A CLASS OF STOCHASTIC RUNGE-KUTTA METHODS FOR WEAK APPROXIMATION OF ITÔ SDE

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ABSTRACT. In the present paper,a class of explicit stochastic Runge-Kutta (SRK) methods for Itô stochastic equation systems w.r.t m-dimentional wiener processes satisfying a commutativity condition is developed. General conditions for the coefficients of the SRK methods assuring convergence with order two in the weak sense are presented. Due to the commutativity condition, no correlated random variables have to be generated for the considered Runge-Kutta methods.

1. Introduction

In many disciplines like engineering or mathematical finance, dynamical systems disturbed by random effects are described by stochastic differential equations (SDEs).Because such differential equations cannot usually be solved analytically, so numerical methods are required and should be designed to perform with a certain order of accuracy.
The paper is organized as follows:In section2,weak approximation is defined, and in section3, a class of SRK methods is introduced.Further more coefficiants for explicit second order SRK schemes are presented.Then,it closes with a numerical example in section4.

2. Weak approximation

We consider a probability space $(\Omega, \Re, P)$ with a filtration $(\Re_t)_{t\geq 0}$.We denote by $(X_t)_{t\in I}$ the solution of the d-dimentional Itô SDE defined by

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \qquad X_{t_0} = x_0, \tag{2.1}$$

with an m-dimentional Wiener process $(W_t)_{t\geq 0}$ and $I = [t_0, T]$.
We assume that Borel-measurable coefficients $a : I \times R^d \to R^d$ and $b : I \times R^d \to R^{d\times m}$ satisfy

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a Lipschitz and a linear growth condition such that the Existence and Uniqueness Theorem[] applied.In the following, let $b^j(t,x) = (b^{i,j}(t,x))_{1\leq i\leq d} \in R^d$ denote the jth column of the diffusion matrix $b(t,x)$ for $j = 1,...,m$.Let a discretization $I_h = t_0, t_1, ..., t_N$ with $t_0 < t_1 < ... < t_N = T$ of the time interval $I = [t_0, T]$ with step sizes $h_n = t_{n+1} - t_n$ for $n = 0, 1, ..., N-1$ be given.Further, define $h = max_{0\leq n\leq N} h_n$ as the maximum step size.Let $C_P^l(R^d, R)$ denote the space of all $g \in C^l(R^d, R)$ fulfilling a polynomial growth condition and let $g \in C_P^{k,l}(I \times R^d, R)$ if $g(.,x) \in C^k(I \times R^d, R)$ and $g(t,.) \in C_P^l(R^d, R)$ for all $t \in I$ and $x \in R^d$.

Definition 2.1. An approximation process Y converges weakly with order p to X as $h \to 0$ at time T if for each functional $f \in C_P^{2(p+1)}(R^d, R)$ exists a constant c_f, which does not depend on h, and a finite $h_0 > 0$ such that

$$|E(f(X_T)) - E(f(Y_T))| \leq c_f h^p \tag{2.2}$$

holds for each $h \in]0, h_0[$.

3. stochastic Runge-Kutta methods

We introduce a class of second order SRK methods for the weak approximation of the solution of the Itô SDE (2.1).We define the d-dimensional approximation process Y with $Y_n = Y(t_n)$ for $t_n \in I$ by the following SRK method with $Y_0 = x_0$ and

$$\begin{aligned} Y_{n+1} = Y_n &+ \sum_{i=1}^{s} \alpha_i a(t_n + c_i^{(0)}, H_i^{(0)}) h_n \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(1)} b^k(t_n + c_i^{(1)}, H_i^{(k)}) \hat{I}_{(k)} \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(2)} b^k(t_n + c_i^{(1)}, H_i^{(k)}) \frac{\hat{I}_{(k,k)}}{\sqrt{h_n}} \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(3)} b^k(t_n + c_i^{(2)}, H_i^{(k)}) \hat{I}_{(k)} \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \beta_i^{(4)} b^k(t_n + c_i^{(2)}, H_i^{(k)}) \sqrt{h_n} \end{aligned} \tag{3.1}$$

for $n = 0, 1, ..., N-1$ with stage values

$$\begin{aligned} H_i^{(0)} &= Y_n + \sum_{j=1}^{s} A_{ij}^{(0)} a(t_n + c_j^{(0)} h_n, H_j^{(0)}) h_n \\ &\quad + \sum_{j=1}^{s}\sum_{l=1}^{m} B_{ij}^{(0)} b^l(t_n + c_j^{(1)} h_n, H_j^{(l)}) \hat{I}_l, \\ H_i^{(k)} &= Y_n + \sum_{j=1}^{s} A_{ij}^{(1)} a(t_n + c_j^{(0)} h_n, H_j^{(0)}) h_n \\ &\quad + \sum_{j=1}^{s} B_{ij}^{(1)} b^k(t_n + c_j^{(1)} h_n, H_j^{(k)}) \sqrt{h_n}, \end{aligned}$$

$$\hat{H}_i^{(k)} = Y_n + \sum_{j=1}^{s} A_{ij}^{(2)} a(t_n + c_j^{(0)} h_n, H_j^{(0)}) h_n$$

$$+ \sum_{j=1}^{s} \sum_{l=1, l\neq k}^{m} B_{ij}^{(1)} b^l (t_n + c_j^{(1)} h_n, H_j^{(l)}) \frac{\hat{I}_{(k,l)}}{\sqrt{h_n}},$$

for the $i = 1, ..., s$ and $k = 1, ..., m$.

The coefficients of such a method can be represented by the usual Butcher-Arrays which take the form

$c^{(0)}$	$A^{(0)}$	$B^{(0)}$	
$c^{(1)}$	$A^{(1)}$	$B^{(1)}$	
$c^{(2)}$	$A^{(2)}$	$B^{(2)}$	
	α^T	${\beta^{(1)}}^T$	${\beta^{(2)}}^T$
		${\beta^{(3)}}^T$	${\beta^{(4)}}^T$

Applying the rooted tree[4] analysis and to all rooted trees up to order 2.5, we can calculate the following complete order two conditions for the SRK methods (2.1).

Theorem 3.1. *if coefficients of the SRK methods (2.1) fulfill the equations*
1)$\alpha^T e = 1$ 2)${\beta^{(4)}}^T e = 0$ 3)${\beta^{(3)}}^T e = 0$
4)$({\beta^{(1)}}^T e)^2 = 1$ 5)${\beta^{(2)}}^T e = 0$ 6)${\beta^{(1)}}^T B^{(1)} e = 0$
7)${\beta^{(4)}}^T A^{(2)} e = 0$ 8)${\beta^{(3)}}^T B^{(2)} e = 0$ 9)${\beta^{(4)}}^T (B^{(2)} e)^2 = 0$ *then the method converges with order 1.0 in the weak sense.In addition,if the equations*
10)$\alpha^T A^{(0)} e = \frac{1}{2}$ 11)$\alpha^T (B^{(0)} e)^2 = \frac{1}{2}$
12)$({\beta^{(1)}}^T e)(\alpha^T B^{(0)} e) = \frac{1}{2}$ 13)$({\beta^{(1)}}^T e)({\beta^{(1)}}^T A^{(1)} e) = \frac{1}{2}$
14)${\beta^{(3)}}^T A^{(2)} e = 0$ 15)${\beta^{(2)}}^T B^{(1)} e = 1$
16)${\beta^{(4)}}^T B^{(2)} e = 1$ 17)$({\beta^{(1)}}^T e)({\beta^{(1)}}^T (B^{(1)} e)^2) = \frac{1}{2}$
18)$({\beta^{(1)}}^T e)({\beta^{(3)}}^T (B^{(2)} e)^2) = \frac{1}{2}$ 19)${\beta^{(1)}}^T (B^{(1)}(B^{(1)} e)) = 0$
20)${\beta^{(3)}}^T (B^{(2)}(B^{(1)} e)) = 0$ 21)${\beta^{(3)}}^T (B^{(2)}(B^{(1)}(B^{(1)} e))) = 0$
22)${\beta^{(1)}}^T (A^{(1)}(B^{(0)} e)) = 0$ 23)${\beta^{(3)}}^T (A^{(2)}(B^{(0)} e)) = 0$
24)${\beta^{(4)}}^T (A^{(2)} e)^2 = 0$ 25)${\beta^{(4)}}^T (A^{(2)}(A^{(0)} e)) = 0$
26)$\alpha^T (B^{(0)}(B^{(1)} e)) = 0$ 27)${\beta^{(2)}}^T A^{(1)} e = 0$
28)${\beta^{(1)}}^T (A^{(1)} e)(B^{(1)} e)) = 0$ 29)${\beta^{(3)}}^T (A^{(2)} e)(B^{(2)} e)) = 0$
30)${\beta^{(4)}}^T (A^{(2)}(B^{(0)} e)) = 0$ 31)${\beta^{(2)}}^T (A^{(1)}(B^{(0)} e)) = 0$
32)${\beta^{(4)}}^T ((B^{(2)} e)^2 (A^{(2)} e)) = 0$ 33)${\beta^{(4)}}^T (A^{(2)}(B^{(0)} e)^2) = 0$
34)${\beta^{(2)}}^T (A^{(1)}(B^{(0)} e)^2) = 0$ 35)${\beta^{(1)}}^T (B^{(1)}(A^{(1)} e)) = 0$
36)${\beta^{(3)}}^T (B^{(2)}(A^{(1)} e)) = 0$ 37)${\beta^{(2)}}^T (B^{(1)} e)^2 = 0$
38)${\beta^{(4)}}^T (B^{(2)}(B^{(1)} e)) = 0$ 39)${\beta^{(2)}}^T (B^{(1)}(B^{(1)} e)) = 0$
40)${\beta^{(1)}}^T (B^{(1)} e)^3 = 0$ 41)${\beta^{(3)}}^T (B^{(2)} e)^3 = 0$
42)${\beta^{(1)}}^T (B^{(1)}(B^{(1)} e)^2) = 0$ 43)${\beta^{(3)}}^T (B^{(2)}(B^{(1)} e)^2) = 0$
44)${\beta^{(4)}}^T (B^{(2)} e)^4 = 0$ 45)${\beta^{(4)}}^T (B^{(2)}(B^{(1)} e))^2 = 0$
46)${\beta^{(4)}}^T ((B^{(2)} e)(B^{(2)}(B^{(1)} e))) = 0$ 47)$\alpha^T ((B^{(0)} e)(B^{(0)}(B^{(1)} e))) = 0$
48)${\beta^{(1)}}^T ((A^{(1)}(B^{(0)} e))(B^{(1)} e)) = 0$ 49)${\beta^{(3)}}^T ((A^{(2)}(B^{(0)} e))(B^{(2)} e)) = 0$
50)${\beta^{(1)}}^T (A^{(1)}(B^{(0)}(B^{(1)} e))) = 0$ 51)${\beta^{(3)}}^T (A^{(2)}(B^{(0)}(B^{(1)} e))) = 0$
52)${\beta^{(4)}}^T ((B^{(2)}(A^{(1)} e))(B^{(2)} e)) = 0$ 53)${\beta^{(1)}}^T (B^{(1)}(A^{(1)}(B^{(0)} e))) = 0$
54)${\beta^{(3)}}^T (B^{(2)}(A^{(1)}(B^{(0)} e))) = 0$ 55)${\beta^{(1)}}^T ((B^{(1)} e)(B^{(1)}(B^{(1)} e))) = 0$

56)$\beta^{(3)^T}((B^{(2)}e)(B^{(2)}(B^{(1)}e))) = 0$ 57)$\beta^{(1)^T}(B^{(1)}(B^{(1)}(B^{(1)}e))) = 0$

58)$\beta^{(4)^T}((B^{(2)}e)(B^{(2)}(B^{(1)}(B^{(1)}e)))) = 0$ 59)$\beta^{(4)^T}((B^{(2)}e)(B^{(2)}(B^{(1)}e)^2)) = 0$

are fulfilled,then the SRK method (2.1) converges with order 2.0 in the weak sense.

Remark 3.2. Due to Theorem 3.1, we have to solve 59 equations for $m > 1$.However, in the case of $m = 1$ and if we choose $A^{(2)}_{ij} = 0$ for $1 \leq i, j \leq s$, then the 59 conditions reduce to 28 conditions. For an explicit SRK method of type (3.1), $s \geq 3$ is needed due to conditions 4,6,and 17,which cannot be fulfilled for $s < 3$.

Considering the order conditions 1-9 of Theorem 3.1,we can easily calculate order two SRK methods converging with order one in the weak sense.For example, the well-known Euler-Maruyama scheme EM belongs to the introduced class of SRK methods having order 1 with $s = 1$ stage and with coefficients $\alpha_1 = \beta^{(1)}_1 = 1, \beta^{(2)}_1 = \beta^{(3)}_1 = \beta^{(4)}_1 = 0, A^{(0)}_{11} = A^{(1)}_{11} = 0$, and $B^{(0)}_{11} = B^{(1)}_{11} = 0$. Further,if we calculate order two SRK methods with $s \geq 3$ stages,there are some degrees of freedom in choosing the coefficients.Especially,it is possible to calculate an SRK methods converging with some higher order if it applied to a deterministic ordinary differential equation.For example,if the weights α_i and the coefficients $A^{(0)}_{ij}$ are fulfilled conditions $\alpha^T(A^{(0)}(A^{(0)}e)) = \frac{1}{6}$ and $\alpha^T(A^{(0)}e)^2 = \frac{1}{3}$,then the SRK method is of order three in the case of $b^j \equiv 0$ for $1 \leq j \leq m$ in SDE (2.1).Therefore,let (p_D, p_S) with $p_D \geq p_S$ denote the order of convergence of the SRK method.

The SRK method RI5 with $p_D = 3$ and $p_S = 2$ presented in Table1.While the SRK scheme RI6 with $p_D = 2$ and $p_S = 2$ presented in Table2.

Table1:SRK method RI5

0									
1	1			$\frac{1}{3}$					
$\frac{5}{12}$	$\frac{25}{144}$	$\frac{35}{144}$		$\frac{-5}{6}$	0				
0									
$\frac{1}{4}$	$\frac{1}{4}$			$\frac{1}{2}$					
$\frac{1}{4}$	$\frac{1}{4}$	0		$\frac{-1}{2}$	0				
0									
0	0			1					
0	0	0		-1	0				
	$\frac{1}{10}$	$\frac{3}{14}$	$\frac{24}{35}$	1	-1	-1	0	1	-1
				$\frac{1}{2}$	$\frac{-1}{4}$	$\frac{-1}{4}$	0	$\frac{1}{2}$	$\frac{-1}{2}$

Table2:SRK method RI6

0									
1	1			1					
0	0	0		0	0				
0									
1	1			1					
1	1	0		-1	0				
0									
0	0			1					
0	0	0		-1	0				
	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{-1}{2}$
				$\frac{-1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{-1}{2}$

4. Simulation study

In the following, the two SRK methods RI5 and RI6 are compared to the Euler-Maruyama scheme (EM) having order 1.
We approximate $E(f(X_T))$ by Monte Carlo simulation.Therefore,we estimate $E(f(Y_T))$ by the sample average of M independently simulated realizations of the approximations $f(Y_{T,k})$,$k = 1, ..., M$,with $Y_{T,k}$ calculated by the scheme under consideration.Then the error is denoted by

$$\hat{\mu} = E(f(X_T)) - \frac{1}{M} \sum_{k=1}^{M} f(Y_{T,k}) \tag{4.1}$$

References

1. A.Rossler, *Runge-Kutta methods for itô stochastic differential equations with scalar noise* , BIT., **46** (2006) 97-110.
2. A.Rossler, *Runge-Kutta methods for the Numerical Solution of Stochastic Differential Equations* , Ph.D.thesis.Darmstadt University of Technology.Shaker Verlag.Aachen,2003.
3. P.E.KLOEDEN and E.PLATEN, *Numerical Solution of Stochastic Differential Equations*, Springer-Veriag,Berlin, 1999.
4. A.Rossler, *Rooted tree analysis for order conditions of stochastic Runge-Kutta methods for the weak approximations of stochastic differential equations* , Stochastic Anal.Appl., **24** (2006) 97-134.
5. K.Debrabant and A.Rossler, *Classification of stochastic Runge-Kutta methods for the weak approximation of stochastic differential equations*, Math.Comput.Simulation., **77** (2008) 408-420.

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