Automatic DR Structural Analysis of Snap-Through and Snap-Back Using Optimized Load Increments

M. Rezaiee-Pajand[1] and J. Alamatian[2]

Abstract: In this paper, new schemes are presented for the dynamic relaxation (DR) method so that the snap-through and the snap-back regions can be traced automatically. These procedures are based on the minimization of the residual force (MRF) and minimization of the residual energy (MRE), and they are capable of updating the load factor in each DR iteration. The suggested techniques are perfectly automatic. Therefore, they do not require any additional parameters such as arc length, incremental displacement, etc. For numerical verification, some frame and truss structures, all possessing geometrical nonlinear behaviors, are analyzed. Tracing the statical path shows that both the MRF and MRE methods can be used successfully in structures with snap-through and snap-back regions. The numerical results indicate that the MRE scheme traces the statical path with a greater number of increments than the MRF. While the jumping probability of the MRE is less than that of the MRF, the analysis time may increase in the MRE. Also, a comparison between the proposed DR methods and arc-length approach shows that the MRF and MRE procedures can present the limit points with higher accuracy.

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Introduction

Finite-element or finite difference formulation leads to a system of simultaneous equations that show the behavior of structures. The stiffness relationships can be written as follows:

$$[S]\{D\} = \{F\} = \{P\} \tag{1}$$

Here, $[S]$ is the structural stiffness matrix and $\{D\}$, $\{F\}$, and $\{P\}$ are displacement and the internal and external force vectors, respectively. Nonlinear effects such as elastic-plastic or large deformation behaviors lead to a complicated system of equations. In these cases, the stiffness matrix or even the external load vector will be a function of displacement. By solving Eq. (1), the displacement vector can be calculated. Other quantities such as strains and stresses can be explicitly calculated based on the displacements. Therefore, the final stage of each analysis can be completed by employing an equation solver. It is important to choose a powerful procedure that can be useful for a variety of problems. In this paper, the dynamic relaxation (DR) technique is used.

In 1965, DR was introduced by Day (Day 1965) and then followed by other researchers such as Otter (Otter 1966). This iterative technique was used for solving a system of simultaneous linear and nonlinear equations. The mentioned procedure may be explained by either mathematical or physical theories. Mathematically, the DR formulation is based on the second-order Richardson role, which was developed by Frankel (Frankel 1950). Physically, the DR scheme can be illustrated by the steady-state response of an artificial dynamic system with a fictitious density (Welsh 1967; Cassell et al. 1968).

The first application to nonlinear problems, postbuckling analysis, and large deflection plates was performed by Rushton (Rushton 1968, 1970, 1972). Brew and Brotton formulated the DR method by using a first-order dynamic equilibrium relationship (Brew and Brotton 1971). Some alternative formulations have been presented for fictitious mass (Wood 1971), critical damping (Bunce 1972), and estimation of steady-state displacement (Alwar et al. 1975). Turvey and Wittrick applied DR method to the postbuckling of laminated composite plates (Turvey and Wittrick 1973). Furthermore, the Gerschgörin circle theory has been used for fictitious mass in nonlinear problems (Cassell and Hobbs 1976). DR scheme has also been used to analysis of membrane structures (Barnes 1975) and form finding analysis (Bames 1977). Moreover, DR method, based on kinetic damping theory, was applied to unstable rock mechanism (Cundall 1976). In other applications, the DR algorithm has been used for nonlinear analysis of plates (Frieze et al. 1978; Turvey 1978, 1979).

The first error analysis of DR iterations was performed by Papadrakakis who described an automatic procedure for the selection of DR parameters (Papadrakakis 1981). Moreover, Underwood presented another interesting formulation for the explicit DR method (Underwood 1983). The implicit DR method has also been formulated (Felippa 1984). Additionally, DR procedure has been used to obtain the nonlinear static response of pretension cable roofs (Lewis et al. 1984) and finite-element analysis of bending plates (Shawi and Mardirosion 1987). In some papers, alternative formulations have been presented for fictitious time and damping (Shizhong 1988) and estimation of steady-state displacement (Zhang and Yu 1989). Other researchers have used the DR algorithm for different engineering problems (Turvey and Salehi 1990; Bardet and Proubet 1991).

Another use of the DR scheme in the postbuckling analysis was performed by Ramesh and Krishnamoorthy, in which they

[1]Professor of Structural Engineering, Dept. of Civil Engineering, Ferdowsi Univ. of Mashhad, Mashhad, Iran (corresponding author). E-mail: mrpajand@yahoo.com

[2]Assistant Professor of Structural Engineering, Dept. of Civil Engineering, Islamic Azad Univ., Mashhad Branch, Mashhad, Iran. E-mail: alamatian@yahoo.com

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independently combined the DR algorithm with the incremental displacement approach and arc-length procedure (Ramesh and Krishnamoorthy 1993, 1994). In the other papers, alternative fictitious damping based on the Rayleigh's principle (Zhang et al. 1994) and DR applications in elastic-plastic and buckling problems of plates have been studied (Kadkhodayan and Zhang 1995; Kadkhodayan et al. 1997). Using the principle of minimum potential energy of surface tension, DR approach was used for form finding of lightweight tension structures that include prestressed cable nets and fabric membranes (Lewis and Lewis 1996).

Several papers in the present decade deal with some well-known DR applications such as shape-finding analysis (Wood 2002; Han and Lee 2003) and the elastoplastic large deflection analysis of annular sector plates (Turvey and Salehi 2005). Furthermore, the DR method has been combined with neural networks to increase model accuracy of tensegrity structures (Domer et al. 2003). In a recent book, Topping and Ivanyi concentrated on the computational aspects of analysis and design of cable membrane structures using the DR method (Topping and Ivanyi 2007). The modified fictitious time step has been formulated based on minimization of the residual force in each DR iteration (Kadkhodayan et al. 2008). Recently, the DR strategy has been used for nonlinear dynamic analysis of structures (Rezaiee-Pajand and Alamatian 2008).

Based on the literature review, a few DR algorithms, such as the one presented by Ramesh, can trace the snap-through and snap-back regions. This paper tries to formulate a new DR method, which can solve the problems with snap-through and snap-back behaviors. For this purpose, a variable external load is considered. In other words, the original contribution of this paper is to optimize the choice of the load increment in each DR iteration. This optimization will improve the ability of the DR method for tracing the snap-through and snap-back regions of the equilibrium path. To reach this goal, the load factor of each DR iteration is calculated by two proposed approaches: minimization of residual force and minimization of residual energy. For numerical verification, some frame and truss structures with nonlinear behavior are analyzed.

DR Method

Both mathematical and physical theories are used in the DR formulation. According to the DR method, an equivalent static system, Eq. (1), is shifted to an assumed dynamic space by adding artificial inertia and damping forces as follows:

$$[\mathrm{M}]^n\{\mathrm{A}\}^n+[\mathrm{C}]^n\{\mathrm{V}\}^n+[\mathrm{S}]^n\{\mathrm{D}\}^n=\{\mathrm{F}\}^n=\{\mathrm{P}\}^n \tag{2}$$

where $\{\mathrm{V}\}^n$ and $\{\mathrm{A}\}^n$=artificial velocity and acceleration vectors and $[\mathrm{M}]^n$ and $[\mathrm{C}]^n$=fictitious mass and damping matrices of the nth iteration of DR, respectively. The steady-state response of this artificial dynamic system is the solution of Eq. (1) when the inertia and damping forces are zero. There are different approaches to derive the DR iterative relationships. In a common formulation, such as the Papadrakakis scheme or the Underwood procedure, mass and damping matrices are assumed to be diagonal, and the explicit central finite difference integration is used. Consequently, DR iterative relationships are obtained as follows (Underwood 1983):

$$v_i^{n+1/2}=\frac{2m_{ii}-\tau^n C_{ii}}{2m_{ii}+\tau^n C_{ii}}v_i^{n-1/2}+\frac{2\tau^n}{2m_{ii}+\tau^n C_{ii}}r_i^n \quad i=1,2,\ldots,q \tag{3}$$

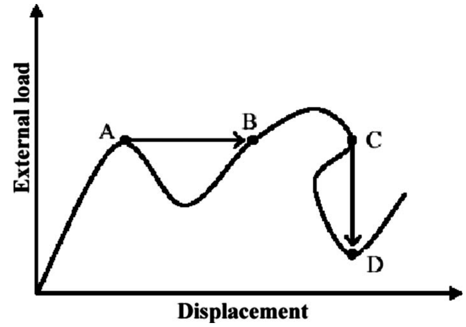

Fig. 1. Schematic behavior of common DR method in the snap-through and snap-back regions

$$D_i^{n+1}=D_i^n+\tau^{n+1}v_i^{n+1/2} \quad i=1,2,\ldots,q \tag{4}$$

where τ^n, m_{ii}, and C_{ii}=fictitious time step, diagonal elements of mass, and damping matrices, respectively. Notation q denotes the number of degrees of freedom and r_i^n is the residual force of the ith degree of freedom, i.e., $\{\mathrm{R}\}^n$ in the nth iteration

$$\{\mathrm{R}\}^n=[\mathrm{M}]^n\{\mathrm{A}\}^n+[\mathrm{C}]^n\{\mathrm{V}\}^n=\{\mathrm{P}\}^n-\{\mathrm{F}\}^n \tag{5}$$

Using Gerschgörin's circle theory, the fictitious mass is obtained as follows (Underwood 1983):

$$m_{ii}^n>\frac{(\tau^n)^2}{4}\sum_{j=1}^{q}|s_{ij}^n| \quad i=1,2,\ldots,q \tag{6}$$

In the explicit DR procedure introduced by Underwood, damping matrix is assumed to be proportional to the diagonal mass matrix as given below

$$[\mathrm{C}]^n=c^n[\mathrm{M}]^n \tag{7}$$

Here, c^n is the damping factor of the nth DR iteration. The theory of critical damping and the Rayleigh's principle (Chopra 2002) are used to estimate fictitious damping factor (Zhang et al. 1994)

$$c^n=2\sqrt{\frac{(\{\mathrm{D}\}^n)^T\{\mathrm{F}\}^n}{(\{\mathrm{D}\}^n)^T[\mathrm{M}]^n\{\mathrm{D}\}^n}} \tag{8}$$

In the most common DR algorithms, constant fictitious time is used ($\tau=1$). This paper tries to trace the snap-through and snap-back regions of statical path which cannot be achieved by the common DR algorithms. To fulfill this goal, two different criteria, i.e., minimization of residual force and minimization of residual energy, are used to calculate the load factor in each DR iteration.

Variable Load in DR Iterations

In the DR iterations, the external load vector is assumed to be constant. Therefore, the common DR method cannot trace the snap-through and snap-back regions and the statical path jump around these points. Fig. 1 shows such behaviors of DR scheme in which statical paths have been transferred from A to B and from C to D. In other words, true curves between A and B and C and D cannot be traced by the common DR iterations.

In general, there are several procedures for tracing the postbuckling path such as the method of artificial springs, the displacement incrementation procedure, the current stiffness parameter scheme, arc-length methods, etc. (Ramesh and Krishnamoorthy 1993). Some of these techniques have been used in the

DR algorithm. The application of DR tactic to the analysis of structures, which exhibit snap through and snap back, is not new. A long ago, Rushton showed how to analyze mode changes using DR strategy (Rushton 1970). Moreover, Turvey presented a DR analysis of the snap buckling of tapered imperfect circular plates (Turvey 1978). On the other hand, Ramesh and Krishnamoorthy developed the displacement incrementation procedure for the DR method (Ramesh and Krishnamoorthy 1993). In this scheme, selection of controlling displacement is difficult for the case of large structures. To mediate this difficulty, they proposed a variable-arc-length procedure. In this method, the square of the total displacement of one degree of freedom (to be selected by the analyst) is set equal to the square of the total arc length. Therefore, a second-order equation is obtained for the load factor.

The total arc-length and incremental displacement methods are not perfectly automatic. They allow the selection of some parameters such as the referenced degree of freedom, which have nonzero external loads, incremental displacement, and arc length. As a result, personal judgment has a considerable effect on the efficiency of the method. The load factor, however, is calculated from a second-order equation. This equation may have two different roots where the appropriate root should be selected based on the approach given by Crisfield (Crisfield 1981). If this second-order equation does not have real root, the load factor should be calculated by other existing techniques.

Based on the above discussion, methods such as incremental displacement and arc length are not attractive enough to use with the DR scheme because they cannot start and run automatically. Here, new formulations are proposed for correcting the load factor of DR iterations in structures with snap-through and snap-back regions. These techniques are based on the minimization of both the residual force and the residual energy of DR iterations. For this purpose, the load factor is introduced as an independent variable as follows:

$$\{\mathrm{R}\}^n = \lambda^n\{\mathrm{P}\}_{\mathrm{ref}} - \{\mathrm{F}\}^n \tag{9}$$

Here, λ^n and $\{\mathrm{P}\}_{\mathrm{ref}}$ are the load factor and the reference load vector, respectively. By this definition, the load factor can be modified in each iteration. A new condition is required for calculating this additional variable. Minimization of residual force and residual energy provides new conditions for calculating the load factor. In the following lines, each condition is formulated separately.

Minimization of Residual Force

The residual or out-of-balance force function is one of the factors that control the convergence of DR iterations. It can be defined as follows:

$$\mathrm{RFF} = \sum_{i=1}^{q} (r_i^n)^2 \tag{10}$$

where RFF=residual force function of the nth DR iteration. Kadkhodayan et al. minimized the residual force of the artificial dynamic system to calculate fictitious time step (Kadkhodayan et al. 2008). This algorithm cannot be used in snap-through and snap-back analyses because the external load is constant in the related formulation. In other words, they minimized the residual force of the artificial dynamic system to calculate the fictitious time step and assumed a constant external load in all iterations. In this study, the external load is variable in each DR iteration and the fictitious time step is constant ($\tau=1$). The load factor in the present paper is calculated by minimizing the residual force.

Substituting Eq. (9) into Eq. (10) leads to the following result:

$$\mathrm{RFF} = \left(\sum_{i=1}^{q} p_{i,\mathrm{ref}}^2\right)(\lambda^n)^2 - 2\left(\sum_{i=1}^{q} p_{i,\mathrm{ref}} f_i^n\right)\lambda^n + \sum_{i=1}^{q} (f_i^n)^2 \tag{11}$$

Here, RFF is a second-order equation of the load factor. The necessary condition for minimization of this function is that its first-order derivative with respect to the load factor is equal to zero

$$\frac{\partial \mathrm{RFF}}{\partial \lambda^n} = 0 \rightarrow \lambda^n = \frac{\sum_{i=1}^{q} p_{i,\mathrm{ref}} f_i^n}{\sum_{i=1}^{q} p_{i,\mathrm{ref}}^2} \tag{12}$$

Based on the second-order derivative test, a sufficient condition for minimization of the RFF is written as follows:

$$\frac{\partial^2 \mathrm{RFF}}{\partial (\lambda^n)^2} > 0 \rightarrow 2\sum_{i=1}^{q} p_{i,\mathrm{ref}}^2 > 0 \tag{13}$$

The last inequality is valid for all step of analyses. Therefore, the load factor formulated from Eq. (12) minimizes the RFF of the structure. This relation can be used in each DR iteration for correcting the load factor so that the nearest position to the equilibrium state is obtained. As a result, the jumping probability is reduced.

Minimization of Residual Energy

In each DR iteration, the residual force creates a displacement increment that approaches zero by converging DR iterations. This incremental displacement is called the out-of-balance displacement and has the following relationship:

$$\{\delta \mathrm{D}\}^{n+1} = \tau^{n+1}\{\mathrm{V}\}^{n+1/2} \tag{14}$$

If the out-of-balance force vector [Eq. (9)] is multiplied by the transpose residual displacement vector [Eq. (14)], the residual energy function (REF) can be defined as follows:

$$\mathrm{REF} = \tau^{n+1}\sum_{i=1}^{q} v_i^{n+1/2} r_i^n \tag{15}$$

Substituting velocity and residual force from Eqs. (3) and (9), respectively, a second-order equation of the load factor is obtained

$$\mathrm{REF} = Z_1^n(\lambda^n)^2 + Z_2^n\lambda^n + Z_3^n \tag{16}$$

Here, Z_1^n, Z_2^n, and Z_3^n are defined as follows:

$$Z_1^n = 2\tau^n\tau^{n+1}\sum_{i=1}^{q} \frac{(p_{i,\mathrm{ref}})^2}{2m_{ii} + \tau^n C_{ii}}$$

$$Z_2^n = \tau^{n+1}\sum_{i=1}^{q} \frac{p_{i,\mathrm{ref}}}{2m_{ii} + \tau^n C_{ii}}[(2m_{ii} - \tau^n C_{ii})v_i^{n-1/2} - 4\tau^n f_i^n]$$

$$Z_3^n = \tau^{n+1}\sum_{i=1}^{q}\frac{f_i^n}{2m_{ii}+\tau^n C_{ii}}[2\tau^n f_i^n - (2m_{ii}-\tau^n C_{ii})v_i^{n-1/2}] \quad (17)$$

The necessary condition for minimization of REF is that its first derivative with respect to the load factor is equal to zero

$$\frac{\partial \text{REF}}{\partial \lambda^n} = 0 \rightarrow 2Z_1^n\lambda^n + Z_2^n = 0 \rightarrow \lambda^n = -\frac{Z_2^n}{2Z_1^n} \quad (18)$$

By substituting Eq. (17) into Eq. (18), the load factor can be obtained as follows:

$$\lambda^n = \frac{\sum_{i=1}^{q}\frac{p_{i,\text{ref}}}{2m_{ii}+\tau^n C_{ii}}[4\tau^n f_i^n - (2m_{ii}-\tau^n C_{ii})v_i^{n-1/2}]}{4\tau^n\sum_{i=1}^{q}\frac{(p_{i,\text{ref}})^2}{2m_{ii}+\tau^n C_{ii}}} \quad (19)$$

It is clear that the REF is minimized when its second-order derivative with respect to the load factor is greater than zero

$$\frac{\partial^2 \text{REF}}{\partial(\lambda^n)^2} > 0 \rightarrow 2Z_1^n = 4\tau^n\tau^{n+1}\sum_{i=1}^{q}\frac{(p_{i,\text{ref}})^2}{2m_{ii}+\tau^n C_{ii}} > 0 \quad (20)$$

The inequality (20) is valid for all analyses. Therefore, the load factor formulated from Eq. (19) minimizes the REF of the structure. It is worth mentioning that this formulation leads to a unique load factor that can be used in each DR iteration.

From a mathematical point of view, the load factors predicted by Eqs. (12) and (19) minimize the residual force and residual energy of each DR iteration, respectively. These relationships are different. If the minimum residual force (MRF) criterion is used, the load factor is only a function of external and internal force vectors [Eq. (12)]. In other words, a fictitious mass, damping, and time step do not have a direct effect on the choice of the load factor. The load factor formulated from the minimum residual energy (MRE) criterion [Eq. (19)], however, is a function of all DR parameters. The main similarity between the MRF and MRE procedures is the load factor, which can be calculated explicitly in both tactics. Moreover, these techniques utilize an unconditional process for evaluating the load factor. Generally, two calculated load factors have different values. For a single degree of freedom system, with $q=1$, and damping factor of $c^n=2/\tau^n$, both MRF and MRE strategies lead to the same value for the load factor, i.e., $\lambda^n=f^n/p_{\text{ref}}$.

DR Algorithm with Variable Load Factor

In the previous section, two groups of formulations have been presented for calculating the load factor of the DR iteration. Based on these formulations, the DR algorithm can be written as follows:

1. Assume values for initial artificial velocity (null vector), initial displacement (null vector or convergence displacement on the previous increment, if available), fictitious time step ($\tau=1$), maximum load factor ($\lambda_{\max}$), and convergence criterion for the out-of-balance force ($e_R=1.0\times10^{-6}$) and the kinetic energy ($e_K=1.0\times10^{-12}$).
2. Construct a tangent stiffness matrix and internal force vector.
3. Calculate a fictitious diagonal mass matrix using Eq. (6).
4. Find a fictitious damping factor by Eq. (8).
5. Correct the load factor from Eq. (12) or Eq. (19) for the MRF or MRE criteria, respectively.

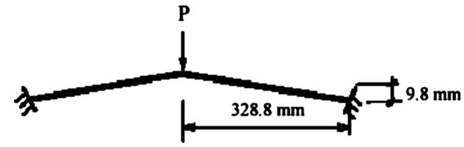

Fig. 2. Toggle frame

6. Calculate an out-of-balance force vector using Eq. (9).
7. If $\sqrt{\sum_{i=1}^{q}(r_i^n)^2}\leq e_R$, go to 12; otherwise, continue to 8.
8. Update the artificial velocity vector using Eq. (3).
9. If $\sum_{i=1}^{q}(v_i^{n+1/2})^2\leq e_K$, go to 12; otherwise, continue to 10.
10. Update the displacement vector using Eq. (4).
11. Go to 2.
12. Print the results of the current increment.
13. If $\lambda>\lambda_{\max}$, stop; otherwise, continue to 14.
14. $\lambda=\lambda+1$
15. Go to 2.

Here, the symbols MRF and MRE are used for the minimization of RFF and REF, respectively. It is clear that the suggested algorithm is perfectly automatic. In other words, a unique load factor is calculated from each scheme. Moreover, the proposed methods do not require any additional parameters such as incremental displacement or arc length. Therefore, personal judgment has no effect on the efficiency of the suggested techniques. It is worth emphasizing that procedures like arc-length approaches do not run perfectly automatic because of golden values such as arc lengths or incremental displacements. Suitable and consistent selection of these values has a great effect on the efficiency of the tactic. The proposed schemes (minimization of residual force and residual energy) eliminate these difficulties.

Numerical Examples

For numerical verification of the suggested methods, some frame and truss structures, with elastic geometrically nonlinear behavior, have been analyzed. To solve these problems, the writers have written in Fortran Power Station software a computer program for numerical studies. In all analyses, maximum load factor was assumed to be 10 ($\lambda_{\max}=10$). It should be noted that maximum load factor depends on the reference and final load of structure as follows:

$$\lambda_{\max} = \sqrt{\frac{\sum_{i=1}^{q}p_i^2}{\sum_{i=1}^{q}p_{i,\text{ref}}^2}} \quad (21)$$

Here, p_i is the final load which should be applied to the ith degree of freedom. As a result, the maximum load factor can be calculated automatically from the structural loading properties.

Toggle Frame

Fig. 2 shows a two-member frame that has been analyzed experimentally and analytically (Wood and Zienkiewicz 1977). This structure is used to verify the ability of the proposed techniques for tracing the statical path. The reference load (P), modulus of elasticity, cross-sectional area, and moment area of frame members are 44.28 N, 71×10^{12} N/mm^2, 118.06 mm^2, and

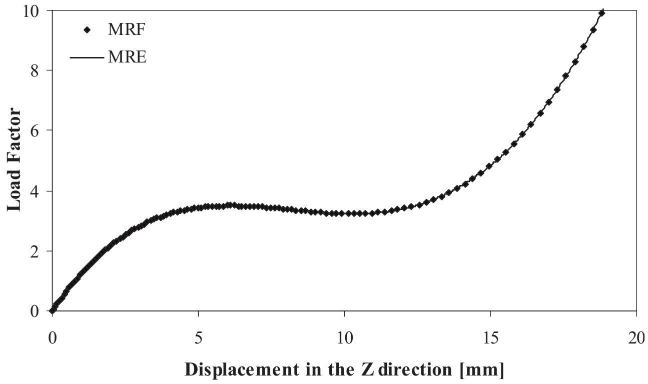

Fig. 3. Load-deflection curve of the toggle frame

374.61 mm^4, respectively. This structure is analyzed by the Felippa's frame element, which was introduced by the corotational finite-element formulation (Felippa 1997). Ten of these elements were selected to analyze toggle frame.

To convince the readers about the accuracy of DR method, a comparison study is necessary between DR results and other well-known numerical techniques such as the Newton-Raphson (NR) strategy. For this purpose, a constant load, i.e., 150 N, is applied to the frame. This structure is analyzed with DR and NR schemes separately. The top displacements of the frame, which is calculated from the DR and NR methods, are 4.6197 and 4.5992 mm, respectively. Results show that the DR method has a suitable accuracy which is perfectly compatible with NR technique. It should be noted that extensive numerical analyses of different structures confirm the accuracy of DR algorithm.

Afterward, the toggle frame is analyzed by applying the variable load factor and using proposed algorithms (MRF and MRE processes). Fig. 3 shows the statical path of the upper node vertical deflection. Both the MRF and MRE procedures present the same and complete equilibrium path. For comparing these techniques, the number of the load increments and total DR iterations have been inserted in Table 1. It is clear that the MRE traces the statical path with more points than the MRF. Therefore, the more consistent equilibrium path can be obtained. By increasing the equilibrium points, however, the total DR iterations increase. Hence, analysis time of the MRE was greater than the MRF.

Space Truss

Fig. 4 shows a space truss with 21 degrees of freedom (Ramesh and Krishnamoorthy 1993). The axial rigidity of bar elements is

Table 1. Number of Load Increments and DR Iterations in the Numerical Analyses

Structure	Method	Number of load increments	Total number of DR iterations
Toggle frame	MRF	120	16,795
	MRE	782	109,172
Star truss under symmetric load	MRF	60	4,162
	MRE	136	11,406
Star truss under asymmetric load	MRF	93	11,325
	MRE	278	44,296
Shallow space truss	MRF	149	61,656
	MRE	454	158,190

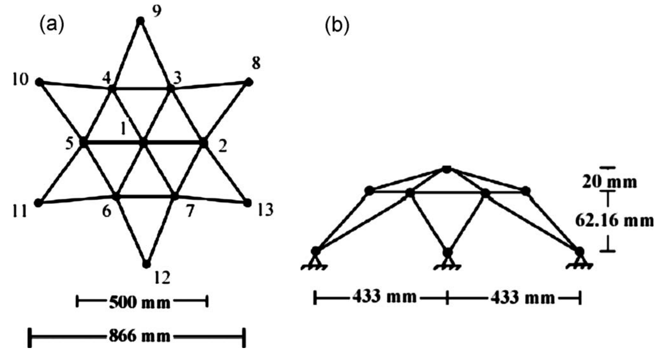

Fig. 4. Space truss: (a) *X-Y* view; (b) *X-Z* view

960,510 N. In this example, Felippa's element, which was introduced by the total LaGrange finite-element formulation, was used (Felippa 1997). Each bar of space truss is considered as one element in the finite-element model. This structure was analyzed for two kinds of loading that have been inserted in Table 2. First, the symmetric loads are applied to the truss. Figs. 5 and 6 show the load-deflection curves of Nodes 1 and 2, respectively. This loading causes snap-back behavior in Node 1. It is clear that when the MRF is used, a jump occurs in the statical path between load factors of 6 and −4. The MRE, however, traces the equilibrium path completely and without any load or displacement jumping. As a result, the MRE procedure may be more efficient than the MRF technique.

To clarify the specification of the proposed techniques, the asymmetric loading was applied to the space truss (Table 2). The load-deflection curves of vertical and horizontal displacements of Node 1 have been plotted in Figs. 7 and 8, respectively. Furthermore, Fig. 9 shows the load-deflection path of the vertical displacement of Node 2. Both the MRF and MRE procedures could

Table 2. Load Cases of the Space Truss

Load case	Node number	Reference load value (N)
Symmetric load	1	50
	2, 3, 4, 5, 6, 7	25
Asymmetric load	1	37.5
	5, 6	75

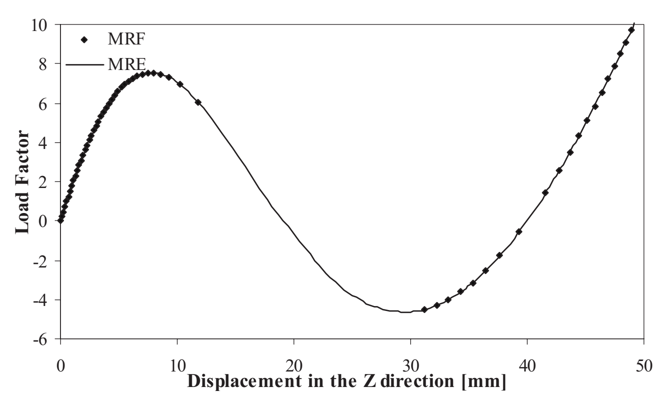

Fig. 5. Load-deflection curve of Node 1 of the space truss under symmetric load

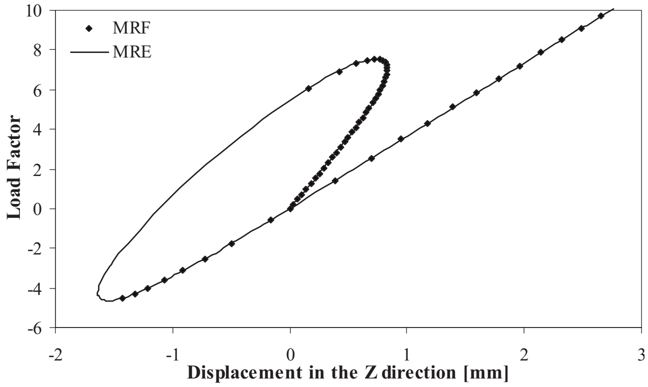

Fig. 6. Load-deflection curve of Node 2 of the space truss under symmetric load

trace the snap-back and snap-through regions. In spite of the same accuracy, the MRE requires more iteration than the MRF. As a result, the MRE method uses more computing time than the MRF scheme.

Shallow Truss Dome

The space truss shown in Fig. 10 has 168 members and 147 degrees of freedom (Powell and Simons 1981). The maximum elevation is 1,790.22 mm. This structure includes many degrees of freedom, and it has highly nonlinear behavior. The axial rigidity of bar elements is 10^5 N. Once again, Felippa's element was used (Felippa 1997). Each bar of space truss is considered as one element in the finite-element model. The reference load applied to the top of the truss is 100 N. This example is employed to compare the proposed MRF and MRE procedures with arc-length method, which is the most common approach for tracing the snap-through and snap-back regions. For arc-length analysis, numbers of necessary iteration (J_D), maximum iteration ($J_{\max}$), acceptable error, and incremental load factor in the prediction step of the first increment ($\Delta\lambda_1^1$) are 6, 20, 10^4, and 1, respectively. The results of arc-length analysis have been taken from the paper by Powell and Simons (Powell and Simons 1981). Fig. 11 shows the variation of tip displacement of shallow truss dome versus the load factor. It is clear that the configuration of statical path obtained from the

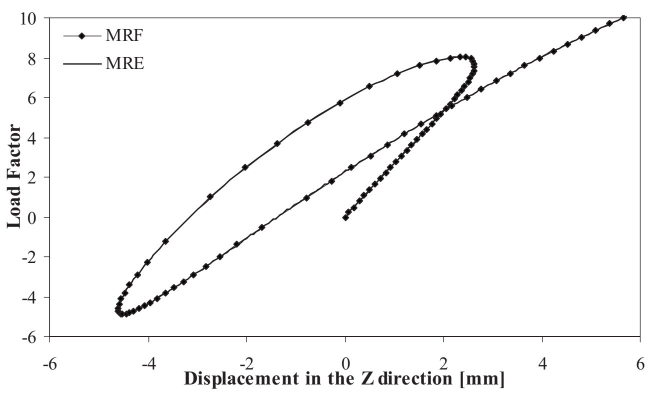

Fig. 7. Load-deflection curve of Node 1 of the space truss under asymmetric load

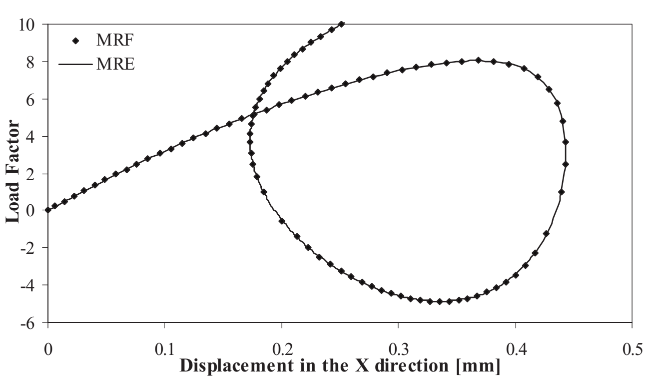

Fig. 8. Load-deflection curve of Node 1 of the space truss under asymmetric load

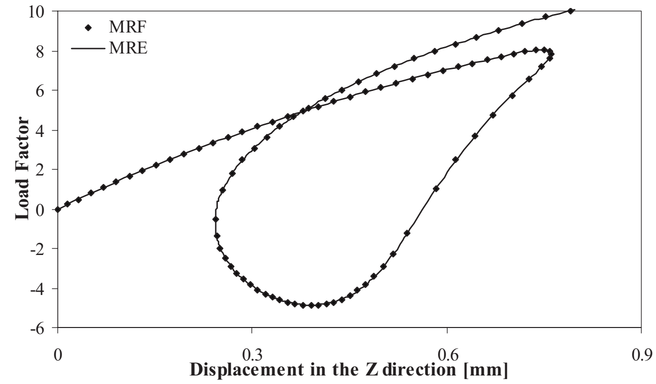

Fig. 9. Load-deflection curve of Node 2 of the space truss under asymmetric load

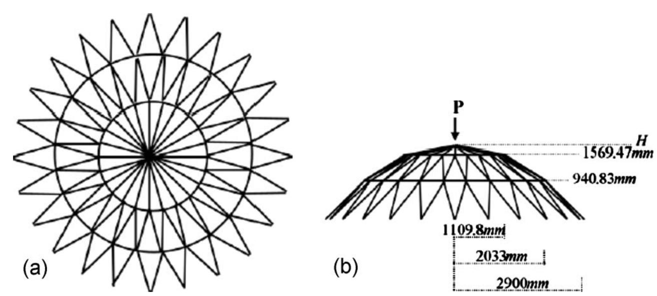

Fig. 10. Shallow truss dome: (a) *X-Y* view; (b) *X-Z* view

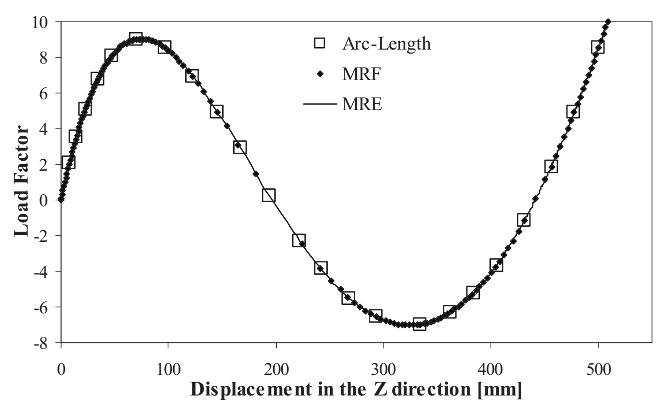

Fig. 11. Load-deflection curve of the tip node of the shallow truss dome

MRF and MRE procedures is the same as the one calculated by the arc-length method. In other words, both of proposed DR techniques and also arc-length approach present the unique curve for the snap-through and snap-back analysis, and their results are perfectly compatible.

It should be added that numbers of load increments for the MRF and MRE techniques are higher than the arc-length method. It seems that the significant weakness of the MRF and MRE procedures compared with the arc-length approach is the very large number of load increments required to trace the deformation path. This weakness can be covered by some unique specifications of proposed DR schemes. As it can be seen in Fig. 11 (also in Figs. 7 and 9), the number of load increment increases around the limit points. This property helps to estimate the limit points with a very suitable accuracy. In other words, the accuracy of the MRF and MRE techniques for predicting the limit points is higher than the arc-length method.

On the other hand, the process of calculating the load factor in the proposed DR methods (MRF and MRE) is perfectly automatic. These formulations do not need any additional parameters to calculate the load factor and trace the snap-through and snap-back regions. However, for performing arc-length analysis, some parameters such as the number of necessary iteration (J_D) and maximum iteration ($J_{\max}$) should be selected based on previous analyses or experiences.

Finally, it seems that another significant weakness of the MRF and MRE procedures compared with arc-length method is the very large number of DR iterations required to trace the deformation path. It is worth emphasizing that DR iterations are performed just by vector operations, while the arc-length process needs matrix calculations in each iteration. Since the matrix calculation is much more than the vector one, the number of iterations is not a suitable parameter to compare the abilities of these two strategies. It should be noted that a system of simultaneous equations should be solved in each arc-length iteration, while each DR iteration is performed by calculating Eqs. (3) and (4).

Concluding Remarks

This paper has developed two new procedures for tracing the snap-through and snap-back regions based on MRF and also MRE criteria for DR method. The mathematical formulation leads to the closed-form equations which are consistent with the DR iterations. The resulted algorithms are perfectly automatic and do not need the analyst intervention. Moreover, the MRF and MRE formulations lead to an unconditional process for calculating the load factor. Generally, two calculated load factors have different values.

A wide range of numerical examples, with geometrical nonlinear behavior, have shown that new techniques can present the snap-through and snap-back regions of the structural statical path. These analyses illustrate that the MRE presents a more accurate and consistent equilibrium path than the MRF. In other words, the MRE scheme can trace the complicated load-deflection paths without any jumping. However, the number of the load increments needed for the MRE is higher than the number needed for the MRF.

Based on comparison study with the other common methods such as arc-length approach, the numbers of increments used by the proposed DR techniques are considerably high. Although this need increases the analysis time, the high number of increments, which is commonly required around the limit points, causes that the proposed MRF and MRE algorithms can present the limit points with a very suitable accuracy. Moreover, because the proposed schemes are perfectly automatic, they eliminate the need for additional or constant parameters such as arc length or incremental displacement.

Notation

The following symbols are used in this paper:

$\{A\}^n$ = artificial acceleration vector of nth iteration;
c^n = artificial damping factor of nth iteration;
$[C]^n$ = artificial damping matrix of nth iteration;
$\{D\}$ = displacement vector;
$\{F\}$ = internal force vector;
$[M]^n$ = artificial mass matrix of nth iteration;
$\{P\}$ = external load vector;
$\{P\}_{\mathrm{ref}}$ = reference load vector;
q = number of degrees of freedom;
$\{R\}$ = unbalanced force vector;
$[S]$ = stiffness matrix;
$\{V\}^n$ = artificial velocity vector of nth iteration;
Z_1^n, Z_2^n, Z_3^n = formulation parameters of nth increment;
λ^n = load factor of nth DR iteration; and
$\lambda_{\max}$ = maximum load factor.

Superscripts

N = iteration number of DR method.

Subscripts

i = each degree of freedom of structure.

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