained,

$$\mathrm{D-DMRL(D-IMRL)} \Longrightarrow \mathrm{D-DVRL(D-IVRL)}.$$

Note that similar results and definitions in this paper, can be verified by using the residual random variable $T_k^* = T - k|T > k$.

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M. KHORASHADIZADEH
A. H. REZAEI ROKNABADI
G. R. MOHTASHAMI BORZADARAN
Department of Statistics
School of Mathematical Sciences
Ferdowsi University of Mashhad
P.O. Box 91775-1159, Mashhad, Iran.