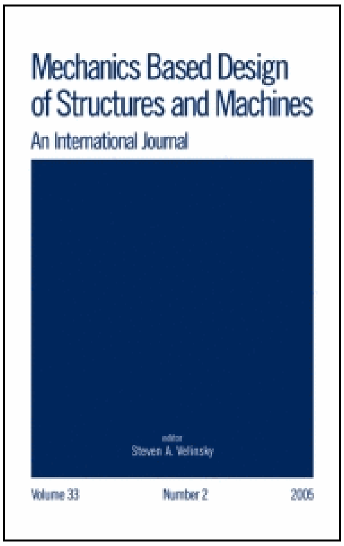

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M. Rezaiee-Pajand[a]; A. R. Naghavi[a]
[a] Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

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ACCURATE SOLUTIONS FOR GEOMETRIC NONLINEAR ANALYSIS OF EIGHT TRUSSES#

M. Rezaiee-Pajand and A. R. Naghavi
Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad, Iran

This paper investigates the analytical solution to eight individual trusses. They are two- and three-dimensional structures. In addition, explicit expressions for the tangent stiffness matrix, critical loads, bifurcation points and limit points as well as equilibrium paths are also assessed. Necessary discussions are provided for different values of effective parameters. As a final objective, the validity of the results obtained by the analytical method is verified by the numerical arc-length technique.

Keywords: Bifurcation point; Critical point; Eigenvalue; Equilibrium path; Explicit solution; Limit point; Stability; Tangent stiffness matrix.

INTRODUCTION

Nonlinear solutions are usually carried out through numerical analysis. Various techniques are available to trace the equilibrium path. Among these, the arc-length method is one of the most robust techniques at hand (Bashir-Ahmed and Xiao-Zu, 2004). However, this method still lacks the ability to analyze certain structures. For example, when encountered with structures, which are unstable at the initial state, the arc-length method fails to provide an acceptable solution. In order to overcome this deficiency and attain an effective analysis technique for trusses, Toklu took advantage of the local search method (Toklu, 2004). This method has the ability to trace any variation of the equilibrium path and can take into account both geometrical and material nonlinearity. However, the computational costs of such analyses are extremely high.

A lot of investigation has been carried analytically in the area of large deformation. Due to space constraint, only a few of the related studies are briefly mentioned here. Ji and Waas determined the exact critical load of a sandwich beam based on the classic theory of elasticity (Ji and Waas, 2007). Pi et al. evaluated the critical load of a curved beam with semi-rigid supports (Pi et al., 2008). Vega-Posadal et al. analyzed the large deformation and post-buckling behavior of a beam with semi-rigid supports (Vega-Posadal et al., 2007). The effect of the longitudinal stress gradient on elastic buckling of thin homogeneous plates with various support

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Correspondence: M. Rezaiee-Pajand, Professor, Department of Civil Engineering, Ferdowsi University of Mashhad, P.O. Box 1111-91775, Mashhad, Iran; E-mail: mrpajand@yahoo.com

conditions was considered by Yu (Yu and Schafer, 2007). Oztorun utilized elements with six degrees of freedom at each node to analyze spatial structures (Oztorun, 2006). In his method, the stiffness matrix of the element can be exactly determined. Luo et al. took advantage of the transformation matrix to explicitly obtain the tangent stiffness matrix and the equilibrium condition (Luo et al., 2007). Pederson proved that in order to determine the tangent stiffness matrix based on Green's strains, there is no need for numerical integration and an explicit solution can be obtained (Pedersen, 2006). Kress et al. resolved the exact characteristic equations for a two-dimensional beam with large deformations (Kress et al., 2006).

Many effective numerical techniques have been developed up to this date and are widely used for nonlinear analysis. However, the pursuit of exact analytical solutions is still in progress. In 2006, Ligaro and Valvo managed to analyze pyramidal trusses and found closed-form solutions (Ligaro and Valvo, 2006). Such problems and their solutions not only have curricular significance, but can also be treated as benchmark problems. Future researches and their formulations, related computer programs and numerical findings can initially be verified via these benchmark problems. This study is concentrated on analytical solutions of eight geometrically nonlinear trusses.

POTENTIAL ENERGY FUNCTION OF A STRUCTURE

If the potential energy function of a small structure is explicitly determined in terms of its nodal displacements, the equilibrium equations and the tangent stiffness matrix would be attainable. Afterwards, the equilibrium path, critical load values, secondary equilibrium paths, and other properties can also be obtained. The potential energy function of a structure is defined in the following form (Felippa, 1999):

$$\Pi = U - V \tag{1}$$

where Π and U are respectively the potential energy and strain energy functions of the structure and V is the potential energy function of the applied loads. The potential energy function of the applied loads, V, is obtained using the following equation:

$$V = p\{q\}^T\{u\} \tag{2}$$

In this equation, p is loading factor, q is the constant load vector and $\{u\}$ is the nodal displacement vector. For a total Lagrangian formulation, the Green–Lagrange strains and the second Pialo–Kirchhoff stresses are employed (Crisfield, 1991). For a single truss element, the Green–Lagrange strain is uniform throughout the element and will attain the following value (Felippa, 1999):

$$e = \frac{L^2 - L_0^2}{2L_0^2} \tag{3}$$

where L and L_0 are the current and initial length of the element, respectively. The current length is calculated according to the nodal displacements which will lead to the strain energy as follows:

$$U = \frac{1}{2} A_0 E L_0 e^2 = \frac{A_0 E}{8 L_0^3} (L^2 - L_0^2)^2 \tag{4}$$

In the above equation, A_0 is the cross-section of the element at the beginning of the analysis and E denotes the elastic modulus. By inserting Eqs. (2) and (4) into Eq. (1), the total potential energy of a single truss element is attained in the following form (Felippa, 1999):

$$\Pi = U - V = \frac{1}{2} \sum_{i=1}^{n} (A_0 E L_0 e^2)_i - p\{q\}^T \{u\} \tag{5}$$

In this equation, n is the number of elements in the structure. Differentiating Eq. (5) with respect to nodal deformations once results in the equilibrium condition and differentiating it for the second time leads to the tangent stiffness matrix. The equilibrium path can be obtained by finding an exact solution to the resulting set of equilibrium equations. In the proceeding text, a number of trusses will be analyzed according to this technique.

CLOSED-FORM SOLUTIONS

The analytical solution to a number of trusses will be obtained in this section. These examples consist of two- and three-dimensional trusses with simple geometrical arrangements and few degrees of freedom. It is obvious that by increasing the degrees of freedom or by rearranging the truss to a more complex structural form, finding an exact solution would be either much more elaborate or even impossible.

Triangular Truss with Three Elements

The first example is illustrated in Fig. 1. The cross-section of all elements is assumed to be A_0 and the initial lengths are determined in multiples of l. These parameters can be given any numerical value.

If the origin of the coordinate system is attached to the point where the elements intersect (point A), the strain energy function of the truss and the potential energy function of the external loads can be determined as follows:

$$U = \frac{A_0}{8l^3} \left[\frac{((x-3l)^2 + (y+4l)^2 - 25l^2)^2}{125} + \frac{((x+3l)^2 + (y+4l)^2 - 25l^2)^2}{125} + \frac{(x^2 + (y+4l)^2 - 16l^2)^2}{64} \right] \tag{6}$$

$$V = p(q_x x + q_y y) \tag{7}$$

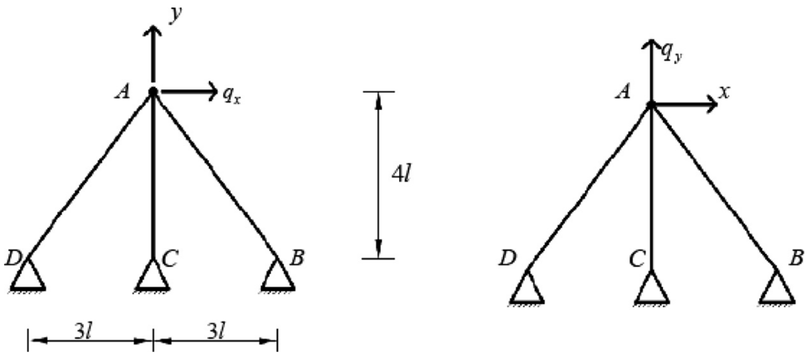


Figure 1 Triangular three-element truss.

In these equations, x and y determine the position of point A after deformation has taken place. The applied load at the point A is denoted by q_x and q_y. By inserting Eqs. (6) and (7) into Eq. (1), the total potential energy function of structure is obtained in the following form:

$$\Pi = \frac{A_0 E}{64000 l^3}(4048 l y(x^2+y^2) + 253(x^2+y^2)^2 + 64 l^2(72x^2 + 253y^2)) - p(q_x x + q_y y). \tag{8}$$

Differentiating this equation with respect to x and$y$ leads to the equilibrium equations as follows:

$$\begin{cases} \dfrac{\partial \Pi}{\partial x} = \dfrac{A_0 E}{64000 l^3}(9216 l^2 x + 8096 l x y + 1012 x(x^2+y^2)) - p q_x = 0 \\ \dfrac{\partial \Pi}{\partial y} = \dfrac{A_0 E}{64000 l^3}(32384 l^2 y + 8096 l y^2 + 4048 l(x^2+y^2) + 1012 y(x^2+y^2)) - p q_y = 0 \end{cases} \tag{9}$$

The second derivative of the potential energy function with respect to the nodal displacements is equal to the tangent stiffness matrix. Therefore, this matrix is obtained in the following form:

$$[K_T] = \frac{A_0 E}{64000 l^3}\begin{bmatrix} 9216 l^2 + 2024 x^2 + 8096 l y + 1012(x^2+y^2) & 8096 l x + 2024 x y \\ 8096 l x + 2024 x y & 32384 l^2 + 24288 l y + 2024 y^2 + 1012(x^2+y^2) \end{bmatrix} \tag{10}$$

In order to analyze the stability of the structure, the eigenvalues of the tangent stiffness matrix must be determined. The points, at which either one or both eigenvalues of the tangent matrix become zero, correspond to structural instability. In other words, any external load capable of translating the node A to these points will cause the structure to buckle. By using the following definitions of the intermediate parameters, c and d:

$$c = 5200l^2 + 506x^2 + 4048ly + 506y^2$$
$$d = \sqrt{\begin{array}{l}8386816l^4 + 2631200l^2x^2 + 64009x^4 + 117230058l^3y + 1024144lx^2y \\ \quad + 5561952l^2y^2 + 128018x^2y^2 + 1024144ly^3 + 64009y^4\end{array}} \tag{11}$$

The eigenvalues of the tangent stiffness matrix can be obtained as follows:

$$\begin{cases} \lambda_1 = \left(\dfrac{A_0E}{16000l^3}\right)(c+d) \\ \lambda_2 = \left(\dfrac{A_0E}{16000l^3}\right)(c-d) \end{cases} \tag{12}$$

In this equation, λ_i represents the ith eigenvalue of the tangent stiffness matrix. Equating $\lambda_1 = 0$ will produce the following results:

$$\begin{cases} x = \pm\dfrac{\sqrt{-5200l^2 - 6072ly - 795y^2 + 16\sqrt{-11967l^4 - 109296l^3y - 13662l^2y^2}}}{\sqrt{759}} \\ x = \pm\dfrac{\sqrt{-5200l^2 - 6072ly - 795y^2 - 16\sqrt{-11967l^4 - 109296l^3y - 13662l^2y^2}}}{\sqrt{759}} \end{cases} \tag{13}$$

Each of the values of x in Eq. (13) can attain a real value. Therefore, four different curves can be obtained by $\lambda_1 = 0$. Although the solution of $\lambda_2 = 0$ will have the same form, the two eigenvalues will not become equal to zero simultaneously. The type of the critical point is determined based on the form of the load vector and the eigenvector of the tangent stiffness matrix. Figure 2 illustrates the curves corresponding to the relations given in Eq. (13). The axes of this figure represent the values of x and y in a dimensionless form, i.e., $\frac{x}{l}$, $\frac{y}{l}$. As can be seen in this figure, each of the eigenvalues will attain a zero value somewhere on the curve. This eigenvalue will have opposite signs inside and outside the curve. Therefore, these curves represent the critical points of the structure. The discontinuities observed at the top and bottom of the curves are the result of the solutions given in Eq. (13). These discontinuities imply the fact that the structure does not have multiple bifurcation points. In the proceeding text, the structural response will be obtained for various loading conditions.

Zero-Loaded Structure By introducing the following load into Eq. (9):

$$q_x = q_y = 0 \tag{14}$$

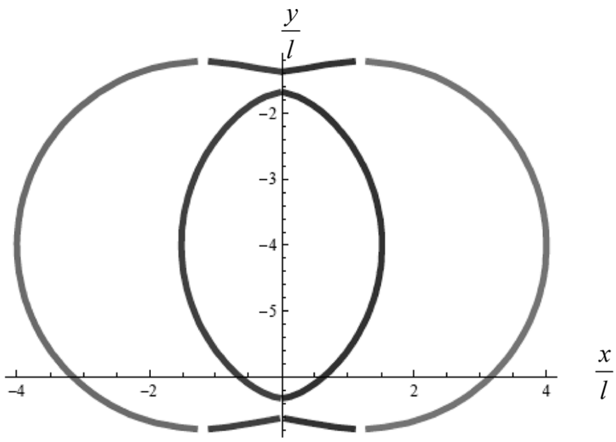


Figure 2 Curves corresponding to zero eigenvalues.

The following equilibrium condition is obtained:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 8096lxy + 1012x(x^2+y^2)) = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 8096ly^2 + 4048l(x^2+y^2) + 1012y(x^2+y^2)) = 0 \end{cases} \tag{15}$$

The solutions of the above equation represent the points at which the zero-loaded structure will be in equilibrium. The critical points are responsible for such states of the structure. The solution to Eq. (15) is as follows:

$$\begin{cases} x = 0 \\ y = \begin{cases} 0 \\ -4l \\ -8l \end{cases} \end{cases}$$
$$\begin{cases} x = \pm 2.62551l \\ y = -4l \end{cases} \tag{16}$$

Vertical Loading By introducing the following values into Eq. (9):

$$q_x = 0, \quad q_y = 1 \tag{17}$$

the equilibrium equations are obtained as follows:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 8096lxy + 1012x(x^2+y^2)) = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 8096ly^2 + 4048l(x^2+y^2) + 1012y(x^2+y^2)) = p \end{cases} \tag{18}$$

In order to determine the response of the structure under this load, the parameter f is defined as follows:

$$f = \frac{64000 l^3 p}{A_0 E} \tag{19}$$

By utilizing this equation, the solutions to the equilibrium equations will take the following form:

$$\begin{cases} x_1 = 0 \\ f = y(32384 l^2 + 12144 l y + 1012 y^2) \end{cases} \tag{20}$$

$$\begin{cases} x_2 = \pm 0.0628695\sqrt{-2304 l^2 - 2024 l y - 253 y^2} \\ f = -36384 l^3 - 9216 l^2 y \end{cases} \tag{21}$$

Since two individual relations exist for the equilibrium path, it can be deduced that the structure has a bifurcation point. Considering the symmetry of the structure and the applied load, the structure will deform in a symmetrical manner. Therefore, the primary deformation path will correspond to $x = 0$. This implies the fact that the primary path is obtained by using Eq. (20). In order to determine the critical load, the results given in Eq. (20) are introduced into Eq. (13). Based on the equity $x = 0$, the critical points are calculated as follows:

$$\begin{cases} x_1 = 0 \\ y_1 = \dfrac{4}{253}(-253 l \pm l\sqrt{27577}) \\ f_1 = \pm 24196.7 l^3 \end{cases} \tag{22}$$

$$\begin{cases} x_2 = 0 \\ y_2 = \dfrac{4}{3}(-3 l \pm l\sqrt{3}) \\ f_2 = \pm 24929.2 l^3 \end{cases} \tag{23}$$

The bifurcation point, the secondary path of the truss is determined by Eq. (21). The primary path becomes unstable at a point corresponding to $y = \frac{4}{253}(-253 l + l\sqrt{27577})$. The structure changes path at this point and traces the secondary path afterwards. Because of structural symmetry, the sign of x on the secondary path can be positive or negative, which is determined by the structural imperfections. The secondary path has a bifurcation point at $y = \frac{4}{253}(-253 l - l\sqrt{27577})$ which the structure returns to the primary path. The only critical point of the secondary path lies at its intersection with the primary path. This path has no other critical points. In case the primary unstable path is traced beyond the bifurcation point, the value of the loading parameter will increase until the structure reaches the limit point. After passing the limit point, the loading parameter decreases until it attains a zero value at$y = 0$, which afterwards the loading parameter changes signs. The primary and secondary equilibrium paths of the structure are illustrated in Fig. 3. In this

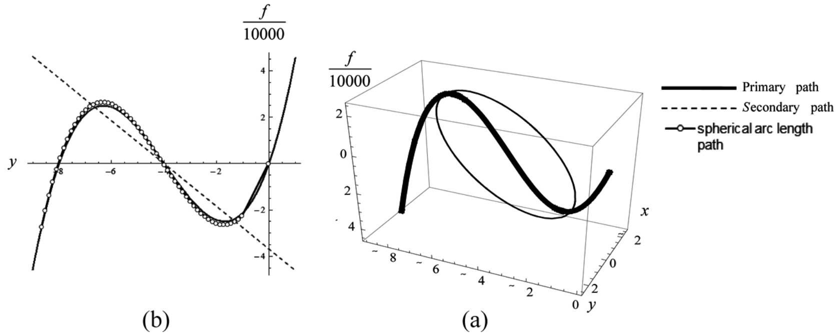


Figure 3 Equilibrium path of the three-element truss under vertical loading.

figure, the vertical axis represents $\frac{f}{10000}$. In order to eliminate unknown factors, the value of l is assumed unity in these curves. Figure 3(b) demonstrates the primary and secondary equilibrium paths of the structure on the $(y - f)$ plane. The dashed line corresponds to the secondary path. Also, the analytically obtained equilibrium path and the one predicted by the spherical arc-length scheme are compared in Fig. 3(b). In this figure, the validity of the suggested closed-form solution is verified by the numerical technique (Bashir-Ahmed and Xiao-Zu, 2004). Figure 3(a) shows the primary and secondary equilibrium paths in the $(x - y - f)$ coordinate system.

Horizontal Loading The equilibrium condition of this loading case can be obtained by introducing the following values into Eq. (9):

$$q_x = 1, \quad q_y = 0 \tag{24}$$

Referring to Eq. (19), the equilibrium equations are determined in the following form:

$$\begin{cases} 9216l^2x + 8096lxy + 1012x(x^2 + y^2) = f \\ 32384l^2y + 8096ly^2 + 4048l(x^2 + y^2) + 1012y(x^2 + y^2) = 0 \end{cases} \tag{25}$$

The solutions to this set of equations are as follows:

$$\begin{cases} y_1 = -4l \\ f = x(-6976l^2 + 1012x^2) \end{cases} \tag{26}$$

$$\begin{cases} y_2 = 0.5\left(-8l - 8\sqrt{l^2 - 0.0625x^2}\right) \\ f = 9216l^2x \end{cases} \tag{27}$$

$$\begin{cases} y_3 = 0.5\left(-8l + 8\sqrt{l^2 - 0.0625x^2}\right) \\ f = 9216l^2x \end{cases} \tag{28}$$

Considering the fact that the primary equilibrium path is the one which passes the original undeformed state of the structure, this path will correspond to Eq. (28). Inserting Eq. (28) into Eq. (13), the following value is obtained for the critical point:

$$\begin{cases} x = \pm 4l \\ y = -4l \\ f = \pm 36864 l^3 \end{cases} \tag{29}$$

These results indicate asymmetrical bifurcation points for the structure. After the structure reaches these states, it will deform according to Eq. (26). The critical points lying on this path are determined by introducing Eq. (26) into Eq. (13). The coordinates of these points are as follows:

$$\begin{cases} x_1 = \pm 4l\sqrt{\dfrac{109}{759}} \\ y = -4l \\ f = \mp 7049.65 l^3 \end{cases} \tag{30}$$

$$\begin{cases} x_2 = \pm 4l \\ y = -4l \\ f = \pm 36864 l^3 \end{cases} \tag{31}$$

The points given in Eq. (30) are limit points. The structure traces its primary deformation path (Eq. (28)) at this state. Therefore, these points do not have any significant value. It should be mentioned that these limit points lie on the secondary path (Eq. (26)). On the other hand, the points indicated by Eq. (31) are bifurcation points. These points lie on the intersection of the primary and secondary paths. Figure 4 illustrates the primary and secondary equilibrium path of the structure. In this figure, similar to Fig. 3, the vertical axis represents $\frac{f}{10000}$. In order to eliminate any influence of unknown variables, the value of l is taken as unity. The projection of the primary and secondary equilibrium path on the $(x - f)$ and $(y - f)$ planes is shown in Figs. 4(a) and 4(b), respectively. Also, Fig. 4(a) demonstrates a comparison between the result obtained by the analytical and the spherical arc-length methods. Figure 4(c) presents these paths in the $(x - y - f)$ coordinate system. In these figures, the primary path is indicated by a solid line, while the dashed line represents the secondary path.

Planar Rotationally Truss

For the pyramidal truss presented by Ligaro and Valvo (2006), if the value of H is set to zero, the structure given in Fig. 5 will be obtained. This structure is a regular truss consisting n elements with the initial length of B. The initial cross-section of each element is assumed to be A_0. The supports were located on a circle which its center is at the intersection of the elements. The angle between each two element is equal to $\theta = \frac{\pi}{n}$. Considering a spatial behavior, the structure is expected to be unstable in its initial undeformed state.

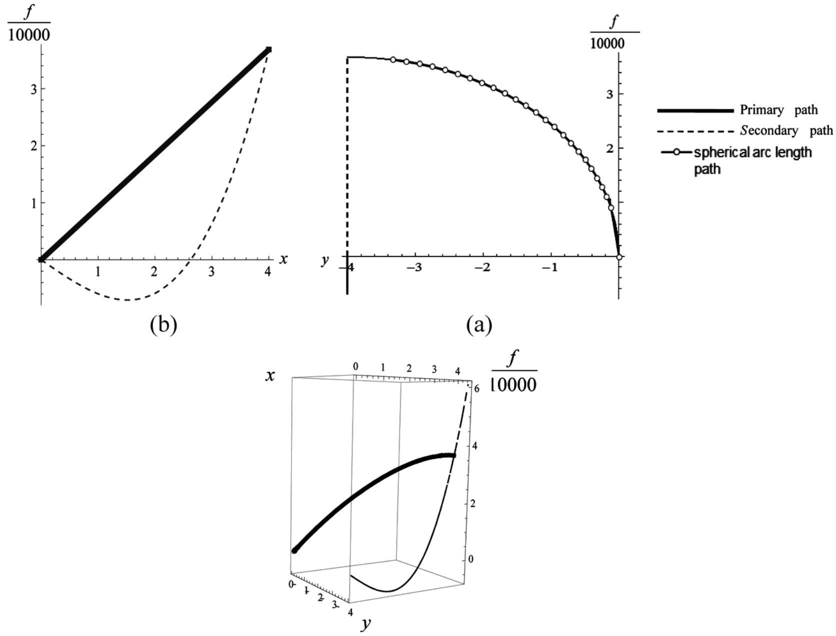


Figure 4 Equilibrium path of the three-element truss under horizontal loading.

The cylindrical coordinate system will be utilized in the analysis of this structure. The origin of this coordinate system is attached to at the intersection of the elements (point O). The potential energy function is written and simplified to the following form:

$$U = \frac{nA_0E}{8B^3}(2B^2r^2 + (r^2 + z^2)^2)$$

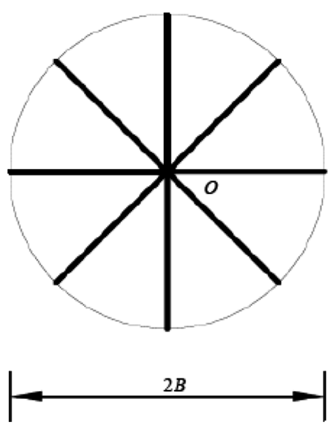


Figure 5 Planar rotationally truss.

$$V = p(q_r r \times \cos(\theta - \theta_p) + q_z z)$$
$$\Pi = U - V = \frac{nA_0E}{8B^3}(2B^2r^2 + (r^2 + z^2)^2) - p(q_r r \times \cos(\theta - \theta_p) + q_z z) \quad (32)$$

In this equation, r, z, and θ represent the position of the node O after deformation. The loading values in the plane of the structure (radial direction) and perpendicular to the plane of the structure are denoted by q_r and q_z, respectively. The angle between the direction of the load and the polar axis is indicated by θ_p. By differentiating Eq. (32), the equilibrium equations are obtained in the following form:

$$\begin{cases} \dfrac{\partial \Pi}{\partial r} = \dfrac{nA_0E}{8B^3}(4B^2r + 4r(r^2 + z^2)) - pq_r \times \cos(\theta - \theta_p) = 0 \\ \dfrac{1}{r}\dfrac{\partial \Pi}{\partial \theta} = pq_r \times \sin(\theta - \theta_p) = 0 \\ \dfrac{\partial \Pi}{\partial z} = \dfrac{nA_0E}{8B^3}(4z(r^2 + z^2)) - pq_z = 0 \end{cases} \quad (33)$$

The tangent stiffness matrix is written as

$$[K_T] = \begin{bmatrix} \Pi_{rr} & \Pi_{r\theta} & \Pi_{rz} \\ \Pi_{\theta r} & \frac{1}{r^2}\Pi_{\theta\theta} & \Pi_{\theta z} \\ \Pi_{zr} & \Pi_{z\theta} & \Pi_{zz} \end{bmatrix}$$
$$= \frac{nA_0E}{8B^3}\begin{bmatrix} 4B^2 + 8r^2 + 4(r^2 + z^2) & 0 & 8rz \\ 0 & 4B^2 + 4(r^2 + z^2) & 0 \\ 8rz & 0 & 8z^2 + 4(r^2 + z^2) \end{bmatrix} \quad (34)$$

In order to analyze the stability of the structure, the eigenvalues of the tangent stiffness matrix must be attained. Mathematical manipulation yields the following results:

$$\begin{cases} \lambda_1 = \dfrac{nA_0E}{8B^3}(4B^2 + 4(r^2 + z^2)) \\ \lambda_2 = \dfrac{nA_0E}{4B^3}(B^2 + 4r^2 + 4z^2 - \sqrt{B^4 + 4B^2r^2 - 4B^2z^2 + 8r^2z^2 + 4z^4}) \\ \lambda_3 = \dfrac{nA_0E}{4B^3}(B^2 + 4r^2 + 4z^2 + \sqrt{B^4 + 4B^2r^2 - 4B^2z^2 + 8r^2z^2 + 4z^4}) \end{cases} \quad (35)$$

Since the square value and the square root of all the terms have a positive sign, the values of λ_1 and λ_3 will always be greater than zero. Therefore, only one of the eigenvalues (λ_2) can become equal to zero implying the fact that no multiple bifurcation points exist for the structure. In order to assess the critical point, the value of λ_2 will be equated to zero in Eq. (35). The solution to the resulting equation is as follows:

$$r = \pm\sqrt{-\frac{B^2}{6} - z \pm \frac{B}{6}\sqrt{B^2 - 24z^2}} \quad (36)$$

The response of the structure under different loading conditions will be discussed in the proceeding text.

Zero-Loaded Structure The equilibrium equations will take the following form for the zero-loaded structure:

$$\begin{cases} \dfrac{nA_0E}{8B^3}(4B^2r + 4r(r^2 + z^2)) = 0 \\ \dfrac{nA_0E}{8B^3}(4z(r^2 + z^2)) = 0 \end{cases} \tag{37}$$

The following values are the solution to the above equation:

$$\begin{cases} r = 0 \\ z = 0 \end{cases} \tag{38}$$

Therefore, the only equilibrium state of the structure is at its original form. Considering the results obtained for the zero-loaded structure, it can be predicted that the structure has no limit points.

Vertical Loading For this case, the equilibrium equations are assessed by introducing the following equations into Eq. (33):

$$q_z = 1, \quad q_r = 0 \tag{39}$$

which leads to the following set of equations:

$$\begin{cases} \dfrac{nA_0E}{8B^3}(4B^2r + 4r(r^2 + z^2)) = 0 \\ \dfrac{nA_0E}{8B^3}(4z(r^2 + z^2)) = p \end{cases} \tag{40}$$

In order to find the solution to this equation, the parameter ((f)) is defined as:

$$f = \frac{8B^3p}{nA_0E} \tag{41}$$

Considering the last equation, the following results are obtained:

$$\begin{cases} r_1 = 0 \\ z_1 = 0.629961f^{\frac{1}{3}} \end{cases} \tag{42}$$

$$\begin{cases} r_2 = \pm\sqrt{-B^2 - 0.0625\dfrac{f^2}{B^4}} \\ z_2 = -0.25\dfrac{f}{B^2} \end{cases} \tag{43}$$

Since the structure is symmetric with respect to the Z axis, the nodal displacements will also have symmetry with respect to the same axis. Therefore, Eq. (42) will correspond to the primary equilibrium path of the structure. It should be noted that Eq. (43) is not an equilibrium path. This is due to the fact that in the expression determined for r_2, the term under the square root sign is a negative number for any value of f. Therefore, r_2 will never be a real number. Considering the fact that only one relation exists for the equilibrium path, it can be deduced that the structure does not have a bifurcation point.

In order to find the instability points and the critical loads, Eq. (42) is introduced into Eq. (36). The value of r is equated to zero in Eq. (36). The only solution to the resulting equation will have the following form:

$$\begin{cases} r = 0 \\ z = 0 \\ f = 0 \end{cases} \tag{44}$$

Therefore, the original configuration of the structure is an unstable state. This point is a limit point at which the structure will go under large deformations if a load is applied. For this loading case (vertical loading), the structure does not have any other critical point. Hence, the entire equilibrium path will be at a stable state. Figure 6 illustrates the equilibrium path of the structure. The vertical axis of this figure represents $\frac{f}{1000}$. In order to eliminate the influence of any unknown variables, the value of B is set to unity.

Since this structure is unstable at its original configuration, the spherical arc-length method is not capable of analyzing it. Therefore, the equilibrium paths cannot be compared to one another.

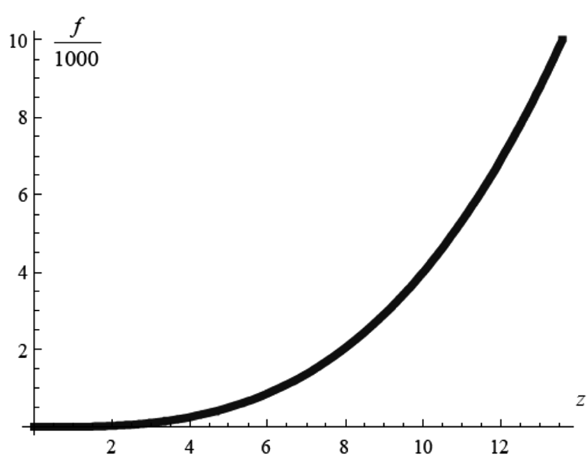


Figure 6 Equilibrium path of the planar rotationally truss (vertical loading).

Radial Loading The equilibrium equations will take the following form for this loading case:

$$\begin{cases} \frac{nA_0E}{8B^3}(4B^2r + 4r(r^2 + z^2)) = p \\ p \times \sin(\theta - \theta_p) = 0 \\ \frac{nA_0E}{8B^3}(4z(r^2 + z^2)) = 0 \end{cases} \tag{45}$$

Utilizing Eq. (41), the solution to the last equation can be obtained as follows:

$$\begin{cases} z = 0 \\ \theta = \theta_p \\ f = r(4B^2 + 4r^2) \end{cases} \tag{46}$$

$$\begin{cases} z = 0 \\ \theta = \theta_p + \pi \\ f = -r(4B^2 + 4r^2) \end{cases} \tag{47}$$

Similar to the last loading case, only one equilibrium path exists. Therefore, the structure will not have a bifurcation point. In order to find the critical load, the value of z is set to zero in Eq. (36). This leads to the following result for the critical point:

$$\begin{cases} r = 0 \\ z = 0 \\ f = 0 \end{cases} \tag{48}$$

The initial configuration is its only unstable state of the structure which is a simple bifurcation point. It should be mentioned that the existence of a bifurcation point at the onset of loading does not have any significance and is only a mathematical property. Figure 7 compares the equilibrium path obtained by the analytical method to the one obtained by the spherical arc-length technique. The values $B = 1$ and $A_0E = 2$ are used in the analysis. The number of elements (n) is set to 4. This figure demonstrates that both strategies yield similar results.

Three-Element Suspended Truss

The third example is illustrated in Fig. 8. The cross-section of the elements is denoted by A_0. It should be noted that the variables A_0 and l can assume any given value. By locating the origin of the coordinate system at node A, the potential energy function of the structure can be attained in the following form:

$$\Pi = \frac{A_0E}{64000l^3}(48ly(x^2 + y^2) + 253(x^2 + y^2)^2 + 64l^2(72x^2 + 253y^2)) - p(q_xx + q_yy) \tag{49}$$

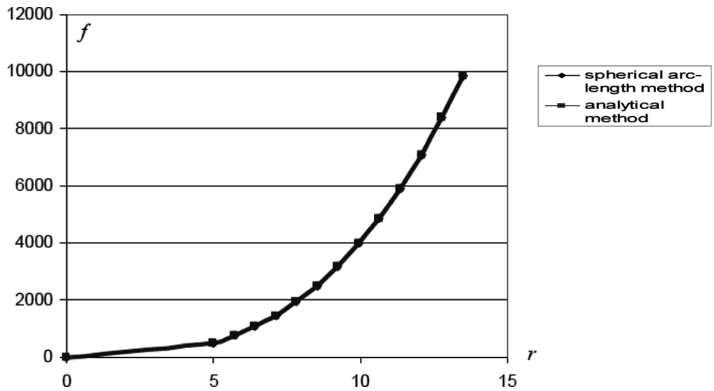


Figure 7 Equilibrium paths obtained by the analytical and spherical arc-length scheme (radial loading).

Differentiating this equation with respect to nodal displacements will yield the following relations for the equilibrium equations:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 96lxy + 1012x(x^2+y^2)) - pq_x = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 96ly^2 + 48l(x^2+y^2) + 1012y(x^2+y^2)) - pq_y = 0 \end{cases} \tag{50}$$

$$[K_T] = \frac{A_0E}{64000l^3}\begin{bmatrix} 9216l^2 + 2024x^2 + 96ly + 1012(x^2+y^2) & 96lx + 2024xy \\ 96lx + 2024xy & 32384l^2 + 288ly + 2024y^2 + 1012(x^2+y^2) \end{bmatrix} \tag{51}$$

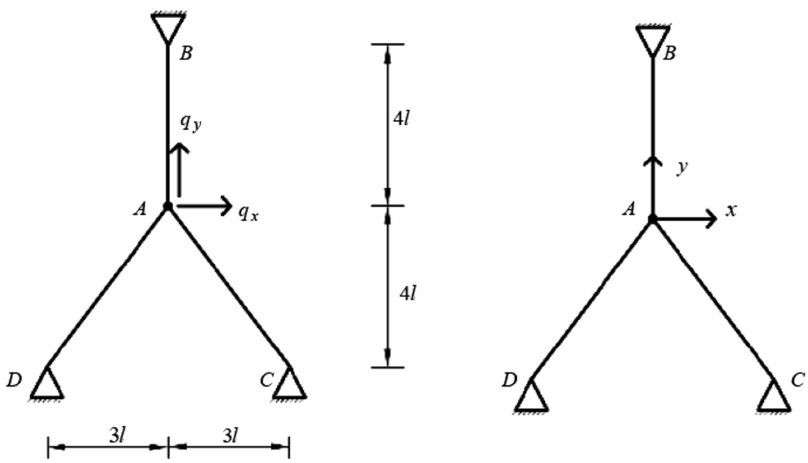


Figure 8 Three-element suspended truss.

The eigenvalues of the tangent stiffness matrix are written in the following form:

$$c = 5200l^2 + 506x^2 + 48ly + 506y^2$$
$$\begin{cases} \lambda_1 = \dfrac{A_0E}{16000l^3}(c-d) \\ \lambda_2 = \dfrac{A_0E}{16000l^3}(c+d) \end{cases} \tag{52}$$

In order to analyze the stability of the structure and determine the critical loads, the roots of the eigenvalues must be evaluated. If the value of λ_1 is equal to zero in Eq. (52), the solutions of the equation are obtained as follows:

$$x = \pm\sqrt{\frac{-1121200l^2}{64009} - \frac{24ly}{253} - y^2 + \frac{16\sqrt{10067565099l^4 + 69500112l^3y + 732647014l^2y^2}}{64009\sqrt{3}}} \tag{53}$$

$$x = \pm\sqrt{\frac{-1121200l^2}{64009} - \frac{24ly}{253} - y^2 - \frac{16\sqrt{10067565099l^4 + 69500112l^3y + 732647014l^2y^2}}{64009\sqrt{3}}} \tag{54}$$

It can be shown that the results of Eqs. (53) and (54) are not real numbers. Therefore, this structure does not have a critical point. Equation (53) is considered for this matter. If the term under the square root (denoted by u) is differentiated with respect to y, the following equation will be obtained:

$$\frac{du}{dy} = -\frac{24l}{253} - 2y + 8\frac{69500112l^3 + 1465294028l^2y}{64009\sqrt{3} \times \sqrt{10067565099l^4 + 69500112l^3y + 732647014l^2y^2}} \tag{55}$$

In order to find the minimum and maximum values of u, its derivative with respect to y is equated to zero. Doing so leads to the following values of u and y:

$$\begin{cases} y = -\dfrac{12l}{253} \\ u = -\dfrac{194256l^2}{64009} \end{cases} \tag{56}$$

Utilizing the second derivative test, it can be proven that this point is a relative maximum point. Since the derivative exists for any value of y, and it does not have any root other than the one mentioned, the relative maximum point will also be the absolute maximum point. Hence, the function u will always have a negative value. This leads to the fact that in Eq. (53) a real value cannot be obtained for x. Figure 9 illustrates the variation of u as a function of y. In this figure, the horizontal and vertical axes represent $\frac{y}{l}$ and $\frac{u}{l^2}$, respectively.

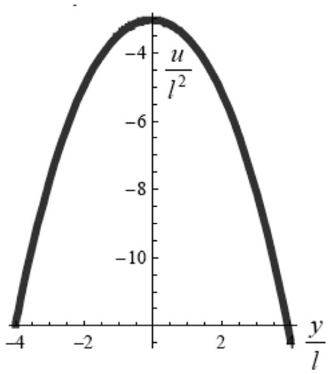


Figure 9 Variation of $\frac{u}{l^2}$ in terms of $\frac{y}{l}$.

Zero-Loaded Structure For this loading condition, the equilibrium equations take the following form:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 96lxy + 1012x(x^2+y^2)) = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 96ly^2 + 48l(x^2+y^2) + 1012y(x^2+y^2)) = 0 \end{cases} \tag{57}$$

which has the following solution:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \tag{58}$$

Vertical Loading The equilibrium equations can be written as follows for this state of loading:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 96lxy + 1012x(x^2+y^2)) = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 96ly^2 + 48l(x^2+y^2) + 1012y(x^2+y^2)) - p = 0 \end{cases} \tag{59}$$

In order to find the solution to this equation, the variable f is defined as follows:

$$f = \frac{64000l^3p}{A_0E} \tag{60}$$

Using Eq. (60), the response of the structure is attained as follows:

$$\begin{cases} x = \pm 0.0628695\sqrt{-2304l^2 - 24ly - 253y^2} \\ f = 0.00395257l^2(-110592l + 5.86035 \times 10^6 y) \end{cases} \tag{61}$$

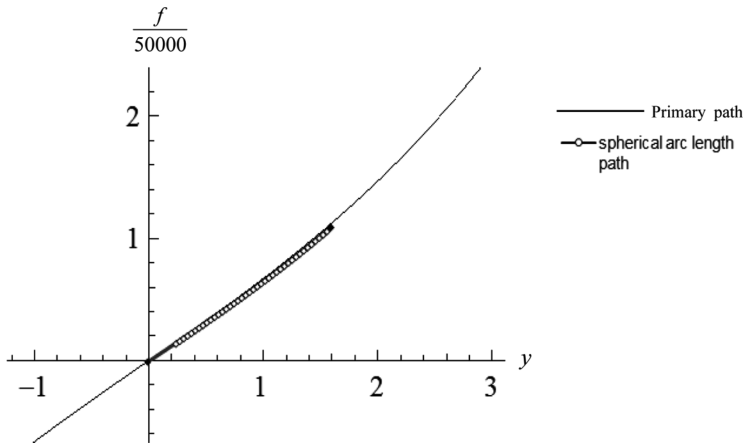


Figure 10 Equilibrium path of the three-element suspended truss under vertical loading.

$$\begin{cases} x = 0 \\ f = y(32384l^2 + 144ly + 1012y^2) \end{cases} \tag{62}$$

A comparison between the equilibrium path obtained by the analytical and the spherical arc-length method is presented in Fig. 10.

Horizontal Loading The following equation represents the equilibrium equation for this loading condition:

$$\begin{cases} \dfrac{A_0E}{64000l^3}(9216l^2x + 96lxy + 1012x(x^2 + y^2)) - p = 0 \\ \dfrac{A_0E}{64000l^3}(32384l^2y + 96ly^2 + 48l(x^2 + y^2) + 1012y(x^2 + y^2)) = 0 \end{cases} \tag{63}$$

The response of the above equation is shown in Fig. 11.

The solution of this equation can be obtained using Eq. (60), leading to the following:

$$c = \sqrt{-8096l^2 - 36ly - 253y^2}$$

$$d = \sqrt{12l + 253y}$$

$$\begin{cases} x = \pm\dfrac{c\sqrt{y}}{d} \\ f = \mp\left(\dfrac{819315 \times 10^6 l^2 y^{\frac{3}{2}} c}{d^3} + \dfrac{36432ly^{\frac{5}{2}}c}{d^3} + \dfrac{256036y^{\frac{7}{2}}c}{d^3} \right. \\ \qquad \left. - \dfrac{9216l^2\sqrt{y}c}{d} - \dfrac{96ly^{\frac{3}{2}}c}{d} - \dfrac{1012y^{\frac{5}{2}}c}{d}\right) \end{cases} \tag{64}$$

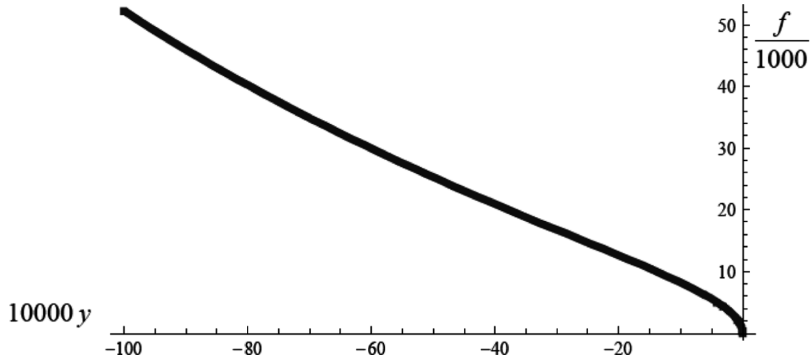


Figure 11 Equilibrium path of the three-element suspended truss under horizontal loading.

Three-Dimensional Bi-Pyramidal Symmetric Truss

The forth example is presented in Fig. 12. This structure consists of two pyramidal trusses joined together at the point A. The properties of these two trusses are similar, i.e. this structure is symmetric with respect to point A. The initial cross section and length of the elements are equal to A_0 and l_0, respectively.

Similar to the pyramidal truss discussed earlier, a cylindrical coordinate system will be used to construct the total potential energy function. The origin of this coordinate system is attached at the center of the circle which the support of one of the pyramids is located. The positive sense of the Z axis is towards the point A.

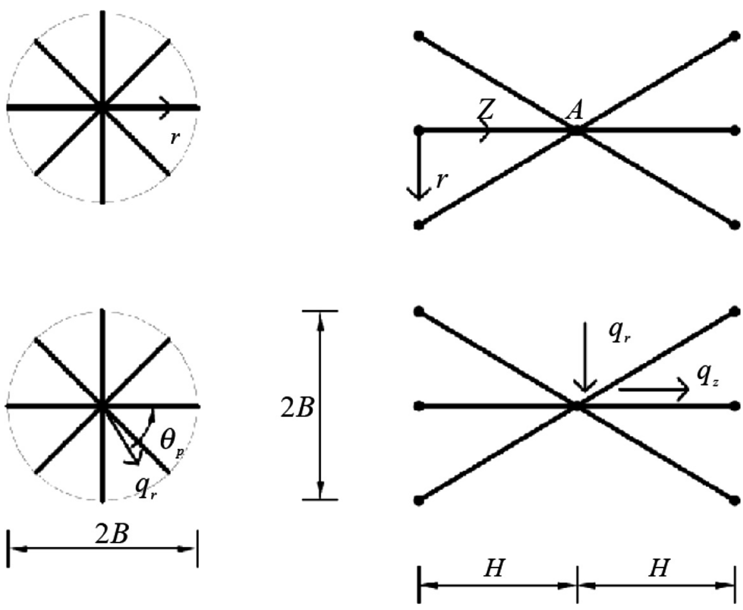


Figure 12 Bi-pyramidal symmetric truss.

After simplifying the relations, the total potential energy function of the structure can be written in the following form:

$$
\begin{aligned}
U &= \frac{nA_0E}{4l_0^3}(5H^4 + 2B^2r^2 - 12H^3z - 4Hz(r^2+z^2) + (r^2+z^2)^2 + 2H^2(r^2+5z^2)) \\
V &= p(q_r Cos(\theta - \theta_p) + q_z(z-H)) \\
\Pi &= U - V
\end{aligned}
\tag{65}
$$

In this equation, r, z and θ represent the location of the point O after deformation. The loading values are respectively q_r and q_z in the radial and Z-axis directions. The angle between the direction of the load and the polar axis is θ_p. Differentiating the total potential energy function of the structure (Eq. (65)) with respect to nodal displacements, the equilibrium equations are obtained in the following form.

$$
\begin{cases}
\dfrac{\partial \Pi}{\partial r} = \dfrac{nA_0E}{l_0^3}(r(B^2 + H^2 + r^2 - 2Hz + z^2)) - pq_r \operatorname{Cos}(\theta - \theta_p) = 0 \\
\dfrac{1}{r}\dfrac{\partial \Pi}{\partial \theta} = pq_r \operatorname{Cos}(\theta - \theta_p) = 0 \\
\dfrac{\partial \Pi}{\partial z} = \dfrac{nA_0E}{l_0^3}(z-H)(3H^2 + r^2 - 2Hz + z^2) - pq_z = 0
\end{cases}
\tag{66}
$$

Considering the equilibrium equations, the tangent stiffness matrix is written as follows:

$$
[K_T] = \left(\frac{nA_0E}{l_0^3}\right)\begin{bmatrix} B^2 + H^2 + 3r^2 - 2Hz + z^2 & 0 & 2r(z-H) \\ 0 & B^2 + H^2 + r^2 - 2Hz + z^2 & 0 \\ 2r(z-H) & 0 & 5H^2 + r^2 - 6Hz + 3z^2 \end{bmatrix}
\tag{67}
$$

The eigenvalues of the tangent stiffness matrix are determined as follows:

$$
\begin{aligned}
c &= B^2 + 6H^2 + 4r^2 - 8hz + 4z^2 \\
\lambda_1 &= \left(\frac{nA_0E}{l_0^3}\right)(B^2 + (H-z)^2 + r^2) \\
\lambda_2 &= \left(\frac{nA_0E}{2l_0^3}\right)(c - d) \\
\lambda_3 &= \left(\frac{nA_0E}{2l_0^3}\right)(c + d)
\end{aligned}
\tag{68}
$$

It is observed that the values of λ_1 and λ_3 are both positive in the last equations. This is deduced by the fact that these eigenvalues are the sum of square rooted and squared terms. On the other hand, λ_2 is the eigenvalue that its value can be equal to zero. Therefore, the truss will not have a multiple bifurcation point.

The following text will discuss λ_2. Setting λ_2 equal to zero will result in Eqs. (70) and (71). An intermediate variable similar to Eq. (69) is utilized to give these results:

$$
\begin{aligned}
a &= -3B^2 - 2H^2 - 6r^2 \\
b &= \sqrt{9B^4 - 12B^2H^2 + 4H^4 + 24B^2r^2 - 48H^2r^2}
\end{aligned} \tag{69}
$$

$$z = \frac{1}{12}(12H \pm 2\sqrt{6} \times \sqrt{a-b}) \tag{70}$$

$$z = \frac{1}{12}(12H \pm 2\sqrt{6} \times \sqrt{a+b}) \tag{71}$$

$$\frac{du}{dr} = -12r + \frac{48B^2r - 96H^2r}{\sqrt{9B^4 - 12B^2H^2 + 4H^4 + 24B^2r^2 - 48H^2r^2}} \tag{72}$$

In order to find the maximum value of u, the above equation is equated to zero and solved for r. The result of this process is given below:

$$\begin{cases} r_1 = 0 \\ r_2 = \pm \dfrac{\sqrt{-5B^4 - 4B^2H^2 + 12H^4}}{2\sqrt{6}\sqrt{B^2 - 2H^2}} \end{cases} \tag{73}$$

In these equations, r_2 is not a correct answer, since introducing it into Eq. (72) will give a non-zero value. The reason that such an answer is obtained lies in the method of solving the equation. In order to solve $\frac{du}{dr} = 0$, the square root term is kept at one side of the equation, while the other terms are taken to the other side. Both sides are powered by two afterwards. In the case where for a specific value of r the two sides have the opposite sign, they will both have positive values after they are powered by two. Any r resulting from such conditions will not be a correct answer to the equation. This is the case for r_2.

Therefore, the only answer to Eq. (72) is $r = 0$. The second derivative test shows that this point is a relative maximum and for the considered domain, the function attains its maximum value at this point. This maximum value is given in the following equation. It can be proven that this value will always be negative.

$$u_{\max} = -3B^2 - 2H^2 + \sqrt{9B^4 - 12B^2H^2 + 4H^4}(74) \tag{74}$$

Hence, this structure will never buckle. Figure 13 shows the variation of $\frac{u}{H^2}$ as a function of $\frac{r}{H}$ and $\frac{B}{H}$. It can be seen in this figure, when $\frac{B}{H}$ and $\frac{r}{H}$ approach zero, the value of $\frac{u}{H^2}$ will also approach zero. The vertical axis represents $\frac{u}{H^2}$ in this figure.

The response of the structure under different loading cases will be discussed in the proceeding text.

Zero-Loaded Structure The solution to this loading condition is the points at which the structure is in equilibrium without the application of any external loads. Since the structure does not have a critical point and a limit point cannot exist, it can be predicted that the only solution to this loading case is the original

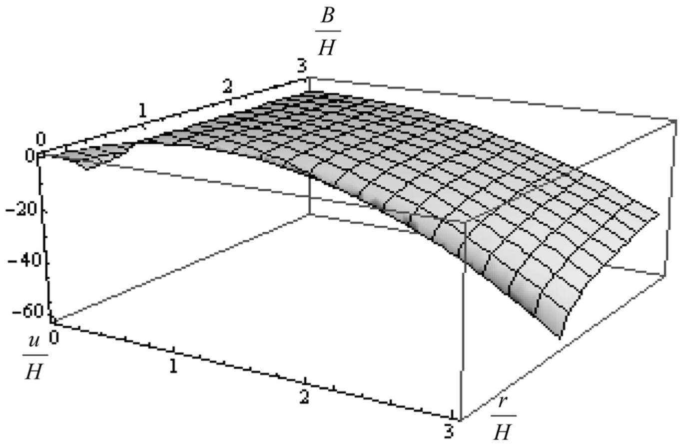


Figure 13 Variation of $\frac{u}{H^2}$ in terms of $\frac{B}{H}$ and $\frac{r}{H}$.

state of the structure. For a truss under zero-loading, the equilibrium equations can be written in the following form:

$$\begin{cases} r(B^2 + H^2 - 2Hz + z^2 + r^2) = 0 \\ (z - H)(3H^2 + r^2 - 2Hz + z^2) = 0 \end{cases} \tag{75}$$

The solution to this equation is as follows:

$$\begin{cases} r = 0 \\ z = H \end{cases} \tag{76}$$

This result is consistent with the previous prediction.

Loading on the ((Z)) Axis Introducing the following loading values into Eq. (66):

$$q_z \neq 0, \quad q_r = 0 \tag{77}$$

will lead to the equilibrium equations as follows:

$$\begin{cases} \dfrac{A_0E}{l_0^3}(r(B^2 + H^2 + r^2 - 2Hz + z^2)) = 0 \\ \dfrac{A_0E}{l_0^3}(z - H)(3H^2 + r^2 - 2Hz + z^2) = p \end{cases} \tag{78}$$

In order to find the solution to this equation, the variable f is defined in the following form:

$$f = \frac{pl_0^3}{nA_0E} \tag{79}$$

Based on this relation, the solution to Eq. (78) takes the following form:

$$\begin{cases} r = 0 \\ f = -3H^3 + 5H^2z - 3Hz^2 + z^3 \end{cases} \tag{80}$$

$$\begin{cases} r = \pm\sqrt{-B^2 - H^2 + 2Hz - z^2} \\ f = B^2H - 2H^3 - B^2z + 2H^2z \end{cases} \tag{81}$$

The primary equilibrium path passes the original point (undeformed state) of the structure. Therefore, this path is determined by Eq. (80). It should be mentioned that equation (81) does not represent an equilibrium path. This is due to the fact that for the given expression of r, the term under the square root is always a negative value, and will not result in a real number. Figure 14 illustrates the equilibrium path of the structure for this loading condition (Eq. (80)). In this figure, the vertical axis represents $\frac{f}{H^3}$. The horizontal axis shows the dimensionless value of $\frac{z}{H}$. As it is seen in the figure, the structure demonstrates a hardening behavior. This hardening characteristic of the structure eliminates the possibility of buckling. Also, this figure compares the equilibrium path predicted by the analytical method to the one obtained by the arc-length technique.

Loading in the Radial Direction The equilibrium equation for this loading case will be as follows:

$$\begin{cases} \dfrac{A_0E}{l_0^3}(r(B^2 + H^2 + r^2 - 2Hz + z^2)) = p\,\mathrm{Cos}(\theta - \theta_p) \\ p\,\mathrm{Cos}(\theta - \theta_p) = 0 \\ \dfrac{A_0E}{l_0^3}(z - H)(3H^2 + r^2 - 2Hz + z^2) = 0 \end{cases} \tag{82}$$

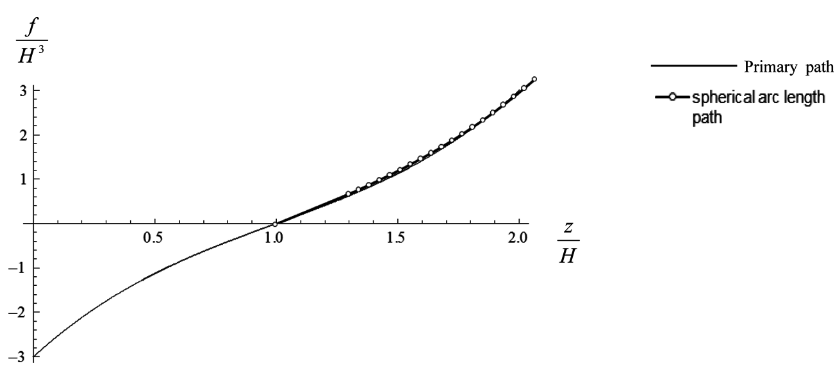


Figure 14 Equilibrium path of the bi-pyramidal symmetric truss due to loading in the ((Z)) direction.

Considering Eq. (79), the solution to this equation is in the following form:

$$\begin{cases} \theta = \theta_p \\ z = .5(2H \pm 2.82843\sqrt{-H^2 - .5r^2}) \\ f = r(B^2 - 2H^2) \end{cases} \tag{83}$$

$$\begin{cases} \theta = \pm\pi + \theta_p \\ z = .5(2H \pm 2.82843\sqrt{-H^2 - .5r^2}) \\ f = -r(B^2 - 2H^2) \end{cases} \tag{84}$$

$$\begin{cases} \theta = \theta_p \\ z = H \\ f = r(B^2 + r^2) \end{cases} \tag{85}$$

$$\begin{cases} \theta = \pm\pi + \theta_p \\ z = H \\ f = -r(B^2 + r^2) \end{cases} \tag{86}$$

It should be mentioned that Eqs. (83) and (84) do not give real values. Hence, the equilibrium path of the structure is presented by Eqs. (85) and (86). The representative curves of these two equations in three-dimensional space are similar. In other words, only one equilibrium path exists for this loading condition, which is the primary path. Figure 15 illustrates the equilibrium path of this truss. In this figure, the horizontal and vertical axes represent the dimensionless values of $\frac{r}{B}$ and $\frac{f}{B^3}$, respectively. In addition, the equilibrium path obtained by the arc-length scheme is compared to the path of Eq. (85) in this figure. It can be mathematically proved that if the cross-section of the elements in one pyramid is at least 10 times the cross-section of the other, the possibility of buckling will exist for the structure. Such a truss is considered in the following text.

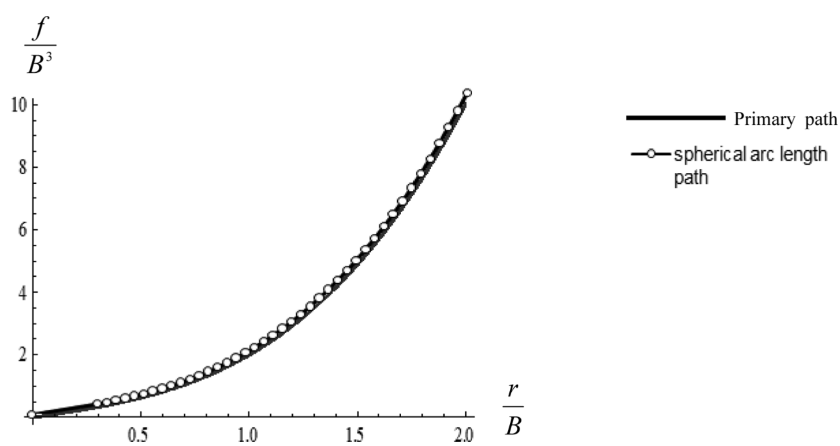


Figure 15 Equilibrium path of the bi-pyramidal symmetric truss under radial loading.

Three-Dimensional Bi-Pyramidal Asymmetric Truss

For the truss illustrated in Fig. 16, if the cross-section of the elements in one pyramid is assumed to be 10 times the cross-section of the other ($A_1 = 0.1A_0$), the structure will no longer be symmetrical and buckling will take place. All the previous assumptions still hold for the current structure. The total potential energy function for this structure will be in the following form after simplification:

$$\begin{aligned} U &= \frac{nA_0E}{8l_0^3}(1.9H^4 + 2.2B^2r^2 + 1.1r^4 - 2.4H^3z + 2.2r^2z^2 + 1.1z^4 \\ &\quad + H^2(-1.4r^2 + 0.2z^2) + H(-0.8r^2z - 0.8z^3)) \\ V &= p(q_r r \operatorname{Cos}(\theta - \theta_p) + q_z z) \\ \Pi &= U - V \end{aligned} \tag{87}$$

Differentiating this equation with respect to displacements leads to the following equilibrium equations:

$$\begin{cases} \dfrac{\partial \Pi}{\partial r} = \dfrac{nA_0E}{8l_0^3}(4.4B^2r - 2.H^2r + 4.4r^3 - 1.6Hrz + 4.4rz^2) - pq_r \operatorname{Cos}(\theta - \theta_p) = 0 \\ \dfrac{1}{r}\dfrac{\partial \Pi}{\partial \theta} = pq_r \operatorname{Sin}(\theta - \theta_p) = 0 \\ \dfrac{\partial \Pi}{\partial z} = \dfrac{nA_0E}{8l_0^3}(-2.4H^3 + 0.4H^2z + 4.4r^2z + 4.4z^3 + H(-0.8r^2 - 2.4z^2)) - pq_z = 0 \end{cases} \tag{88}$$

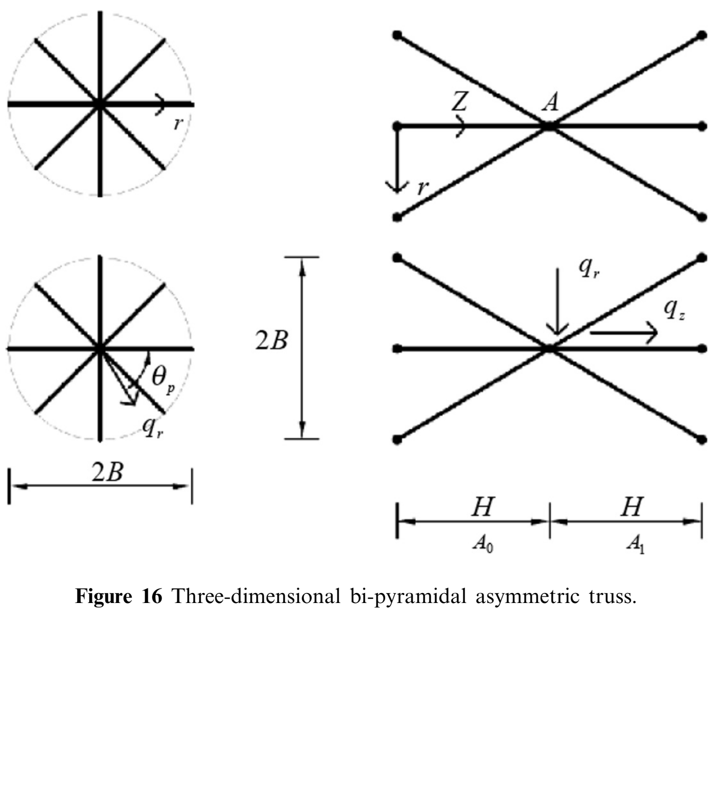


Figure 16 Three-dimensional bi-pyramidal asymmetric truss.

If the last equation is differentiated with respect to nodal displacements, the tangent stiffness matrix will be attained as follows:

$$[K_T] = \left(\frac{nA_0E}{8l_0^3}\right)\begin{bmatrix} 4.4B^2 - 2.8H^2 + 13.2r^2 - 1.6Hz + 4.4z^2 & 0 & -1.6Hr + 8.8rz \\ 0 & 4.4B^2 - 2.H^2 + 4.4r^2 - 1.6Hz + 4.4z^2 & 0 \\ -1.6Hr + 8.8rz & 0 & 0.4H^2 + 4.4r^2 - 4.8Hz + 13.2z^2 \end{bmatrix} \tag{89}$$

With the definition of the variables c and d, the eigenvalues of the tangent stiffness matrix are obtained as follows:

$$c = 4.4B^2 - 2.4H^2 + 17.6r^2 - 6.4Hz + 17.6z^2$$

$$d = \sqrt{\begin{array}{l} B^4 - 1.45455B^2H^2 + 0.528926H^4 + 1.45455B^2Hz - 1.05785H^3z \\ \quad - 4B^2z^2 + 3.43802H^2z^2 - 2.90909Hz^3 + 4z^4 \end{array}}$$

$$\begin{cases} \lambda_1 = \dfrac{nA_0E}{8l_0^3}(4.4B^2 - 2.8H^2 + 4.4r^2 - 1.6Hz + 4.4z^2) \\ \lambda_2 = \dfrac{nA_0E}{8l_0^3}\left(\dfrac{c - 4.4d}{2}\right) \\ \lambda_3 = \dfrac{nA_0E}{8l_0^3}\left(\dfrac{c + 4.4d}{2}\right) \end{cases} \tag{90}$$

All of these eigenvalues can be equal to zero. In the proceeding text, the roots of each eigenvalue will be evaluated for different loading conditions.

Zero-Loaded Structure The equilibrium equations for this loading case will take the following form:

$$\begin{cases} \dfrac{nA_0E}{8l_0^3}(4.4B^2r - 2.H^2r + 4.4r^3 - 1.6Hrz + 4.4rz^2) = 0 \\ \dfrac{nA_0E}{8l_0^3}(-2.4H^3 + 0.4H^2z + 4.4r^2z + 4.4z^3 + H(-0.8r^2 - 2.4z^2)) = 0 \end{cases} \tag{91}$$

The solution to this equation is as follows:

$$\begin{cases} r = 0 \\ z = H \end{cases} \tag{92}$$

Vertical Loading The following relations are the equilibrium equations resulting from this loading case:

$$\begin{cases} \dfrac{nA_0E}{8l_0^3}(4.4B^2r - 2.H^2r + 4.4r^3 - 1.6Hrz + 4.4rz^2) = 0 \\ \dfrac{nA_0E}{8l_0^3}(-2.4H^3 + 0.4H^2z + 4.4r^2z + 4.4z^3 + H(-0.8r^2 - 2.4z^2)) = p \end{cases} \tag{93}$$

In order to solve this equation, the variable f is defined in the following form:

$$f = \frac{8l_0^3 p}{nA_0E} \tag{94}$$

The solution is obtained as follows:

$$\begin{cases} r = 0 \\ f = 0.2(-12H^3 + 2H^2z - 12Hz^2 + 22z^3) \end{cases} \tag{95}$$

$$\begin{cases} r = \pm 0.301511\sqrt{-11B^2 + 7H^2 + 4Hz - 11z^2} \\ f = 0.2(4B^2H - 14.5455H^3 - 22B^2z + 14.5455H^2z) \end{cases} \tag{96}$$

Since the primary equilibrium path of the structure passes through the origin point, Eq. (95) will be presenting it. If the equalities relevant to the primary path are introduced into Eq. (90), the critical point will be assessed in the following form:

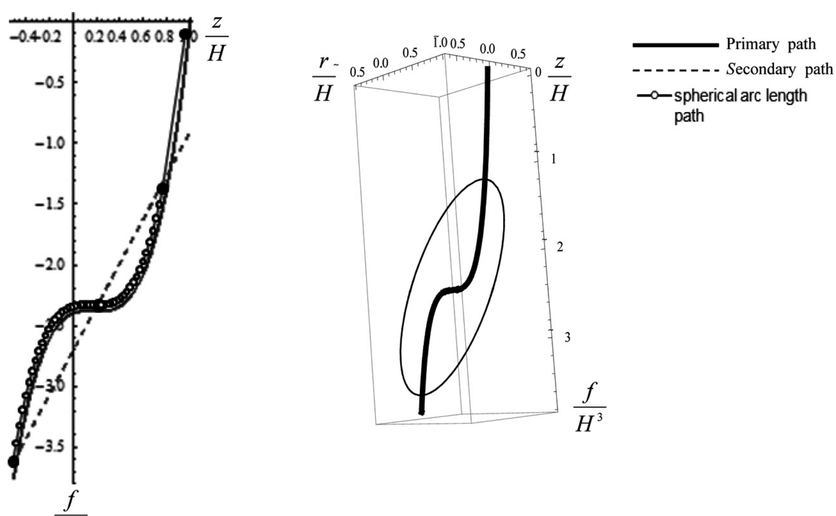
$$\begin{cases} r = 0 \\ z = 0.113636\left(1.6H + 7.2\sqrt{-1.49383B^2 + H^2}\right) \\ f = -2.38017H^3 - 3.6B^2\sqrt{-1.49383B^2 + H^2} \\ \qquad + 2.38017H^2\sqrt{-1.49383B^2 + H^2} \end{cases} \tag{97}$$

$$\begin{cases} r = 0 \\ z = 0.113636\left(1.6H - 7.2\sqrt{-1.49383B^2 + H^2}\right) \\ f = -2.38017H^3 + 3.6B^2\sqrt{-1.49383B^2 + H^2} \\ \qquad - 2.38017H^2\sqrt{-1.49383B^2 + H^2} \end{cases} \tag{98}$$

$$\begin{cases} r = 0 \\ z = 0.234305H \\ f = -2.38144H^3 \end{cases} \tag{99}$$

$$\begin{cases} r = 0 \\ z = 0.129332H \\ f = -2.37889H^3 \end{cases} \tag{100}$$

Eqs. (97) and (98) represent multiple bifurcation points. If the geometry of the structure had the condition $B \leq H\sqrt{0.66942}$, these points would lie on the equilibrium path. In this case, the stable path of the structure would be according to Eq. (96) after bifurcation. It should be mentioned that the value of θ in Eq. (96) may take any value. If in the case $B = H\sqrt{0.66942}$, the two bifurcation points will coincide and the secondary equilibrium path will become a single point. Eqs. (99) and (100) correspond to limit points. For any ratio of $\frac{B}{H}$, these points will be on the equilibrium path. For trusses which have bifurcation points, bifurcation may take place before or after the limit point, depending on the geometry of the structure. In addition, these points may even coincide with each other and form a triple-bifurcation point. With the condition $B = \sqrt{\frac{2}{3}}H$, a triple-bifurcation point will exist. Figure 17 illustrates the equilibrium path of a truss where the ratio $\frac{B}{H}$ is equal to ((0.5)). Hence, the multiple bifurcation point lies on the equilibrium path. In this figure, the vertical axis represents the dimensionless value $\frac{f}{H^3}$. The secondary path is indicated by a dashed line. Figure 17(a) shows the projection of the equilibrium path on the $(f - z)$ plane, while this path in the three-dimensional $(f - z - r)$ space is given in Fig. 17(b). In these figures, the axes r and z correspond to dimensionless values $\frac{r}{H}$ and $\frac{z}{H}$, respectively. Also, Fig. 17(a) compares the equilibrium path obtained by the analytical and the arc-length methods.



(a) (b)

Figure 17 Equilibrium path of the asymmetric bi-pyramidal truss under vertical loading (a) $(((f - z)))$ plane and (b) $(((f - z - r)))$ space.

Radial Loading For this loading case, the equilibrium equations will take the following form:

$$\begin{cases} \dfrac{nA_0E}{8l_0^3}(4.4B^2r - 2.H^2r + 4.4r^3 - 1.6Hrz + 4.4rz^2) - p\,\mathrm{Cos}(\theta - \theta_p) = 0 \\ p\,\mathrm{Sin}(\theta - \theta_p) = 0 \\ \dfrac{nA_0E}{8l_0^3}(-2.4H^3 + 0.4H^2z + 4.4r^2z + 4.4z^3 + H(-0.8r^2 - 2.4z^2)) = 0 \end{cases} \tag{101}$$

Based on Eq. (94) and by defining the intermediate parameters c and d, the solution will be as follows:

$$c = \sqrt{-6H^3 + H^2z - 6Hz^2 + 11z^3}$$
$$d = \sqrt{2H - 11z}$$
$$\begin{cases} r = \pm\dfrac{c}{d} \\ \theta = \theta_p \\ f = \pm 0.2\left(\dfrac{-132H^3c}{d^3} + \dfrac{22B^2c}{d} - \dfrac{14H^2c}{d} + \dfrac{22H^2zc}{d^3} - \dfrac{8Hzc}{d}\right. \\ \qquad \left. - \dfrac{132Hz^2c}{d^3} + \dfrac{22z^2c}{d} + \dfrac{242z^3c}{d^3}\right) \end{cases} \tag{102}$$

It can be proven that the eigenvalues of the tangent stiffness matrix are positive for the entire equilibrium path. Therefore, this structure will never buckle under radial loading. At the onset of loading, the z component of the free node (node A) has the value of H. This value will decrease with the application of the load. In the respective relation for d in Eq. (102), the term under the square root will become negative when $z > \frac{2}{11}H$. Therefore, the term under the square root in the respective relation for c should also be negative, which happens when $z < H$. The value of f becomes extremely large as z approaches $\frac{2}{11}H$. The rate at which f increases is greater than the z component. In other words, the structure has a hardening behavior. This hardening characteristic eliminates the possibility of buckling. Figure 18 shows the equilibrium path of this structure under radial loading. Figure 18(a) illustrates the projection of the equilibrium path on the $(z - f)$ axis, while the projection of this path on the $(r - f)$ plane is given in Fig. 18(b). Figure 18(c) illustrates the equilibrium path of the structure in three-dimensional space.

It has been shown that this truss has a very specific behavior. Its response under vertical loading is similar to the regular pyramidal truss. While under horizontal loading, it behaves alike the three-member suspended truss. In the proceeding text, the characteristics of some other two-dimensional trusses will be discussed briefly.

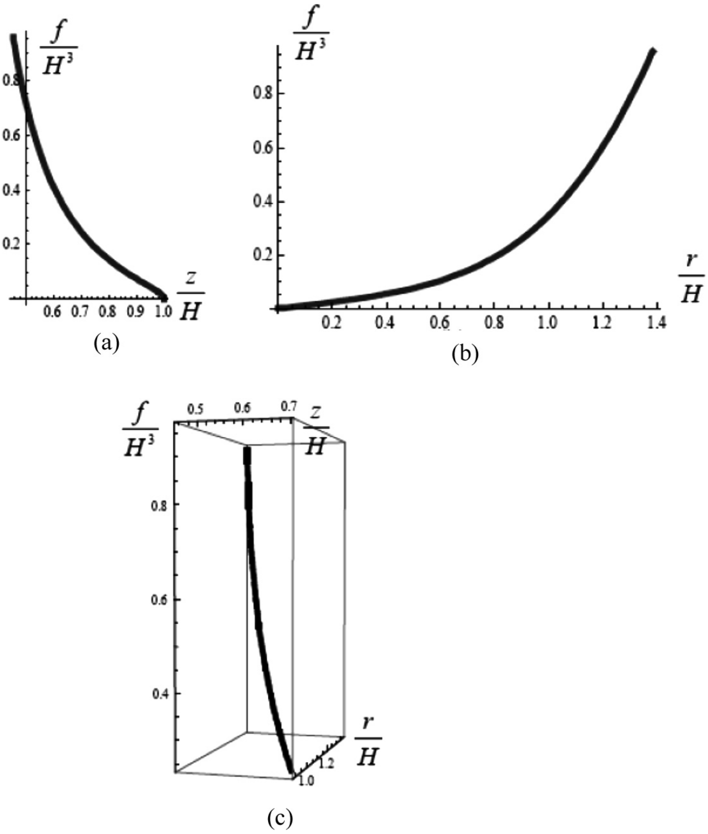


Figure 18 Equilibrium path of the asymmetric bi-pyramidal truss under radial loading.

Semi-Circle Truss

The geometry of the semi-circle truss is given in Fig. 19. This truss consists of n members at $\frac{\pi}{n-1}$ intervals. The radius of the circle is denoted by r. The only free node is located at the center of the circle where the origin of the coordinate system will be attached.

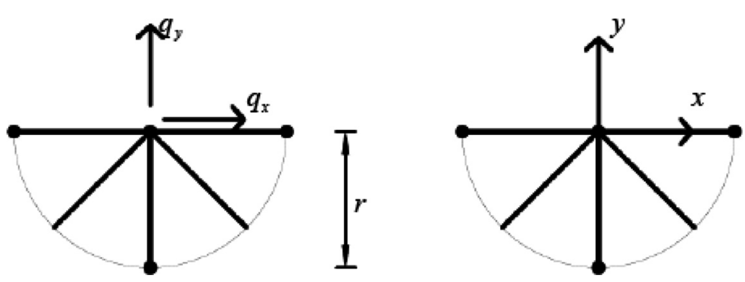


Figure 19 Semi-circle truss with (n) members.

The total potential energy function of this truss is presented by the following equation:

$$U = \frac{A_0 E}{8r^3} \sum_{i=1}^{n} \left(\left(x - rCos\left(\frac{i-1}{n-1}\pi \right) \right)^2 + \left(y + rSin\left(\frac{i-1}{n-1}\pi \right) \right)^2 - r^2 \right)^2 \tag{103}$$

The analytical results of this example with $n = 3$ and $n = 5$ are given in the following.

Three-Member Semi-Circle Truss Figure 20 shows the three-member semi-circle truss.

For this case, the total potential energy function takes the following form:

$$\Pi = \frac{A_0 E}{8r^3}(4ry(x^2 + y^2) + 3(x^2 + y^2)^2 + 4r^2(2x^2 + y^2)) - p(q_x x + q_y y) \tag{104}$$

The equilibrium equations are obtained as follows:

$$\begin{cases} \dfrac{\partial \Pi}{\partial x} = \dfrac{A_0 E}{8r^3}(16r^2 x + 8rxy + 12x(x^2 + y^2)) - pq_x = 0 \\ \dfrac{\partial \Pi}{\partial y} = \dfrac{A_0 E}{8r^3}(8r^2 y + 8ry^2 + 4r(x^2 + y^2) + 12y(x^2 + y^2)) - pq_y = 0 \end{cases} \tag{105}$$

The tangent stiffness matrix is assessed as follows:

$$[K_T] = \frac{A_0 E}{8r^3} \begin{bmatrix} 16r^2 + 24x^2 + 8ry + 12(x^2 + y^2) & 8rx + 24xy \\ 8rx + 24xy & 8r^2 + 24ry + 24y^2 + 12(x^2 + y^2) \end{bmatrix} \tag{106}$$

It can be proven that this structure will never buckle. In order to find the solution to the structure, the variable f is defined as follows:

$$f = \frac{8r^3 p}{A_0 E} \tag{107}$$

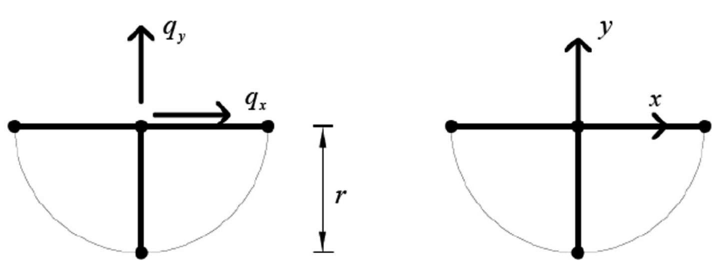


Figure 20 Three-member semi-circle truss.

The response of the structure under different conditions is provided in the following.

1. Zero-loading

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \tag{108}$$

2. Vertical loading

$$\begin{cases} x = 0 \\ f = 8r^2 + 12ry + 12y^2 \end{cases} \tag{109}$$

3. Horizontal loading

$$c = \sqrt{-2r^2 - 3ry - 3y^2}$$
$$d = \sqrt{r + 3y}$$
$$\begin{cases} x = \pm \dfrac{c\sqrt{y}}{d} \\ f = \mp\left(\dfrac{24r^2y^{\frac{3}{2}}c}{d^3} - \dfrac{36ry^{\frac{5}{2}}c}{d^3} - \dfrac{36y^{\frac{7}{2}}c}{d} + \dfrac{16r^2c\sqrt{y}}{d} + \dfrac{8ry^{\frac{3}{2}}c}{d} + \dfrac{12y^{\frac{5}{2}}c}{d}\right) \end{cases} \tag{110}$$

In the last equation, y must be positive in order for x and f to have real values.

Five-Member Semi-Circle Truss Figure 21 shows this structure.
For this structure, the total potential energy function is attained as follows:

$$\Pi = \frac{A_0E}{8r^3}(4(1+\sqrt{2})(x^2+y^2)ry + 5(x^2+y^2)^2 + 4r^2(3x^2+2y^2)) - p(q_xx + q_yy) \tag{111}$$

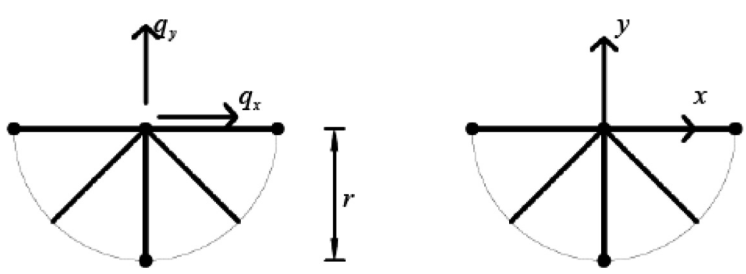


Figure 21 Five-member semi-circle truss.

The equilibrium equations take the following form:

$$\begin{cases} \dfrac{\partial \Pi}{\partial x} = \dfrac{A_0 E}{8r^3}(24r^2 x + 8rxy(1+\sqrt{2}) + 20x(x^2+y^2)) - pq_x = 0 \\ \dfrac{\partial \Pi}{\partial y} = \dfrac{A_0 E}{8r^3}(16r^2 y + 8ry^2(1+\sqrt{2}) + 4r(1+\sqrt{2})(x^2+y^2) + 20y(x^2+y^2)) - pq_y = 0 \end{cases} \tag{112}$$

The tangent stiffness matrix is obtained as follows:

$$[K_T] = \frac{A_0 E}{8r^3}\begin{bmatrix} 24r^2 + 40x^2 + 8ry(1+\sqrt{2}) + 20(x^2+y^2) & 8rx(1+\sqrt{2}) + 40xy \\ 8rx(1+\sqrt{2}) + 40xy & 16r^2 + 24ry(1+\sqrt{2}) + 40y^2 + 20(x^2+y^2) \end{bmatrix} \tag{113}$$

This truss will never buckle as well. According to Eq. (106), the response of the structure under different loading conditions will be assessed in the following.

1. Zero-loading

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \tag{114}$$

2. Vertical loading

$$\begin{cases} x = 0 \\ f = y(16r^2 + 28.9706ry + 20y^2) \end{cases} \tag{115}$$

3. Horizontal loading

$$c = \sqrt{-1.65685r^2 - 3ry - 2.07107y^2}$$
$$d = \sqrt{r + 2.07107y}$$
$$\begin{cases} x = \pm \dfrac{c\sqrt{y}}{d} \\ f = \mp\left(\dfrac{33.1371r^2 y^{\frac{3}{2}} c}{d^3} - \dfrac{60ry^{\frac{5}{2}} c}{d^3} - \dfrac{41.4214y^{\frac{7}{2}} c}{d^3} + \dfrac{24r^2 c\sqrt{y}}{d} + \dfrac{19.3137ry^{\frac{3}{2}} c}{d} + \dfrac{20y^{\frac{5}{2}} c}{d}\right) \end{cases} \tag{116}$$

In the last equation, similar to Eq. (110), y must be positive.

Two-Member Truss

Figure 22 illustrates the two-member truss. The initial cross-section of the elements is equal to A_0 and l can be given any value. The origin of the coordinate

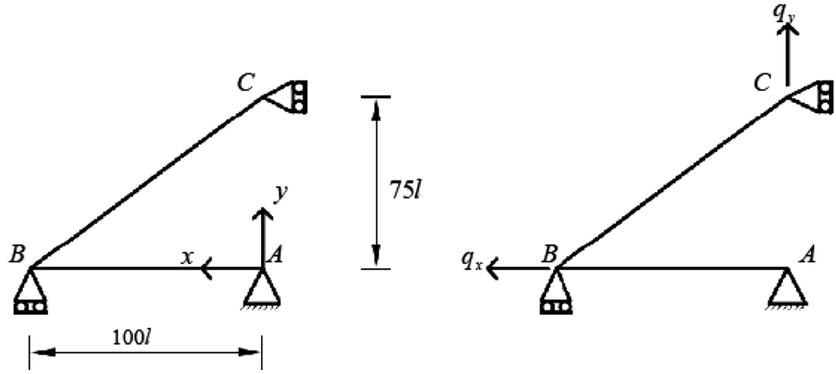


Figure 22 Geometry of the two-member truss.

system is positioned at A. The location of B is denoted by x and the distance of node C from the point A is equal to y. The applied loads at B and C are equal to q_x and q_y, respectively.

The total potential energy function of this structure is assessed as follows:

$$\Pi = \frac{A_0 E}{10^9 l^3}(125(-10000l^2 + x^2)^2 + 64(-15625l^2 + x^2 + y^2)^2) - p(q_x x + q_y y) \quad (117)$$

The equilibrium equations take the following form:

$$\begin{cases} \dfrac{\partial \Pi}{\partial x} = \dfrac{A_0 E}{10^9 l^3}(500x(-10000l^2 + x^2) + 256x(-15625l^2 + x^2 + y^2)) - pq_x = 0 \\ \dfrac{\partial \Pi}{\partial y} = \dfrac{A_0 E}{10^9 l^3}(256y(-15625l^2 + x^2 + y^2)) - pq_y = 0 \end{cases} \quad (118)$$

The tangent stiffness matrix is determined as follows:

$$[K_T] = \frac{A_0 E}{10^9 l^3}\begin{bmatrix} 1512x^2 + 500(-10000l^2 + x^2) + 256(-15625l^2 + x^2 + y^2) & 512xy \\ 512xy & 512y^2 + 256(-15625l^2 + x^2 + y^2) \end{bmatrix} \quad (119)$$

Critical points can be obtained after the structure is solved for each loading condition. In order to attain the response of the structure, the variable f is defined as follows:

$$f = \frac{10^9 l^3 p}{A_0 E} \quad (120)$$

1. Zero-loading

$$\begin{cases} x = 0 \\ y = 0 \end{cases}$$
$$\begin{cases} x = \pm 109.109l \\ y = 0 \end{cases}$$
$$\begin{cases} x = \pm 100l \\ y = \pm 75l \end{cases} \tag{121}$$
$$\begin{cases} x = 0 \\ y = \pm 125l \end{cases}$$

2. Vertical loading
   The primary equilibrium path is presented by the following equation:

$$\begin{cases} x = 0.0727393\sqrt{2.25 \times 10^6 l^2 - 64y^2} \\ f = 0.00529101y(-1.8 \times 10^8 l^2 + 32000y) \end{cases} \tag{122}$$

The secondary path will be according to:

$$\begin{cases} x = 0 \\ f = y(-4 \times 10^6 l^2 + 256y^2) \end{cases} \tag{123}$$

The critical points of the primary path are determined as follows:

$$B : \begin{cases} x = 0 \\ y = \pm 187.5l \\ f = \pm 9375 \times 10^5 l^3 \end{cases}$$
$$L : \begin{cases} x = 106.15951l \\ y = \pm 43.30127l \\ f = \mp 27492894.43l^3 \end{cases} \tag{124}$$

This path has two symmetric bifurcation points marked by B. Two limit points also exist and are denoted by L. The secondary path, aside a bifurcation point, has a limit point as well. The location of this limit point is as given below:

$$L : \begin{cases} x = 0 \\ y = \pm \dfrac{125}{\sqrt{3}} l \\ f = \mp 192450089.7l^3 \end{cases} \tag{125}$$

It should be mentioned that at this point, the deformation of the structure lies on the primary equilibrium path.

3. Horizontal loading
   The primary equilibrium path is expressed as follows:

$$\begin{cases} y = \sqrt{15625l^2 - x^2} \\ f = x(-5 \times 10^6 l^2 + 500x^2) \end{cases} \tag{126}$$

The secondary equilibrium path is presented by the following equation:

$$\begin{cases} y = 0 \\ f = x(-9 \times 10^6 l^2 + 756x^2) \end{cases} \tag{127}$$

The critical points of the primary path are given below:

$$B: \begin{cases} x = \pm 125l \\ y = 0 \\ f = \pm 351562500l^3 \end{cases}$$
$$L: \begin{cases} x = \pm 57.73517l \\ y = 110.86771l \\ f = \mp 192450089.7l^3 \end{cases} \tag{128}$$

With the exclusion of the bifurcation point, which is at the intersection of the primary and secondary paths, the critical point of the secondary equilibrium path is as follows:

$$L: \begin{cases} x = \pm \dfrac{500l}{3\sqrt{7}} \\ y = 0 \\ f = \mp 377964473l^3 \end{cases} \tag{129}$$

It should be reminded that all limit points lying on the secondary path have no significance. This is due to the fact that at the occurrence of these points, the structure deforms on the primary path. In other words, the structure will never take the geometric shape attributed to these points.

CONCLUSION

This paper investigates analytical solutions to important geometrically nonlinear benchmark problems. A Series of structural trusses under various loading conditions are considered in this study. Stress-strain linearity, large deformations and ideal hinges are the primary assumptions in the development of the closed-form solutions. Post-buckling and equilibrium paths, as well as critical points, are obtained through the mathematical procedures. In addition, some equilibrium paths of structures are also determined by this technique. Necessary discussions are provided for different values of effective parameters. In contrast to initial analyst guesses, and based on this investigation, it is possible to design structural trusses to have both hardening and geometric nonlinear behavior. These classes of structures are very stiff and stable under any severe loading.

In order to determine the validity of the results, the equilibrium paths calculated by the analytical scheme are compared to the ones obtained by the spherical arc-length technique. It has been demonstrated that both strategies produce the same response, and therefore, the validities of the closed-form solutions are certified. The structural problems considered in this paper and their corresponding solutions can be considered as benchmark problems in the field of geometric nonlinear analysis. By utilizing the same analytical procedure, some other complex static paths can be found. In general, the standard numerical schemes could not pass these kinds of the curves. It should be emphasized that one of the difficult numerical techniques is calculating the secondary path in bifurcation points. This study has found some of the mentioned paths.

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