Journal of Advanced Research in Pure Mathematics
Online ISSN: 1943-2380

Vol. 3, Issue. 1, 2011, pp. 75-86
doi: 10.5373/jarpm.445.052610

Generalized higher derivations are sequences of generalized derivations

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. In this paper we give a characterization of generalized higher derivations (Jordan generalized higher derivations) from $\mathcal{A}$ to $\mathcal{B}$ in terms of generalized derivations (Jordan generalized derivations) on $\mathcal{B}$ under some certain conditions.

Keywords: Algebra; Derivation; Higher derivation; d_0-derivation; Jordan derivation; Jordan higher derivation; Generalized derivation; Generalized higher derivation.

Mathematics Subject Classification 2010: 16W25.

1 Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras, $\mathcal{X}$ be a $\mathcal{B}$-bimodule and $\sigma : \mathcal{A} \to \mathcal{B}$ be a linear mapping. A linear mapping $\delta : \mathcal{A} \to \mathcal{X}$ is called a *σ-derivation* if it satisfies the Leibniz rule $\delta(ab) = \delta(a)\sigma(b) + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{A}$. In the case $\mathcal{A} = \mathcal{B} = \mathcal{X}$ and $\sigma = I_{\mathcal{A}}$, the identity mapping on $\mathcal{A}$, a σ-derivation is called a *derivation.* For other approaches to generalized notions of derivations and their applications see [1, 2, 5, 11, 12] and references therein. A sequence $\{f_n\}_{n\in\mathbb{N}\cup\{0\}}$ of linear mappings from $\mathcal{A}$ to $\mathcal{B}$ is called a *higher derivation* if $f_n(ab) = \sum_{k=0}^{n} f_k(a)f_{n-k}(b)$ for each $a, b \in \mathcal{A}$ and each non-negative integer n. Higher derivations were introduced by Hasse and Schmidt [6], and algebraists sometimes call them Hasse-Schmidt derivations. For an account of higher derivations the reader is referred to the book [4].

In [10], the first named author gave a characterization of higher derivations from an algebra $\mathcal{A}$ into itself, in terms of derivations on $\mathcal{A}$, provided that f_0 is the identity

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[†]Received: 26 May 2010, revised: 24 July 2010, accepted: 12 August 2010.

mapping on $\mathcal{A}$. In [13] we characterized all higher derivations from an algebra $\mathcal{A}$ into another algebra $\mathcal{B}$ in terms of derivations on $\mathcal{B}$, provided that f_0 is onto and $\ker(f_0) \subseteq \ker(f_n)$ $(n \in \mathbb{N})$. Here, continuing the previous work [13], we characterize generalized higher derivations in terms of generalized derivations and Jordan generalized higher derivations in terms of Jordan generalized derivations. The importance of our work is to transfer the problems such as automatic continuity (see [8] and [9]) of generalized higher derivations (Jordan generalized higher derivations) into the same problems concerning generalized derivations (Jordan generalized derivations). A discussion on Jordan higher derivations in Banach ternary algebras can be found in [7].

Let $\{f_n\}$ be a higher derivation. Then f_0 is a homomorphism and f_1 is a f_0-derivation. Thus if f_0 is onto then $\tilde{f_0} : \mathcal{A}/\ker(f_0) \to \mathcal{B}$ defined by $\tilde{f_0}(a + \ker(f_0)) = f_0(a)$ is an isomorphism. Moreover, for each $n \in \mathbb{N}$, $\tilde{f_n} : \mathcal{A}/\ker(f_0) \to \mathcal{B}$ defined by $\tilde{f_n}(a+\ker(f_0)) = f_n(a)$ is a well-defined linear mapping provided that $\ker(f_0) \subseteq \ker(f_n)$.

During the paper we use the following result proved in [13]. In order to make this paper self contained, we state it with its proof here.

Lemma 1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras (associative or not) and $\{f_n\}$ be a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ with $f_0(\mathcal{A}) = \mathcal{B}$ and $\ker(f_0) \subseteq \ker(f_n)$ $(n \in \mathbb{N})$. Then there is a sequence $\{\gamma_n\}$ of derivations on $\mathcal{B}$ such that for each non-negative integer n

$$(n+1)\tilde{f}_{n+1} = \sum_{k=0}^{n} \gamma_{k+1}\tilde{f}_{n-k}.$$

Proof. We use induction on n. For $n = 0$, let $\gamma_1 : \mathcal{B} \to \mathcal{B}$ be defined by $\gamma_1 = \tilde{f_1}\tilde{f_0}^{-1}$ and $b_1, b_2 \in \mathcal{B}$. Since $\tilde{f_0}$ is an isomorphism, there exist $a_1, a_2 \in \mathcal{A}$ such that $f_0(a_1) = b_1$ and $f_0(a_2) = b_2$. We thus have

$$\begin{aligned} \gamma_1(b_1b_2) &= \tilde{f_1}\tilde{f_0}^{-1}(f_0(a_1)f_0(a_2)) \\ &= f_1(a_1a_2) \\ &= f_0(a_1)f_1(a_2) + f_1(a_1)f_0(a_2) \\ &= f_0(a_1)\tilde{f_1}\tilde{f_0}^{-1}(f_0(a_2)) + \tilde{f_1}\tilde{f_0}^{-1}(f_0(a_1))f_0(a_2) \\ &= b_1\gamma_1(b_2) + \gamma_1(b_1)b_2. \end{aligned}$$

So γ_1 is a derivation. Note that $\tilde{f_1} = \gamma_1\tilde{f_0}$.

Now suppose that γ_k is defined and is a derivation for $k \leq n$ satisfying the result. Putting $\gamma_{n+1} = [(n+1)\tilde{f}_{n+1} - \sum_{k=0}^{n-1}\gamma_{k+1}\tilde{f}_{n-k}]\tilde{f_0}^{-1}$, we show that γ_{n+1} is a derivation.

For $b_1, b_2 \in \mathcal{B}$ there are $a_1, a_2 \in \mathcal{A}$ such that $f_0(a_1) = b_1$ and $f_0(a_2) = b_2$. Hence

$$\begin{aligned}
\gamma_{n+1}(b_1 b_2) &= [(n+1)\tilde{f}_{n+1} - \sum_{k=0}^{n-1} \gamma_{k+1} \tilde{f}_{n-k}] \tilde{f_0}^{-1}(f_0(a_1) f_0(a_2)) \\
&= [(n+1)\tilde{f}_{n+1} - \sum_{k=0}^{n-1} \gamma_{k+1} \tilde{f}_{n-k}](a_1 a_2 + \ker(f_0)) \\
&= (n+1) f_{n+1}(a_1 a_2) - \sum_{k=0}^{n-1} \gamma_{k+1} f_{n-k}(a_1 a_2) \\
&= (n+1) \sum_{k=0}^{n+1} f_k(a_1) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1} \gamma_{k+1} \left(\sum_{\ell=0}^{n-k} f_\ell(a_1) f_{n-k-\ell}(a_2) \right).
\end{aligned}$$

Since $\gamma_1, \ldots, \gamma_n$ are derivations,

$$\begin{aligned}
\gamma_{n+1}(b_1 b_2) &= \sum_{k=0}^{n+1} k f_k(a_1) f_{n+1-k}(a_2) + \sum_{k=0}^{n+1} f_k(a_1)(n+1-k) f_{n+1-k}(a_2) \\
&\quad - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \left[\gamma_{k+1}(f_\ell(a_1)) f_{n-k-\ell}(a_2) + f_\ell(a_1) \gamma_{k+1}(f_{n-k-\ell}(a_2)) \right].
\end{aligned}$$

Writing

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k f_k(a_1) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \gamma_{k+1}(f_\ell(a_1)) f_{n-k-\ell}(a_2), \\
L &= \sum_{k=0}^{n+1} f_k(a_1)(n+1-k) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} f_\ell(a_1) \gamma_{k+1}(f_{n-k-\ell}(a_2))
\end{aligned}$$

we have $\gamma_{n+1}(b_1 b_2) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k}$ we have $0 \leq k + \ell \leq n$ and $k \neq n$. Thus if we put $r = k + \ell$ then we can write it as the form $\sum_{r=0}^{n} \sum_{k+\ell=r, k\neq n}$. Putting $\ell = r - k$ we indeed have

$$\begin{aligned}
K &= \sum_{k=0}^{n+1} k f_k(a_1) f_{n+1-k}(a_2) - \sum_{r=0}^{n} \sum_{0 \leq k \leq r, k \neq n} \gamma_{k+1}(f_{r-k}(a_1)) f_{n-r}(a_2) \\
&= \sum_{k=0}^{n+1} k f_k(a_1) f_{n+1-k}(a_2) - \sum_{r=0}^{n-1} \sum_{k=0}^{r} \gamma_{k+1}(f_{r-k}(a_1)) f_{n-r}(a_2) - \sum_{k=0}^{n-1} \gamma_{k+1}(f_{n-k}(a_1)) f_0(a_2).
\end{aligned}$$

Putting $r+1$ instead of k in the first summation we have

$$\begin{aligned}
& K+\sum_{k=0}^{n-1}\gamma_{k+1}(f_{n-k}(a_1))f_0(a_2) \\
= & \sum_{r=0}^{n}(r+1)f_{r+1}(a_1)f_{n-r}(a_2)-\sum_{r=0}^{n-1}\sum_{k=0}^{r}\gamma_{k+1}(f_{r-k}(a_1))f_{n-r}(a_2) \\
= & \sum_{r=0}^{n-1}\left[(r+1)f_{r+1}(a_1)-\sum_{k=0}^{r}\gamma_{k+1}(f_{r-k}(a_1))\right]f_{n-r}(a_2)+(n+1)f_{n+1}(a_1)f_0(a_2) \\
= & \sum_{r=0}^{n-1}\left[(r+1)\tilde{f}_{r+1}-\sum_{k=0}^{r}\gamma_{k+1}\tilde{f}_{r-k}\right](\tilde{f}_0^{-1}f_0(a_1))f_{n-r}(a_2)+(n+1)f_{n+1}(a_1)f_0(a_2).
\end{aligned}$$

By our assumption

$$(r+1)\tilde{f}_{r+1}-\sum_{k=0}^{r}\gamma_{k+1}\tilde{f}_{r-k}=0$$

for $r=0,\dots,n-1$. We can therefore deduce that

$$\begin{aligned}
K &= \left[(n+1)f_{n+1}(a_1)-\sum_{k=0}^{n-1}\gamma_{k+1}(f_{n-k}(a_1))\right]f_0(a_2) \\
&= \left[(n+1)\tilde{f}_{n+1}-\sum_{k=0}^{n-1}\gamma_{k+1}\tilde{f}_{n-k}\right](\tilde{f}_0^{-1}f_0(a_1))f_0(a_2) \\
&= \gamma_{n+1}(b_1)b_2.
\end{aligned}$$

By a similar argument we have

$$\begin{aligned}
L &= f_0(a_1)\left[(n+1)\tilde{f}_{n+1}-\sum_{k=0}^{n-1}\gamma_{k+1}\tilde{f}_{n-k}\right](\tilde{f}_0^{-1}f_0(a_2)) \\
&= b_1\gamma_{n+1}(b_2).
\end{aligned}$$

Thus

$$\gamma_{n+1}(b_1b_2)=K+L=\gamma_{n+1}(b_1)b_2+b_1\gamma_{n+1}(b_2).$$

Whence γ_{n+1} is a derivation on $\mathcal{A}$. □

2 The results

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. We recall some definitions from [3].

Definition 2.1. A linear mapping $d:\mathcal{A}\to\mathcal{A}$ is said to be

i. a *Jordan derivation* if $d(a^2) = d(a)a + ad(a)$ for every $a \in \mathcal{A}$;

ii. a *generalized derivation* if there exists a derivation $f : \mathcal{A} \to \mathcal{A}$ such that $d(ab) = d(a)b + af(b)$ for every $a, b \in \mathcal{A}$;

iii. a *Jordan generalized derivation* if there exists a derivation $f : \mathcal{A} \to \mathcal{A}$ such that $d(a^2) = d(a)a + af(a)$ for every $a \in \mathcal{A}$.

The following notions are given in [3] for the case that the $\{d_n\}$ is a sequence of linear mappings on $\mathcal{A}$ and $d_0 = I_{\mathcal{A}}$. These notions also can be defined whenever the $\{d_n\}$ is a sequence of linear mappings from $\mathcal{A}$ into $\mathcal{B}$ and $d_0 : \mathcal{A} \to \mathcal{B}$ is an arbitrary homomorphism.

Definition 2.2. Let $\{d_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a family of linear mappings from $\mathcal{A}$ into $\mathcal{B}$. Then we say that $\{d_n\}$ is

i. a *Jordan higher derivation* if $d_n(a^2) = \sum_{k=0}^{n} d_k(a)d_{n-k}(a)$ for each $a \in \mathcal{A}$ and each non-negative integer n;

ii. a *generalized higher derivation* if there exists a higher derivation $\{f_n\}_{n\in\mathbb{N}\cup\{0\}}$ from $\mathcal{A}$ into $\mathcal{B}$ such that $d_n(ab) = \sum_{k=0}^{n} d_k(a)f_{n-k}(b)$ for each $a, b \in \mathcal{A}$ and each non-negative integer n;

iii. a *Jordan generalized higher derivation* if there exists a higher derivation $\{f_n\}$ from $\mathcal{A}$ into $\mathcal{B}$ such that $d_n(a^2) = \sum_{k=0}^{n} d_k(a)f_{n-k}(a)$ for each $a \in \mathcal{A}$ and each non-negative integer n.

Let $\{d_n\}$ be a generalized higher derivation from $\mathcal{A}$ into $\mathcal{B}$. We define $\tilde{d_n}$ from $\mathcal{A}/\ker(d_0)$ to $\mathcal{B}$ with $\tilde{d_n}(a + \ker(d_0)) = d_n(a)$ when $d_0(\mathcal{A}) = \mathcal{B}$ and $\ker(d_0) \subseteq \ker(d_n)$.

Theorem 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras (associative or non-associative), $\{d_n\}$ be a generalized higher derivation from $\mathcal{A}$ into $\mathcal{B}$ and $\{f_n\}$ be a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ such that $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)f_{n-k}(a_2)$ for each $a_1, a_2 \in \mathcal{A}$ and each non-negative integer n. Moreover, suppose that $f_0(\mathcal{A}) = d_0(\mathcal{A}) = \mathcal{B}$, $\ker(f_0) \subseteq \ker(f_n)$ and $\ker(d_0) \subseteq \ker(d_n)$ for each non-negative integer n. Then there is a sequence $\{\delta_n\}$ of generalized derivations on $\mathcal{B}$ associated with a sequence $\{\gamma_n\}$ of derivations such that for each non-negative integer n

$$(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n} \delta_{k+1}\tilde{d}_{n-k} \quad (*)$$

Moreover, Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras (associative or non-associative), $\{d_n\}$ be a generalized higher derivation from $\mathcal{A}$ into $\mathcal{B}$ and $\{f_n\}$ be a higher derivation from $\mathcal{A}$ into $\mathcal{B}$ such that $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)f_{n-k}(a_2)$ for each $a_1, a_2 \in \mathcal{A}$ and each non-negative integer n. Furthermore,

$$d_n = \sum_{i=1}^{n} \left(\sum_{\sum_{j=1}^{i} r_j = n} \left(\prod_{j=1}^{i} \frac{1}{r_j + \ldots + r_i} \right) \delta_{r_1} \ldots \delta_{r_i} d_0 \right),$$

where the inner summation is taken over all positive integers $r_1, \dots, r_n$ with $\sum_{j=1}^{i} r_j = n$.

Proof. We use Lemma 1.1 and induction on n. For $n = 0$, let $\delta_1 : \mathcal{B} \to \mathcal{B}$ be defined by $\delta_1 = \tilde{d}_1 \tilde{d}_0^{-1}$. Let $b_1, b_2 \in \mathcal{B}$. Since d_0 and f_0 are surjective, there exist $a_1, a_2 \in \mathcal{A}$ such that $d_0(a_1) = b_1$ and $f_0(a_2) = b_2$. We thus have

$$\begin{aligned}
\delta_1(b_1 b_2) &= \tilde{d}_1 \tilde{d}_0^{-1}(d_0(a_1) f_0(a_2)) \\
&= \tilde{d}_1 \tilde{d}_0^{-1}(d_0(a_1 a_2)) \\
&= d_1(a_1 a_2) \\
&= d_0(a_1) f_1(a_2) + d_1(a_1) f_0(a_2) \\
&= d_0(a_1) \tilde{f}_1 \tilde{f}_0^{-1}(f_0(a_2)) + \tilde{d}_1 \tilde{d}_0^{-1}(d_0(a_1)) f_0(a_2) \\
&= b_1 \gamma_1(b_2) + \delta_1(b_1) b_2.
\end{aligned}$$

Since γ_1 is a derivation, δ_1 is a generalized derivation. Note that $\tilde{d}_1 = \delta_1 \tilde{d}_0$.

Now suppose that δ_k is defined and is a generalized derivation for $k \leqslant n$ satisfying $(*)$ such that

$$\delta_k(b_1 b_2) = \delta_k(b_1) b_2 + b_1 \gamma_k(b_2)$$

for each $b_1, b_2 \in \mathcal{B}$. Putting $\delta_{n+1} = [(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k}] \tilde{d}_0^{\,-1}$, we show that δ_{n+1} is a generalized derivation such that $\delta_{n+1}(b_1 b_2) = \delta_{n+1}(b_1) b_2 + b_1 \gamma_{n+1}(b_2)$ for all $b_1, b_2 \in \mathcal{B}$. Let $b_1, b_2 \in \mathcal{B}$. Take $a_1, a_2 \in \mathcal{A}$ such that $d_0(a_1) = b_1$ and $f_0(a_2) = b_2$. Hence

$$\begin{aligned}
\delta_{n+1}(b_1 b_2) &= [(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k}] \tilde{d}_0^{\,-1}(d_0(a_1) f_0(a_2)) \\
&= [(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1} \tilde{d}_{n-k}](a_1 a_2 + \ker(d_0)) \\
&= (n+1) d_{n+1}(a_1 a_2) - \sum_{k=0}^{n-1} \delta_{k+1} d_{n-k}(a_1 a_2) \\
&= (n+1) \sum_{k=0}^{n+1} d_k(a_1) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1} \delta_{k+1} \left(\sum_{\ell=0}^{n-k} d_\ell(a_1) f_{n-k-\ell}(a_2) \right).
\end{aligned}$$

Since $\delta_k(b_1 b_2) = \delta_k(b_1) b_2 + b_1 \gamma_k(b_2)$ for $k \leq n$,

$$\begin{aligned}
\delta_{n+1}(b_1 b_2) = &\sum_{k=0}^{n+1} k d_k(a_1) f_{n+1-k}(a_2) + \sum_{k=0}^{n+1} d_k(a_1)(n+1-k) f_{n+1-k}(a_2) \\
&- \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k} \left[\delta_{k+1}(d_\ell(a_1)) f_{n-k-\ell}(a_2) + d_\ell(a_1) \gamma_{k+1}(f_{n-k-\ell}(a_2)) \right].
\end{aligned}$$

Writing

$$\begin{aligned} K &= \sum_{k=0}^{n+1} k d_k(a_1) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k} \delta_{k+1}(d_\ell(a_1)) f_{n-k-\ell}(a_2), \\ L &= \sum_{k=0}^{n+1} d_k(a_1)(n+1-k) f_{n+1-k}(a_2) - \sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k} d_\ell(a_1)\gamma_{k+1}(f_{n-k-\ell}(a_2)) \end{aligned}$$

we have $\delta_{n+1}(b_1b_2) = K + L$. Let us compute K and L. In the summation $\sum_{k=0}^{n-1}\sum_{\ell=0}^{n-k}$ we have $0 \leq k+\ell \leq n$ and $k \neq n$. Thus if we put $r = k+\ell$ then we can write it as the form $\sum_{r=0}^{n}\sum_{k+\ell=r, k\neq n}$. Putting $\ell = r-k$ we indeed have

$$\begin{aligned} K &= \sum_{k=0}^{n+1} k d_k(a_1) f_{n+1-k}(a_2) - \sum_{r=0}^{n} \sum_{0\leq k\leq r, k\neq n} \delta_{k+1}(d_{r-k}(a_1)) f_{n-r}(a_2) \\ &= \sum_{k=0}^{n+1} k d_k(a_1) f_{n+1-k}(a_2) \\ &\quad - \sum_{r=0}^{n-1}\sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1)) f_{n-r}(a_2) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1)) f_0(a_2). \end{aligned}$$

Putting $r+1$ instead of k in the first summation we have

$$\begin{aligned} & K + \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1)) f_0(a_2) \\ = & \sum_{r=0}^{n} (r+1) d_{r+1}(a_1) f_{n-r}(a_2) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1)) f_{n-r}(a_2) \\ = & \sum_{r=0}^{n-1} \left[(r+1) d_{r+1}(a_1) - \sum_{k=0}^{r} \delta_{k+1}(d_{r-k}(a_1)) \right] f_{n-r}(a_2) + (n+1) d_{n+1}(a_1) f_0(a_2) \\ = & \sum_{r=0}^{n-1} \left[(r+1)\tilde{d}_{r+1} - \sum_{k=0}^{r} \delta_{k+1}\tilde{d}_{r-k} \right] (\tilde{d}_0^{-1} d_0(a_1)) f_{n-r}(a_2) + (n+1) d_{n+1}(a_1) f_0(a_2). \end{aligned}$$

By our assumption

$$(r+1)\tilde{d}_{r+1} - \sum_{k=0}^{r} \delta_{k+1}\tilde{d}_{r-k} = 0$$

for $r = 0, \ldots, n-1$. We can therefore deduce that

$$\begin{aligned} K &= \left[(n+1)d_{n+1}(a_1) - \sum_{k=0}^{n-1} \delta_{k+1}(d_{n-k}(a_1))\right] f_0(a_2) \\ &= \left[(n+1)\tilde{d}_{n+1} - \sum_{k=0}^{n-1} \delta_{k+1}\tilde{d}_{n-k}\right] (\tilde{d}_0^{-1} d_0(a_1)) f_0(a_2) \\ &= \delta_{n+1}(b_1)b_2. \end{aligned}$$

Now for computing L, we put $n-r$ instead of k and l in the first and second summation of L, respectively. We have

$$\begin{aligned} L &= \sum_{r=0}^{n} d_{n-r}(a_1)(r+1)f_{r+1}(a_2) - \sum_{k=0}^{n-1}\sum_{r=k}^{n} d_{n-r}(a_1)\gamma_{k+1}(f_{r-k}(a_2)) \\ &= \sum_{r=0}^{n} d_{n-r}(a_1)(r+1)f_{r+1}(a_2) \\ &\quad - \sum_{r=0}^{n-1}\sum_{k=0}^{r} d_{n-r}(a_1)\gamma_{k+1}(f_{r-k}(a_2)) - \sum_{k=0}^{n-1} d_0(a_1)\gamma_{k+1}(f_{n-k}(a_2)). \end{aligned}$$

And so

$$\begin{aligned} & L + d_0(a_1)\sum_{k=0}^{n-1}\gamma_{k+1}(f_{n-k}(a_2)) \\ = & \sum_{r=0}^{n} d_{n-r}(a_1)(r+1)f_{r+1}(a_2) - \sum_{r=0}^{n-1}\sum_{k=0}^{r} d_{n-r}(a_1)\gamma_{k+1}(f_{r-k}(a_2)) \\ = & \sum_{r=0}^{n-1} d_{n-r}(a_1)\left[(r+1)f_{r+1}(a_2) - \sum_{k=0}^{r}\gamma_{k+1}(f_{r-k}(a_2))\right] + d_0(a_1)(n+1)f_{n+1}(a_2) \\ = & \sum_{r=0}^{n-1} d_{n-r}(a_1)\left[(r+1)\tilde{f}_{r+1} - \sum_{k=0}^{r}\gamma_{k+1}(\tilde{f}_{r-k})\right](\tilde{f}_0^{-1}f_0(a_2)) + d_0(a_1)(n+1)f_{n+1}(a_2). \end{aligned}$$

For $r = 0, \ldots, n$, Lemma 1.1 implies that

$$(r+1)\tilde{f}_{r+1} - \sum_{k=0}^{r}\gamma_{k+1}\tilde{f}_{r-k} = 0.$$

We therefore deduce that

$$\begin{aligned} L &= d_0(a_1)\left[(n+1)\tilde{f}_{n+1} - \sum_{k=0}^{n-1}\gamma_{k+1}(\tilde{f}_{n-k})\right](\tilde{f}_0^{-1}f_0(a_2)) \\ &= b_1\gamma_{k+1}(b_2). \end{aligned}$$

Whence δ_{n+1} is a generalized derivation on $\mathcal{B}$ such that

$$\delta_{n+1}(b_1b_2) = K + L = \delta_{n+1}(b_1)b_2 + b_1\gamma_{n+1}(b_2).$$

For the last part of the result we show that if d_n is of the mentioned form then $\tilde{d}_n$ satisfies the recursive relation $(*)$. Since the solution of the recursive relation is unique, this proves the theorem.

Simplifying the notation we put $a_{r_1,\dots,r_i} = \prod_{j=1}^{i} \frac{1}{r_j+\dots+r_i}$. Note that if $r_1 + \dots + r_i = n+1$ then $(n+1)a_{r_1,\dots,r_i} = a_{r_2,\dots,r_i}$. Moreover, $a_{n+1} = \frac{1}{n+1}$.

Now for each $a \in \mathcal{A}$ we have

$$\begin{aligned}
(n+1)\tilde{d}_{n+1}(a+\ker(d_0)) &= (n+1)d_{n+1}(a) \\
&= \sum_{i=2}^{n+1}\left(\sum_{\sum_{j=1}^{i} r_j = n+1} (n+1)a_{r_1,\dots,r_i}\delta_{r_1}\dots\delta_{r_i}d_0\right)(a) \\
&\quad +\delta_{n+1}d_0(a) \\
&= \sum_{i=2}^{n+1}\left(\sum_{r_1=1}^{n+2-i}\delta_{r_1}\sum_{\sum_{j=2}^{i} r_j = n+1-r_1} a_{r_2,\dots,r_i}\delta_{r_2}\dots\delta_{r_i}d_0\right)(a) \\
&\quad +\delta_{n+1}d_0(a) \\
&= \sum_{r_1=1}^{n}\delta_{r_1}\sum_{i=2}^{n-(r_1-2)}\left(\sum_{\sum_{j=2}^{i} r_j = n-(r_1-2)} a_{r_2,\dots,r_i}\delta_{r_2}\dots\delta_{r_i}d_0\right)(a) \\
&\quad +\delta_{n+1}d_0(a) \\
&= \sum_{r_1=1}^{n}\delta_{r_1}d_{n-(r_1-1)}(a) + \delta_{n+1}d_0(a) \\
&= \sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}(a+\ker(d_0)).
\end{aligned}$$

We therefore have $(n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}$. □

Theorem 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras (associative or non-associative), $\{f_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a higher derivations from $\mathcal{A}$ into $\mathcal{B}$ with $f_0(\mathcal{A}) = \mathcal{B}$ and $\ker(f_0) \subseteq \ker(f_n)$ $(n \in \mathbb{N})$ and $\{\gamma_n\}$ be the sequence of derivations such that

$$f_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j+\dots+r_i}\right)\gamma_{r_1}\dots\gamma_{r_i}f_0\right).$$

Suppose also that $\sigma : \mathcal{A} \to \mathcal{B}$ be a surjective generalized homomorphism associated with f_0, i.e., $\sigma(a_1a_2) = \sigma(a_1)f_0(a_2)$, Δ is the set of all sequences $\{\delta_n\}_{n\in\mathbb{N}}$ of generalized derivations on $\mathcal{B}$ such that $\delta_n(b_1b_2) = \delta_n(b_1)b_2 + b_1\gamma_n(b_2)$ for each $b_1, b_2 \in \mathcal{B}$ and D be the set of all generalized higher derivations $\{d_n\}_{n\in\mathbb{N}\cup\{0\}}$ from $\mathcal{A}$ into $\mathcal{B}$ with $d_0 = \sigma$, $\ker(d_0) \subseteq \ker(d_n)$ $(n \in \mathbb{N})$ and $d_n(a_1a_2) = \sum_{k=0}^{n} d_k(a_1)f_{n-k}(a_2)$ for each $a_1, a_2 \in \mathcal{A}$. Then there is a one to one correspondence $\varphi : \Delta \to D$ defined by $\varphi(\{\delta_n\}) = \{d_n\}$, where

$$d_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \ldots + r_i}\right)\delta_{r_1}\ldots\delta_{r_i}d_0\right). \tag{2.1}$$

Proof. Let $\{\delta_n\} \in \Delta$ and $\{\gamma_n\}$ is the sequence of derivations on $\mathcal{B}$ such that $\delta_n(b_1b_2) = \delta_n(b_1)b_2 + b_1\gamma_n(b_2)$ for each $b_1, b_2 \in \mathcal{B}$. Define $d_n : \mathcal{A} \to \mathcal{B}$ by $d_0 = \sigma$ and

$$d_n = \sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i} r_j = n}\left(\prod_{j=1}^{i}\frac{1}{r_j + \ldots + r_i}\right)\delta_{r_1}\ldots\delta_{r_i}d_0\right).$$

We show that $\{d_n\} \in D$. By Lemma 1.1 the sequences $\{\tilde{d}_n\}$ and $\{\tilde{f}_n\}$ satisfy the following relations

$$(r+1)\tilde{f}_{r+1} = \sum_{k=0}^{r}\gamma_{k+1}\tilde{f}_{r-k} \quad , \quad (n+1)\tilde{d}_{n+1} = \sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}.$$

To show that $\{d_n\}$ is a generalized higher derivation we use induction on n. For $n = 0$ we have $d_0(a_1a_2) = \sigma(a_1a_2) = d_0(a_1)f_0(a_2)$. Let us assume $d_k(a_1a_2) = \sum_{i=0}^{k} d_i(a_1)f_{k-i}(a_2)$ for $k \le n$. Thus we have

$$\begin{aligned}
& (n+1)d_{n+1}(a_1a_2) \\
= \ & (n+1)\tilde{d}_{n+1}(a_1a_2 + \ker(d_0)) \\
= \ & \sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}(a_1a_2 + \ker(d_0)) \\
= \ & \sum_{k=0}^{n}\delta_{k+1}d_{n-k}(a_1a_2) \\
= \ & \sum_{k=0}^{n}\delta_{k+1}\sum_{i=0}^{n-k}d_i(a_1)f_{n-k-i}(a_2) \\
= \ & \sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}\delta_{k+1}d_{n-k-i}(a_1)\right)f_i(a_2) + \sum_{i=0}^{n}d_i(a_1)\left(\sum_{k=0}^{n-i}\gamma_{k+1}f_{n-k-i}(a_2)\right) \\
= \ & \sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}\delta_{k+1}\tilde{d}_{n-k-i}(a_1 + \ker(d_0))\right)f_i(a_2) \\
& + \sum_{i=0}^{n}d_i(a_1)\left(\sum_{k=0}^{n-i}\gamma_{k+1}\tilde{f}_{n-k-i}(a_2 + \ker(d_0))\right).
\end{aligned}$$

Using our assumption, we can write

$$\begin{aligned}
& (n+1)d_{n+1}(a_1a_2) \\
= & \sum_{i=0}^{n}(n-i+1)\tilde{d}_{n-i+1}(a_1+\ker(d_0))f_i(a_2)+\sum_{i=0}^{n}d_i(a_1)(n-i+1)\tilde{f}_{n-i+1}(a_2+\ker(d_0)) \\
= & \sum_{i=0}^{n}(n-i+1)d_{n-i+1}(a_1)f_i(a_2)+\sum_{i=0}^{n}d_i(a_1)(n-i+1)f_{n-i+1}(a_2) \\
= & \sum_{i=1}^{n+1}id_i(a_1)f_{n+1-i}(a_2)+\sum_{i=0}^{n}(n+1-i)d_i(a_1)f_{n+1-i}(a_2) \\
= & (n+1)\sum_{k=0}^{n+1}d_k(a_1)f_{n+1-k}(a_2).
\end{aligned}$$

Thus $\{d_n\}\in D$. Note that for each $n\in\mathbb{N}$, $\ker(d_0)\subseteq\ker(d_n)$.

Conversely, suppose that $\{d_n\}\in D$. Define $\delta_n:\mathcal{B}\to\mathcal{B}$ by $\delta_1=\tilde{d}_1\tilde{d}_0^{-1}$ and

$$\delta_n=\left[n\tilde{d}_n-\sum_{k=0}^{n-2}\delta_{k+1}\tilde{d}_{n-1-k}\right]\tilde{d}_0^{-1}\qquad(n\geqslant 2).$$

Then Theorem 2.3 ensures us that $\{\delta_n\}\in\Delta$.

Now define $\varphi:\Delta\to D$ by $\varphi(\{\delta_n\})=\{d_n\}$, where

$$d_n=\sum_{i=1}^{n}\left(\sum_{\sum_{j=1}^{i}r_j=n}\left(\prod_{j=1}^{i}\frac{1}{r_j+\ldots+r_i}\right)\delta_{r_1}\ldots\delta_{r_i}d_0\right). \tag{2.2}$$

Then φ is clearly surjective, we show that it is injective. Let $\{d_n\}=\varphi(\{\delta_n\})=\varphi(\{\delta'_n\})=\{d'_n\}$. We use induction on n. For $n=1$ we have

$$\delta_1=\tilde{d}_1\tilde{d}_0^{-1}=\tilde{d'}_1\tilde{d'}_0^{-1}=\delta'_1$$

Now suppose that $\delta_k=\delta'_k$ for $k\leqslant n$. Similar to the proof of Theorem 2.3 the following relations is obtained

$$(n+1)\tilde{d}_{n+1}=\sum_{k=0}^{n}\delta_{k+1}\tilde{d}_{n-k}\quad,\quad(n+1)\tilde{d'}_{n+1}=\sum_{k=0}^{n}\delta'_{k+1}\tilde{d'}_{n-k}$$

for $n\geqslant 0$. So

$$\delta_{n+1}=[(n+1)\tilde{d}_{n+1}-\sum_{k=0}^{n-1}\delta_{k+1}\tilde{d}_{n-k}]\tilde{d_0}^{-1}=[(n+1)\tilde{d'}_{n+1}-\sum_{k=0}^{n-1}\delta'_{k+1}\tilde{d'}_{n-k}]\tilde{d'_0}^{-1}=\delta'_{n+1}.$$

Thus $\{\delta_n\}=\{\delta'_n\}$ and φ is one to one.

□

Remark 2.5. Modifying the above arguments we have similar results concerning Jordan higher derivations and generalized Jordan higher derivations.

Acknowledgments

This research was supported by a grant from Ferdowsi University of Mashhad; No. MP89141MIZ. The authors wish to acknowledge the referees for their valuable comments and suggestions.

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