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NORM CONTINUITY OF WEAKLY QUASI-CONTINUOUS MAPPINGS

BY

ALIREZA KAMEL MIRMOSTAFAEE (Mashhad)

Abstract. Let $\mathcal{Q}$ be the class of Banach spaces X for which every weakly quasi-continuous mapping $f : A \to X$ defined on an α-favorable space A is norm continuous at the points of a dense G_δ subset of A. We will show that this class is stable under c_0-sums and ℓ^p-sums of Banach spaces for $1 \le p < \infty$.

1. Introduction. In 1974, I. Namioka [16] proved that every weakly continuous mapping f from a countably Čech-complete space A into a Banach space X is norm continuous at the points of a dense G_δ subset of A. It was conjectured that Namioka's result remains valid for any α-favorable space A. However, in 1985, M. Talagrand [19] gave an example of a weakly continuous nowhere norm continuous mapping defined on an α-favorable space. Therefore the following problem naturally arises:

Under what conditions on a Banach space X, every weakly quasi-continuous mapping from an α-favorable A into X is norm continuous at each point of a dense G_δ subset of A?

During the past four decades similar problems have been considered by several mathematicians: see e.g. [1, 4, 11, 12], [14]–[19] and the references therein.

A Banach space X is said to have the *property* $\mathcal{Q}$ if every quasi-continuous mapping f defined on an α-favorable space A into (X, weak) is norm-continuous on a dense G_δ subset of A. It is known that ℓ^∞ and ℓ^∞/c_0 do not have the property $\mathcal{Q}$ [9]. However, the class of Banach spaces with the property $\mathcal{Q}$ properly contains all Banach spaces which are weakly σ-fragmentable [5, 8]. It follows that this class includes all weakly Lindelöf Banach spaces and Banach spaces with an equivalent Kadec norm [6, 9, 13].

In [8] and [9] a game characterization of Banach spaces X with the property $\mathcal{Q}$ was obtained. In fact it was shown that the absence of winning

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strategy for one of the players in the fragmenting game on a Banach space X guarantees the property $\mathcal{Q}$, and that ℓ^∞ is not in the class $\mathcal{Q}$.

In this paper, we use this characterization to show that if each member of a family $\{X_\gamma\}_{\gamma\in\Gamma}$ of Banach spaces satisfies the property $\mathcal{Q}$, then so do the c_0-sum and ℓ^p-sum of the family for $1 \le p < \infty$.

2. Preliminaries. Let A and X be topological spaces. Following Kempisty [7], a mapping $f : A \to X$ is said to be *quasi-continuous at a point* $a_0 \in A$ if for every open neighborhood U of $f(a_0)$, there exists an open set $V \subset A$ such that $a_0 \in \overline{V}$ (the closure of V in A) and $f(V) \subset U$.

The mapping f is called *quasi-continuous* if it is quasi-continuous at each point of A.

Next, we introduce the following topological game on a topological space A, which is a version of the classical Banach–Mazur game [2, 3]:

Two players α and β alternately select nonempty open subsets of A. Player β begins the game by selecting a nonempty open set $V_1 \subset A$. In response, α selects some nonempty open subset $W_1 \subset V_1$. Inductively, player β's nth move is a nonempty open subset $V_n \subset W_{n-1}$ followed by α's nth move, a nonempty open subset W_n of V_n. Proceeding in this fashion, the players generate a sequence $(V_n, W_n)_{n=1}^\infty$ which is called a *play*. Player α *wins* the play $(V_n, W_n)_{n=1}^\infty$ if $\bigcap_{n\ge1} V_n = \bigcap_{n\ge1} W_n \neq \emptyset$. Otherwise β wins. A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* s for player α is a rule which determines α's move at each stage based on the game played so far. An *s-play* is a play in which α selects his moves according to the strategy s. The strategy s is said to be a *winning strategy* for the player α if every s-play is won by α. A strategy for β can be defined similarly by switching the sides. A is called *α-favorable* if there exists a winning strategy for α in A.

Krom [10] and Raymond [18] have shown independently that a topological space is Baire if and only if it is β-unfavorable. Hence every α-favorable space is a Baire space.

Let τ and τ' be two topologies on a set X. The topological game $G(X, \tau, \tau')$ is played by two players Σ and Ω as follows:

Σ starts a game by taking a nonempty subset A_1 of X. Then Ω selects a nonempty relatively τ-open subset B_1 of A_1. In general if the selection B_n of player Ω is already specified, Σ makes the next move by choosing an arbitrary nonempty set A_{n+1} contained in B_n. Continuing, the players produce a sequence of nonempty sets $A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq B_n \supseteq \cdots$, which is called a *play* and will be denoted by $p := (A_i, B_i)_{i\ge1}$. The winning rule is connected with the topology τ'. Player Ω is said to *win* a play $p := (A_i, B_i)_{i\ge1}$ if the set $\bigcap_{n\ge1} A_n$ is either empty or contains exactly one point

x and for every τ'-open neighborhood U of x, there is some positive n with $B_n \subset U$. Otherwise Σ wins.

A *partial play* is a finite sequence which consists of the first few moves $A_1 \supseteq B_1 \supseteq A_2 \supseteq \cdots \supseteq B_n$ of the players. A strategy for either of the players in $G(X, \tau, \tau')$ can be defined as in the Banach–Mazur game.

The game $G(X, \tau, \tau')$ (or the space X) is called *Σ-unfavorable* if there does *not* exist a winning strategy for player Σ.

We have the following connection between the topological game and dense ~~τ'-continuity~~ of τ-quasi-continuous mappings:

THEOREM 2.1 ([8, 9]). *Let τ, τ' be two T_1 topologies on a set X. Suppose that for every τ'-open set U and every point $x \in U$ there exists a τ'-neighborhood V of x such that $\overline{V}^{\tau} \subset U$. Then the following conditions are equivalent:*

(i) *The game $G(X, \tau, \tau')$ is Σ-unfavorable.*
(ii) *Every quasi-continuous mapping $f : Z \to (X, \tau)$ from an α-favorable space Z into (X, τ) is τ'-continuous at all points of a subset which is of second category in every nonempty open subset of Z.*

In particular, when τ' is a metrizable topology, the set of τ'-continuity points is a dense G_δ subset of Z.

Let X be a Banach space. By applying the above result when τ' is the norm topology and τ is the weak topology on X, we get the following result:

COROLLARY 2.2. *The following assertions are equivalent:*

(a) *X does not have the property $\mathcal{Q}$.*
(b) *There exists a strategy σ for player Σ in the game $G(X, \mathrm{weak}, \|\cdot\|)$ such that for each σ-play $(A_i, B_i)_{i \geq 1}$,*

$$\bigcap_{n \geq 1} A_n \neq \emptyset \quad \textit{and} \quad \text{norm-diam}(A_n) > \varepsilon \textit{ for each } n \in \mathbb{N}$$

for some $\varepsilon > 0$.

3. c_0-sums of Banach spaces and the property $\mathcal{Q}$. Let $\{(X_\gamma, \|\cdot\|_\gamma) : \gamma \in \Gamma\}$ be a family of Banach spaces. The *c_0-sum* of this family, denoted by $c_0\{X_\gamma : \gamma \in \Gamma\}$, is the set of all $x \in \prod_{\gamma \in \Gamma} X_\gamma$ such that for each $\varepsilon > 0$, the set $\{\gamma : \|x(\gamma)\|_\gamma \geq \varepsilon\}$ is finite. This set equipped with the norm

$$\|x\|_\infty = \sup\{\|x(\gamma)\|_\gamma : \gamma \in \Gamma\}$$

is a Banach space. Throughout this section, we will assume that X is the Banach space $c_0\{X_\gamma : \gamma \in \Gamma\}$, where $\{(X_\gamma, \|\cdot\|_\gamma) : \gamma \in \Gamma\}$ is a family of Banach spaces.

LEMMA 3.1. *Given $\varepsilon > 0$, player Ω has a strategy s in X such that for every s-play $(A_i, B_i)_{i\geq 1}$ either $\bigcap_{i=1}^{\infty} A_i = \emptyset$ or for some $n_0 \in \mathbb{N}$ and a finite subset $F \subset \Gamma$,*

$$\{\gamma \in \Gamma : \|x(\gamma)\| > \varepsilon\} \subset F \quad \textit{for all } x \in A_{n_0}. \tag{3.1}$$

Proof. Let A_1 be the first choice of player Σ. Then the following cases may happen:

(i$_1$) For each $x \in A_1$ and $\gamma \in \Gamma$, $\|x(\gamma)\|_\gamma \leq \varepsilon$. In this case put $F_1 = \emptyset$ and define $B_1 = s(A_1) = A_1$.

(ii$_1$) For some $x_1 \in A_1$ and $\gamma_1 \in \Gamma$, $\|x_1(\gamma_1)\|_{\gamma_1} > \varepsilon$. Then define

$$B_1 = s(A_1) = \{x \in A_1 : \|x(\gamma_1)\|_{\gamma_1} > \varepsilon\}$$

as the first move of Ω and let $F_1 = \{\gamma_1\}$.

In step $n \geq 2$, when the partial play $p_n = (A_1, \ldots, A_n)$ and finite subsets $F_1 \subseteq \cdots \subseteq F_{n-1}$ of Γ have already been selected, we consider the following possibilities:

(i$_n$) For each $x \in A_n$,

$$\{\gamma \in \Gamma : \|x(\gamma)\|_\gamma > \varepsilon\} \subset F_{n-1}.$$

In this situation, we define $s(A_1, \ldots, A_n) = B_n$ and $F_n = F_{n-1}$.

(ii$_n$) There are some $x_n \in A_n$ and $\gamma_n \in \Gamma - F_{n-1}$ such that $\|x_n(\gamma_n)\|_{\gamma_n} > \varepsilon$. Let

$$B_n = s(A_1, \ldots, A_n) = \{x \in A_n : \|x(\gamma_n)\|_{\gamma_n} > \varepsilon\}$$

be the next move of Ω and define $F_n = F_{n-1} \cup \{\gamma_n\}$.

In this way, by induction on n, a strategy s for player Ω is defined. If for some $n_0 \in \mathbb{N}$, (i$_{n_0}$) is satisfied, then for $F = F_{n_0}$, (3.1) holds. Suppose that for each $n \in \mathbb{N}$, (ii$_n$) holds. We will show that $\bigcap_{n\geq 1} A_n = \emptyset$. On the contrary, let $x \in \bigcap_{n\geq 1} A_n$. Define $F_x = \{\gamma \in \Gamma : \|x(\gamma)\|_\gamma > \varepsilon\}$. Since $x \in A_n$ for all $n \in \mathbb{N}$,

$$\|x(\gamma_n)\|_{\gamma_n} > \varepsilon \quad (n \in \mathbb{N}).$$

Therefore F_x contains the infinite set $\bigcup_{n\geq 1} F_n$. However, by the definition, F_x is finite. This contradiction shows that in this case $\bigcap_{n\geq 1} A_n = \emptyset$. ∎

LEMMA 3.2. *Suppose that $X_1, \ldots, X_n$ are Banach spaces with the property $\mathcal{Q}$. Then $G(\prod_{i=1}^{n} X_i, \mathrm{weak}, \|\cdot\|_\infty)$ is Σ-unfavorable.*

Proof. Let f be a quasi-continuous mapping from an α-favorable space A to $\prod_{i=1}^{n} X_i$. Then for each $1 \leq i \leq n$, $\pi_i \circ f : A \to X_i$ is quasi-continuous, where π_i denotes the canonical projection map to the ith coordinate. Since each X_i has the property $\mathcal{Q}$, there is a dense G_δ subset D_i of A such that

$\pi_i \circ f|_{D_i}$ is norm continuous. Define $D = \bigcap_{i=1}^n D_i$. Clearly f is $\|\cdot\|_\infty$-continuous on the dense G_δ set D. Hence the result follows from Theorem 2.1. ∎

THEOREM 3.3. *Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of Banach spaces with the property $\mathcal{Q}$. Then $c_0\{X_\gamma : \gamma \in \Gamma\}$ has the property $\mathcal{Q}$ as well.*

Proof. On the contrary, suppose that $c_0\{X_\gamma : \gamma \in \Gamma\}$ does not have the property $\mathcal{Q}$. Then by Corollary 2.2, player Σ has a strategy σ such that for each σ-play $(A_i, B_i)_{i\ge 1}$,

$$\bigcap_{n\ge 1} A_n \neq \emptyset \quad \text{and} \quad \text{norm-diam}(A_n) > \varepsilon \tag{3.2}$$

for some $\varepsilon > 0$ and all $n \ge 1$. By Lemma 3.1, Ω has a strategy s such that for each s-play $(A_i, B_i)_{i\ge 1}$, either $\bigcap_{n\ge 1} A_n = \emptyset$ or there is a finite subset F of Γ such that

$$\|x(\gamma)\|_\gamma \le \varepsilon \quad \text{for all } x \in A_{n_0} \text{ and } \gamma \in \Gamma - F. \tag{3.3}$$

However, by (3.2), $\bigcap_{n\ge 1} A_n \neq \emptyset$. Therefore, we may assume that (3.3) holds. We define a strategy s' for player Ω as follows:

For $1 \le n < n_0$, let $s'(A_1,\dots,A_n) = s(A_1,\dots,A_n)$. Suppose that $n \ge n_0$ and the partial play $p_n = (A_1,\dots,A_n)$ is specified. Let $\pi_F : c_0\{X_\gamma : \gamma \in \Gamma\} \to \prod_{\gamma\in F} X_\gamma$ be the canonical projection $\pi_F(\{x_\gamma\}_{\gamma\in\Gamma}) = \{x_\gamma\}_{\gamma\in F}$. Choose a relatively open subset B_n of A_n such that $\pi_F(B_n)$ is the response of player Ω to the partial play $(\pi_F(A_1),\dots,\pi_F(A_n))$ according to the strategy whose existence is guaranteed by Lemma 3.2, and define $s'(A_1,\dots,A_n) = B_n$.

In this way, a strategy s' for Ω in $c_0\{X_\gamma : \gamma \in \Gamma\}$ is defined. By Lemma 3.2, $G(\prod_{\gamma\in F} X_\gamma, \text{weak}, \|\cdot\|_\infty)$ is Σ-unfavorable, hence there is a play $(\pi_F(A_n), \pi_F(B_n))_{n\ge 1}$ such that either $\bigcap_{n=1}^\infty \pi_F(A_n) = \emptyset$ or for some $n_0 \in \mathbb{N}$, $\|x(\gamma) - y(\gamma)\|_\gamma < \varepsilon$ for all $x, y \in A_{n_0}$ and $\gamma \in F$. Hence either $\bigcap_{n=1}^\infty A_n = \emptyset$ or $\|\cdot\|_\infty$-diam$(A_{n_0}) \le \varepsilon$, by (3.3). This contradiction proves the theorem. ∎

4. The property $\mathcal{Q}$ for ℓ^p-sums of Banach spaces. For $1 \le p < \infty$, we use $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ to denote the Banach space of all $x \in \prod_{\gamma\in\Gamma} X_\gamma$ for which the norm series

$$\|x\|_p = \Big\{\sum_{\gamma\in\Gamma} \|x(\gamma)\|_\gamma^p\Big\}^{1/p}$$

converges.

LEMMA 4.1. *Let $\{X_\gamma : \gamma \in \Gamma\}$ be a family of Banach spaces and $\varepsilon > 0$. Then player Ω has a strategy s in $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ such that for each s-play*

$(A_i, B_i)_{i\geq 1}$, *either* $\bigcap_{i=1}^{\infty} A_i = \emptyset$ *or there are some* $n_0 \in \mathbb{N}$ *and a finite subset* F *of* Γ *such that*

$$\sum_{\gamma\in\Gamma - F} \|x(\gamma)\|_\gamma^p \leq \varepsilon \quad \textit{for all } x \in A_{n_0}.$$

Proof. Let player Σ start a game with a nonempty subset A_1 of $\ell^p\{X_\gamma : \gamma \in \Gamma\}$. Then we distinguish the following two possibilities:

(i_1) For each $x \in A_1$, $\sum_{\gamma\in\Gamma} \|x(\gamma)\|_\gamma^p \leq \varepsilon$. In this case, put $F_1 = \emptyset$ and define $B_1 = A_1$ as the first choice of Ω.

(ii_1) There is $x_1 \in A_1$ such that $\sum_{\gamma\in\Gamma} \|x_1(\gamma)\|_\gamma^p > \varepsilon$. By the definition, there is a finite subset $F_1 \subset \Gamma$ such that $\sum_{\gamma\in F_1} \|x_1(\gamma)\|_\gamma^p > \varepsilon$. We can assume $x_1(\gamma) \neq 0$ for each $\gamma \in F_1$, and choose some $\delta_1 > 0$ such that $\|x_1(\gamma)\|_\gamma^p > \delta_1$ for all $\gamma \in F_1$ and $\sum_{\gamma\in F_1} \|x_1(\gamma)\|_\gamma^p > \varepsilon + n_1\delta_1$, where $|F_1| = n_1$. Define

$$\begin{aligned} B_1 &= s(A_1) \\ &= \{x \in A_1 : \|x(\gamma)\|_\gamma > (\|x_1(\gamma)\|_\gamma^p - \delta_1)^{1/p} \text{ for all } \gamma \in F_1\}. \end{aligned}$$

Then for each $x \in B_1$, we have

$$\sum_{\gamma\in F_1} \|x(\gamma)\|_\gamma^p > \sum_{\gamma\in F_1} \|x_1(\gamma)\|_\gamma^p - n_1\delta_1 > \varepsilon.$$

In step k, when $A_1, \ldots, A_k$ together with finite subsets $F_1, \ldots, F_{k-1}$ of Γ have already been specified, we consider the following possibilities:

(i_k) $\sum_{\gamma\in\Gamma-\bigcup_{i=1}^{k-1} F_i} \|x(\gamma)\|_\gamma^p \leq \varepsilon$ for each $x \in A_k$. In this situation, let $F_k = F_{k-1}$ and define $B_k = s(A_k) = A_k$ as the next move of player Ω.

(ii_k) There is some $x_k \in A_k$ such that $\sum_{\gamma\in\Gamma-\bigcup_{i=1}^{k-1} F_i} \|x_k(\gamma)\|_\gamma^p > \varepsilon$. By the definition, we can find a finite subset $F_k \subset \Gamma - \bigcup_{i=1}^{k-1} F_i$ such that $\sum_{\gamma\in F_k} \|x_k(\gamma)\|_\gamma^p > \varepsilon$. As before, we can assume that $x_k(\gamma) \neq 0$ for each $\gamma \in F_k$. Let $|F_k| = n_k$ and select $\delta_k > 0$ such that $\|x_k(\gamma)\|_\gamma^p > \delta_k$ for all $\gamma \in F_k$ and $\sum_{\gamma\in F_k} \|x_k(\gamma)\|_\gamma^p > \varepsilon + n_k\delta_k$. Define

$$\begin{aligned} B_k &= s(A_1, \ldots, A_k) \\ &= \{x \in A_k : \|x(\gamma)\|_\gamma > (\|x_k(\gamma)\|_\gamma^p - \delta_k)^{1/p} \text{ for all } \gamma \in F_k\} \end{aligned}$$

as the response of Ω to the partial play $(A_1, \ldots, A_k)$. A similar argument to the one in (ii_1) can be used to prove that for each $x \in B_k$, $\sum_{\gamma\in F_k} \|x(\gamma)\|_\gamma^p > \varepsilon$.

In this way, by induction on k, a strategy s for Ω is defined. The following two cases may happen:

(1) There is some $n_0 \in \mathbb{N}$ such that (i_{n_0}) happens. In this situation, for $F = \bigcup_{i=1}^{n_0} F_i$, we have

$$\sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \le \varepsilon \quad \text{for all } x \in A_{n_0}.$$

(2) For each $n \in \mathbb{N}$, (ii_n) holds. In this case, we claim that $\bigcap_{i=1}^{\infty} A_i = \emptyset$. On the contrary, suppose that $x \in \bigcap_{i=1}^{\infty} A_i$. Then for each $n \in \mathbb{N}$, we have

$$\sum_{\gamma \in \Gamma} \|x(\gamma)\|_\gamma^p \ge \sum_{\gamma \in \bigcup_{1 \le i \le n} F_i} \|x(\gamma)\|_\gamma^p = \sum_{i=1}^{n} \sum_{\gamma \in F_i} \|x(\gamma)\|_\gamma^p > n\varepsilon.$$

Hence $x \notin \ell^p\{X_\gamma : \gamma \in \Gamma\}$. This contradiction proves our claim in this case. ∎

The proof of the following lemma is similar to the proof of Lemma 3.2, hence it is omitted.

LEMMA 4.2. *Let $X_1, \ldots, X_n$ be Banach spaces with the property $\mathcal{Q}$. Then $G(\ell^p\{X_i : 1 \le i \le n\}, \mathrm{weak}, \|\cdot\|_p)$ is Σ-unfavorable.*

THEOREM 4.3. *If $\{X_\gamma : \gamma \in \Gamma\}$ is a family of Banach spaces with the property $\mathcal{Q}$, then $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ has the property $\mathcal{Q}$.*

Proof. Let $\varepsilon > 0$. By Lemma 4.1, player Ω has a strategy s such that for each s-play $(A_i, B_i)_{i \ge 1}$, either $\bigcap_{i \ge 1} A_i = \emptyset$ or there is $n_0 \in \mathbb{N}$ such that for some finite subset F of Γ,

$$\sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \le \frac{\varepsilon^p}{2^{p+1}} \quad \text{for all } x \in A_{n_0}. \tag{4.1}$$

According to Corollary 2.2, we may assume that (4.1) holds. We define a strategy s' for player Ω as follows:

For each $1 \le n < n_0$, let $s'(A_1, \ldots, A_n) = s(A_1, \ldots, A_n)$. Suppose that for $n \ge n_0$, $A_1, \ldots, A_n$ have been selected. Let $\pi_F : \ell^p\{X_\gamma : \gamma \in \Gamma\} \to \prod_{\gamma \in F} X_\gamma$ be the canonical map and $B_n = s'(A_1, \ldots, A_n)$ be a relatively open subset of A_n such that $\pi_F(B_n)$ is the answer of player Ω to $(\pi_F(A_1), \ldots, \pi_F(A_n))$ according to the strategy whose existence is guaranteed by Lemma 4.2. In this way, a strategy s' for Ω is determined.

By Lemma 4.2, $G(\ell^p\{X_\gamma : \gamma \in F\}, \mathrm{weak}, \|\cdot\|_p)$ is Σ-unfavorable, so that there is a play $(\pi_F(A_i), \pi_F(B_i))_{i \ge 1}$ such that either $\bigcap_{i \ge 1} \pi_F(A_i) = \emptyset$ or norm-diam $\pi_F(A_{n_0}) < \varepsilon / 2^{1/p}$. In the first case $\bigcap_{i \ge 1} A_i = \emptyset$. In the latter case for each $x, y \in A_{n_0}$, we have

$$\|x - y\|_p^p \le \sum_{\gamma \in F} \|x(\gamma) - y(\gamma)\|_\gamma^p + \sum_{\gamma \in \Gamma - F} \|x(\gamma) - y(\gamma)\|_\gamma^p. \tag{4.2}$$

Since for each $x, y \in A_{n_0}$,

$$\sum_{\gamma \in F} \|x(\gamma) - y(\gamma)\|_\gamma^p \leq \{\text{norm-diam}\, \pi_F(A_{n_0})\}^p < \frac{\varepsilon^p}{2}$$

and

$$\Big\{ \sum_{\gamma \in \Gamma - F} \|x(\gamma) - y(\gamma)\|_\gamma^p \Big\}^{1/p} \leq \Big\{ \sum_{\gamma \in \Gamma - F} \|x(\gamma)\|_\gamma^p \Big\}^{1/p} + \Big\{ \sum_{\gamma \in \Gamma - F} \|y(\gamma)\|_\gamma^p \Big\}^{1/p}$$
$$\leq \frac{\varepsilon}{2^{(p+1)/p}} + \frac{\varepsilon}{2^{(p+1)/p}} = \frac{\varepsilon}{2^{1/p}},$$

by (4.2), $\|x - y\|_p \leq \varepsilon$ for each $x, y \in A_{n_0}$. Therefore norm-diam$(A_{n_0}) \leq \varepsilon$. Corollary 2.2 implies that $\ell^p\{X_\gamma : \gamma \in \Gamma\}$ has the property $\mathcal{Q}$. ∎

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Alireza Kamel Mirmostafaee
Center of Excellence in Analysis on Algebraic Structures
Department of Pure Mathematics
Ferdowsi University of Mashhad
P.O. 1159 Mashhad, Iran
E-mail: mirmostafaei@ferdowsi.um.ac.ir

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