Elasto-Plastic Analysis of Trusses by Virtual Loads

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Abstract: *In this research a new approach is proposed for elasto-plastic analysis of structures with truss elements. This method covers both perfectly plastic and hardening properties. The Proposed technique uses substituting virtual loads instead of modifying the stiffness matrix. To solve this kind of problems, complementary programming is utilized. Numerical examples demonstrate that elasto-plastic analysis by this approach has very good convergence, rapidity, and accuracy.*

Keywords: *elasto-plastic, truss, hardening, complementary programming, virtual loads*

1. Introduction

Because of the special nature of plasticity problems, some of the researchers have tried to use mathematical programming methods for solving them. In most of these researches, elastic-perfectly-plastic state is studied. Also these methods are suitable for path independent problems which are called "Holonomic Behavior".

One of the mathematical approaches is based on the maximum plastic work principle. In this technique, stresses are limited by yield surface and increment of plastic strain work is maximized simultaneously. Writing Kohn-Tacker conditions, the flow rule and the yield criteria appear [2]. In the mentioned approach, unknown quantities consist of stresses, Lagrangian coefficients and plastic strains. In these cases, the usage of mathematical methods is not economical, because there are too many unknown quantities. Therefore, in comparison with general iterative approaches such as Newton-Raphson, the mathematical programming methods are seldom taken into consideration [1].

Another method is deformation theory. It solves plastic problems by piecewise yield constraints and the associated flow rule [4,3]. This technique has been used in reliability analysis of elasto-plastic truss systems [5]. In this method, only an elasto-plastic analysis using total loads is done and the history of displacements is not important. In addition, unknown parameters are nodal displacements and plastic ratio vectors. Related equations can be solved by quadratic or complementary programming methods. However, in this approach, many variables exist, and the run time is very long. Nonetheless, the stated approaches are able to pass through singular points of equilibrium path.

2. Nonlinear Analysis of Structures

In incremental methods of nonlinear analysis, loads are applied in several steps and the stiffness matrix is modified at each stage. In case there are only a few increments, these methods are not accurate at all.

In the mentioned method, imbalanced loads are calculated by considering the nonlinear behavior of elements. Afterwards, imbalanced loads are applied to the structure. This process is continued until the
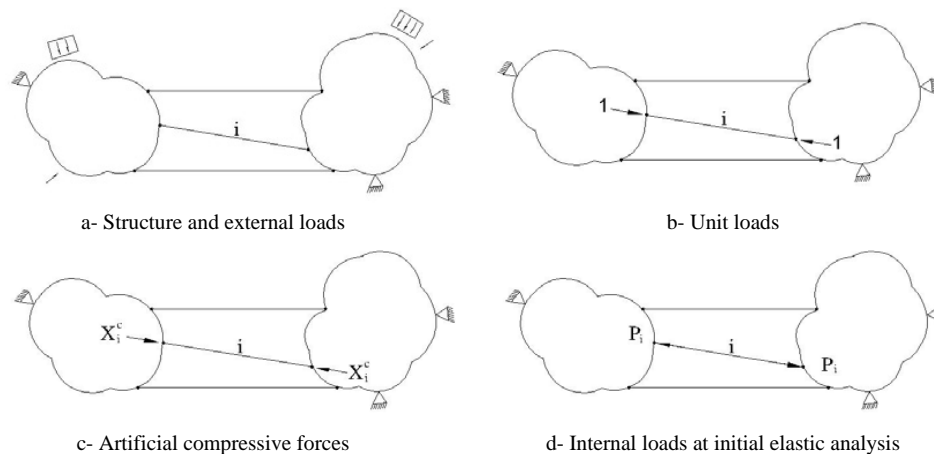
a- Structure and external loads   b- Unit loads

c- Artificial compressive forces   d- Internal loads at initial elastic analysis

Fig.1 A structure with limited compression capacity members

imbalanced loads become very small.

Some methods such as Newton-Raphson modify the stiffness of the structure at each iteration. This procedure sometimes results in numerical instability, and it can not pass through the singular points.

The other techniques of nonlinear analysis are structural variation theories presented by Majid, et al for topological design of plane trusses used instead of reanalysis. They also used these theories for bending frames [6]. The researches showed that the analysis of a structure can be done by considering the results of another similar structure analysis. The effects of variation or elimination of members can also be studied easily. It is also shown that a simple elastic analysis is sufficient to present the complete behavior of a frame up to collapse. Indeed, this method is remarkable because of its ability in altering the material properties of elements [10]. In this type of analysis, a group of virtual loads is used instead of altering material properties or eliminating. In the mentioned method, stiffness matrix is not modified directly, so it is approximately similar to initial stiffness method.

Recently, a new method has presented for

elasto-plastic analysis of trusses in Tarbiat Modarres University [9]. It is shown that method is suitable for analyzing the structures including compression-only and tension-only members [7]. This technique is similar to structural variation theories in the application of stiffness variation effects; nevertheless, the proposed method has high accuracy. It can pass through the singular points of equilibrium path easily using mathematical programming. Additionally, in comparison with the other mathematical techniques, very few variables enter in the proposed programming [9]. In the following part, this method is reviewed and then improved for solving the path dependent problems in next parts. Also, the complementary programming is utilized instead of quadratic programming.

3. Elastic – Perfect Plastic Analysis by Virtual Loads

First, the method is explained for elements with limited compression capacity, and then it will be developed for the other cases.

Figure (1-a) shows a structure with truss members including limited compression capacity.

It is supposed that the member i has limited compression capacity equal to R_i^c. If an ordinary analysis, under the applied loads, is done, forces acting on both ends may be the same as those in Fig (1-d). The element will not be able to accept them if these forces exceed the compression capacity of the member. In this case, the value of stiffness of the element should be modified by replacing it with zero. Otherwise, the surplus stiffness must be discarded. The surplus stiffness can be removed by applying a pair of self-equilibrium X_i^c forces shown in Fig (1.c). It is equivalent to the stiffness modification of the member i.

Calculating the virtual force X_i^c, one obtains the internal force of the element, and there is no need of an iterative process.

It is assumed that P_i is the internal force of the element i due to external loads. In addition, q_i is the internal force of that element corresponding to unit loads illustrated in Fig (1.b). When the virtual load X_i^c and external loads act on the structure simultaneously, the internal force of the element i is achieved by the following equation:

$$\overline{P}_i = q_i.x_i^c + p_i \tag{1}$$

In order to eliminate the stiffness effect of element i, the sum of the virtual force X_i^c and $\overline{P}_i$ must be equal to the capacity limit of that element. Hence, for the case of compressive yielding this equation can be written:

$$\overline{P}_i + X_i^c = R_i^c \tag{2}$$

Substituting in Eqn (1) results:

$$X_i^c + q_i.X_i^c + P_i = R_i^c \tag{3}$$

The above equation shows that the sum of

forces applying to the ends of element i is equal to compression capacity. The virtual force X_i^c is a nodal load, and after applying it, the structure will be in its actual equilibrium condition. On the other hand, the modified stiffness of a member is compensated by some virtual loads. Using Eqn (3), the real force of member i is obtained for compression yield condition.

$$P_i^R = X_i^c + q_i.X_i^c + P_i = R_i^c \tag{4}$$

Wherein, P_i^R is the internal force value of the member i. It should be noted that if the force of the member i reaches its capacity limit, the member is eliminated. In this case, the stiffness effect of the element is removed by applying X_i^c to its both ends. This means that Eqn (4) is fulfilled. In contrast with that, if the member i does not yield, there will be no need to apply X_i^c to the ends of the element. In this case, P_i^R will be greater than R_i^c.

$$P_i^R = X_i^c + q_i.X_i^c + P_i > R_i^c \tag{5}$$

Combining equations (4) and (5), the following result is available:

$$P_i^R = X_i^c + q_i.X_i^c + P_i \geq R_i^c \tag{6}$$

It is essential to know that in Eqn (4), X_i^c has a nonzero quantity when P_i^R is equal to the capacity limit. In contrast, P_i^R in Eqn (5) is the internal force of member and is greater than. In this case, there is no need to apply X_i^c. Thus, X_i^c equals to zero. Therefore, the following condition derived from the problem nature is added to the governing equation.

$$X_i^c.(P_i^R - R_i^C) = 0 \tag{7}$$

Substituting P_i^R in equation (4) leads to:

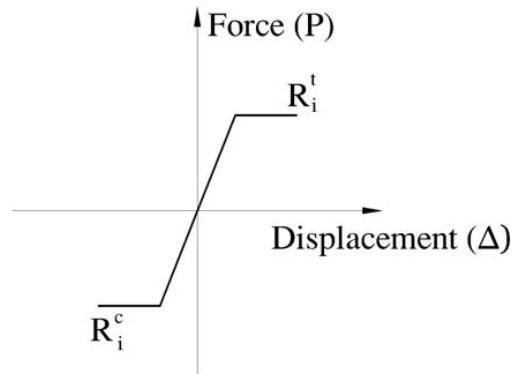$$X_i^c.(X_i^c + q_i.X_i^c + P_i - R_i^c) = 0 \tag{8}$$

Fig.2 Force-displacement diagram of truss members including limited capacity

These equations are used where the structure has only one member with the compression capacity limit. The equations can be developed for the cases in which many members with compression and tension capacity limit contribute. The force-displacement relationship of a truss member with limited capacity is illustrated in Fig (2).

For structures with more than one limited capacity member, the following equations are obtained.

$$P_i^R = X_i^c - X_i^t + \sum_{j=1}^{n} q_{ij}.(X_i^c - X_i^t) + P_i \geq R_i^c$$

$$(9)$$

$$P_i^R = X_i^c - X_i^t + \sum_{j=1}^{n} q_{ij}.(X_i^c - X_i^t) + P_i \leq R_i^t$$

$$(10)$$

$$X_i^c.(P_i^R - R_i^C) = 0 \qquad (11)$$

$$X_i^t.(P_i^R - R_i^t) = 0 \qquad (12)$$

$$X_i^c.X_i^t = 0 \qquad (13)$$

By solving the above programming problem, unknown quantities of X_i^t and X_i^c are obtained [9]. Here, X_i^t is a virtual force similar to X_i^c that should be applied in the opposite direction of X_i^c for removing surplus tension stiffness. Using Eqn (9) and the values of X_i^t

and X_i^c, the real force of the member i is calculated.

4. Solving Problem by Mathematical Programming

The aforementioned equations can be solved by quadratic programming [7, 9]. In this case, the left sides of equations (11) to (13) are equal to zero. So, the sum of the equations must also be equal to zero. Nonetheless, the minimum value of this summation equals zero. Beside the minimization of the mentioned value, satisfying the conditions (9) and (10) is necessary. Therefore, the problem can be written as a quadratic programming below.

$$Min \sum_{i=1}^{n} \left\{ (X_i^c - X_i^t). \left[2((X_i^c - X_i^t) \right. \right.$$

$$\left. + \sum_{j=1}^{n} q_{ij}.(X_i^c - X_i^t) + p_i) - R_i^c - R_i^t \right] + X_i^c + X_i^t \right\}$$

$$P_i^R = X_i^c - X_i^t + \sum_{j=1}^{n} q_{ij}.(X_i^c - X_i^t) + P_i \geq R_i^c$$

$$P_i^R = X_i^c - X_i^t + \sum_{j=1}^{n} q_{ij}.(X_i^c - X_i^t) + P_i \leq R_i^t$$

$$X_i^C, X_i^t \geq 0 \qquad (14)$$

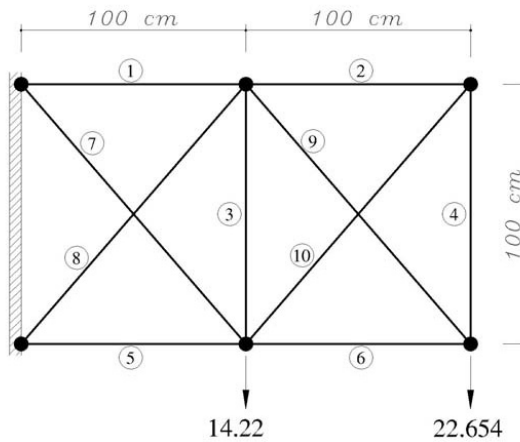Here, n is the number of truss elements with

Fig.3 Ten bar truss

limited capacity.

There are some techniques for solving the quadratic problems. In these methods, a lot of additional variables should be found.

Complementary programming is another technique for solving above equations. Because of linear constraints, design space is convex. So, one can apply a linear programming method such as simplex technique and check the constraints (11) to (13) in every steps. This method is nominated complementary programming. The results of complementary programming method are more accurate while the method is also a time-saving approach. The following example shows the advantage of the complimentary programming method to quadratic programming technique.

Example 1

Consider the ten-member truss shown in Fig. (3). All of the members have limited tension and compression capacity, as they are path independent elements. This example has been solved in deformation theory [5]. In the presented method, the quadratic programming result is obtained with 28 iterations, while the complementary programming converges with three

iterations. Member properties are shown in Table (1). The results are given in Table (2).

5. Truss Elements with Hardening

Fig (4) shows the behavior of a tension member possessing hardening behavior. Internal force of the aforementioned member due to external and unit loads is obtained from equation (15).

$$\overline{P}_i = -q_i . X_i^t + P_i \qquad (15)$$

The real force of member can be obtained from equation (16).

$$P_i^R = -X_i^t - q_i X_i^t + P_i \leq \overline{R}_i^t \qquad (16)$$

The after yielding tension capacity of the member, $\overline{R}_i^t$, is calculated as follows.

$$\overline{R}_i^t = R_i^t + \frac{K_i^t}{K_i}\left[\overline{P}_i - R_i^t\right] \qquad (17)$$

Here, k_i and K_i^t are the axial initial stiffness of member i and the after yielding stiffness, respectively. In addition, R_i^t is the force related to stiffness alteration point of member i. Substituting equation (17) in (16) and simplifying the result, these expressions are obtained:

Table 1 Member properties

Bar	E (KN/Cm^2)	Section area (cm^2)	Axial force at yielding	
			Tension (KN)	Compression (KN)
1	2.06×10^4	10.0	235.368	-235.368
2	2.06×10^4	5.0	117.684	-117.684
3	2.06×10^4	2.0	47.074	-34.325
4	2.06×10^4	10.0	47.074	-34.325
5	2.06×10^4	5.0	235.368	-235.368
6	2.06×10^4	1.5	117.684	-117.684
7	2.06×10^4	1.5	35.305	-23.537
8	2.06×10^4	1.5	35.305	-23.537
9	2.06×10^4	1.0	23.537	-11.768
10	2.06×10^4	1.0	23.537	-11.768

Table 2 Results of ten bar analysis

Bar	1	2	3	4	5	6	7	8	9	10
Axial force	39.377	8.276	2.325	8.276	-42.834	-14.388	28.525	-23.636	20.348	-11.704
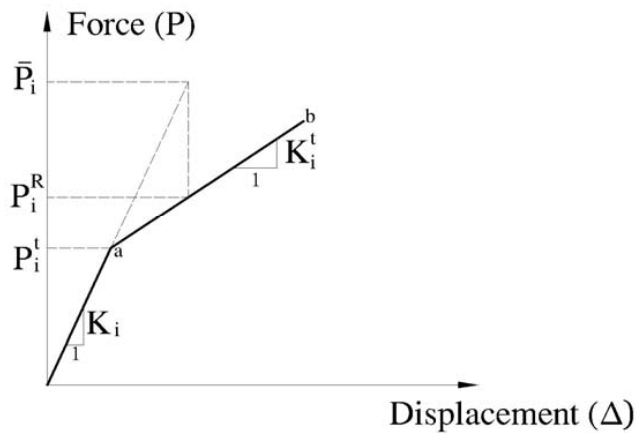
Fig.4 Force -displacement of truss element with strain hardening

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a. Loading variation

b.force-displacement diagram of member

c. Modified diagram at point 1

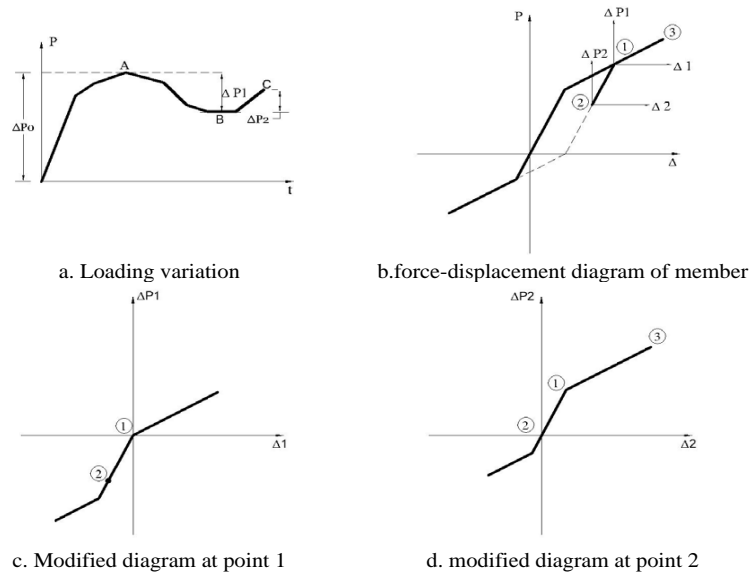d. modified diagram at point 2

Fig.5 Incremental behavior of elasto-plastic truss member

$$-X_i^t + m_i^t(-q_i X_i^t + P_i) \leq m_i^t.R_i^t \qquad (18)$$

$$m_i^t = \frac{k_i - k_i^t}{k_i} \qquad (19)$$

The behavior constraint is written below.

$$X_i^t\left[-X_i^t + m_i^t(-q_i X_i^t + P_i) - m_i^t R_i^t\right] = 0.0 \quad (20)$$

Equations (18) and (20) are obtained, as the structure has only one member with the stated properties. Analogous equations can be developed for the general case wherein there are many truss members with strain hardening properties. Suppose that member i in compression part is similar to tension part and has yielding point R_i^c. In addition, the stiffness is altered to k_i^c after compressive yielding. Thus, the following programming form is used for the problem's solution.

$$X_i^c - X_i^t + m_i^t\left[\sum_{j=1}^{n} q_{ij}(X_j^c - X_j^t) + P_i\right] + f_i^t = m_i^t R_i^t$$

$$(21)$$

$$X_i^c - X_i^t + m_i^c\left[\sum_{j=1}^{n} q_{ij}(X_j^c - X_j^t) + P_i\right] + f_i^c = m_i^c R_i^c$$

$$(22)$$

$$X_i^t.f_i^t = 0 \qquad (23)$$

$$X_i^C.f_i^c = 0 \qquad (24)$$

$$X_i^c.X_i^t = 0 \qquad (25)$$

In which, f_i^t and f_i^c are slack and surplus variables respectively [9]. All of the above equations are useable for path independent problems. These types of problems are called holonomic elasto-plastic problems. If loading is path dependent, the load can be applied to the structure incrementally, and the stiffness modification is used in every step. It should be noticed that, the magnitude of steps of the present method could be greater than that of the other incremental techniques. On the other hand, applying the peaks of the loading diagram to the structure will give excellent approximation.

The loading variation of a single degree of freedom is similar to Fig (5.a). Three points of 1, 2, and 3 on the force-displacement diagram of Fig (5.b) correspond to points A, B, and C, respectively. After applying the first increment, (ΔP_0), and finding the results
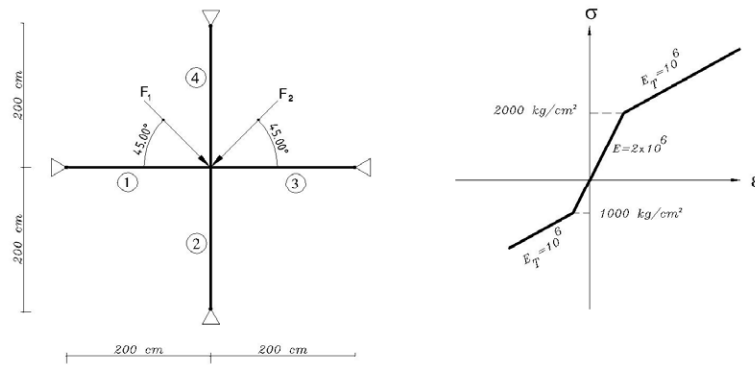
Fig.6 Four bar truss and material behavior

by the mathematical programming, forces and displacements of point 1 are obtained .The results are saved and the problem can be supposed as a new problem now. After that, ΔP_1 is applied to the structure. In this case, force-displacement diagram is changed to a new form shown in Fig. (5.c). This procedure is applicable at every step.

Analyzing the structure with the load ΔP_1, one can achieve the increment of stresses and displacements of this step. It is clear that the total forces and displacements will be computed by the summation of increments and the application of Fig. (5). This sequence is repeated until the last step, and the results of the analysis are obtained. Considering k as the number of load increments, equations (26) to (30) can be rewritten below:

$$ (26) $$

$$ {}^kX_i^c - {}^kX_i^t + {}^km_i^t\left[\sum_{j=1}^{n}q_{ij}.({}^kX_j^c - {}^kX_j^t) + P_i^k\right] + {}^kf_i^t = {}^km_i^t.{}^kR_i^t $$

$$ (27) $$

$$ {}^kX_i^c - {}^kX_i^t + {}^km_i^c\left[\sum_{j=1}^{n}q_{ij}.({}^kX_j^c - {}^kX_j^t) + P_i^k\right] + {}^kf_i^c = {}^kmc.{}^kR_i^c $$

$$ {}^kX_i^t . {}^kf_i^t = 0 \qquad (28) $$

$$ {}^kX_i^c . {}^kf_i^c = 0 \qquad (29) $$

$$ {}^kX_i^c . {}^kX_i^t = 0 \qquad (30) $$

Finally, the real forces and displacements are computed in these forms.

$$ P_i^R = \sum_{k=1}^{m} {}^kP_i^R \qquad (31) $$

$$ U_i^R = \sum_{k=1}^{m} {}^kU_i^R \qquad (32) $$

A computer program is written for elasto-plastic analysis of structures. Two examples are solved by that program, and their results are presented.

Example 2

A four-member truss is shown in Fig (6). The cross section areas of the members are $1cm^2$, and F_1 is applied to the structure before F_2.

After analyzing, the internal force vector will be $\{250, -3750, 250, 4250\}^T$. Furthermore, the displacement vector of the central node equals to $\{.05, -. 65\}^T$.

The results obtained by this method and Newton-Raphson approach are alike.

Example 3

Fig.(7) shows a navigation tower and its properties of members. External loads of Table (3) are applied to the structure. This example is also solved in reference [8]. There, the properties of a plastic member are modified at every step. This way is called "Celernomic analysis". The result is
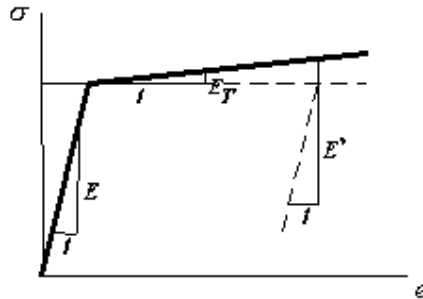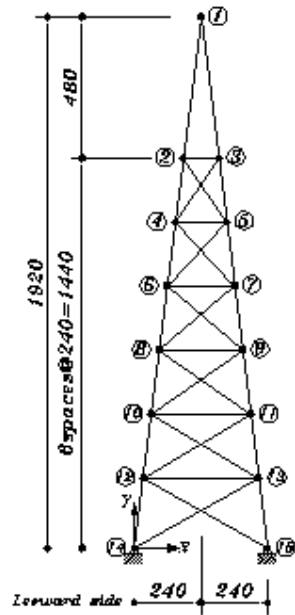
$$\sigma_0 = 36000$$
$$E = E' = 30 \times 10^6$$
$$E_T = 1 \times 10^6$$

Element	1,2	1,3	2,4	3,5	4,6	5,7	6,8	7,9	8,10	9,11	10,12
Area	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.44	1.44	1.44

Element	11,13	12,14	13,15	2,3	4,5	6,7	8,9	10,11	12,13	Others
Area	1.44	1.44	1.44	1.21	1.21	1.21	1.21	1.21	1.21	0.484

Fig.7 Navigation tower

Table-3 Navigation tower external loading

Node No.	Vertical Load	Lateral Load	Node No.	Vertical Load	Lateral Load
1	-308	-9650	8	-304	-87.5
2	-273	-50	9	-304	-438
3	-273	-250	10	-374	-100
4	-292	-62.5	11	-374	-500
5	-292	-312	12	-398	-112
6	-294	-75	13	-378	-563
7	-294	-275			

Table 4 Navigation tower internal loading

Rod	Applied Load		Unloading		Residual	
	Celernomic Meth.	Proposed Meth.	Celernomic Meth.	Proposed Meth.	Celernomic Meth.	Proposed Meth.
1,2	- 39100	-39056	-39100	-39056	0	0
1,3	38700	38745	38700	38745	0	0
2,4	-39100	-39149	-39400	-39433	300	284
3,5	39000	39053	38800	38770	200	283
4,6	-39600	-39711	-40200	-40199	600	288
5,7	39400	39471	39000	38984	400	487
6,8	-40400	-40419	-41100	-41099	700	680
7,9	40000	39970	39300	39292	700	678
8,10	-42300	-42178	-42200	-42139	-100	-39
9,11	39700	39612	39800	39654	-100	-42
10,12	-43500	-43350	-43500	-433354	0	4
11,13	40300	40110	40300	10108	0	2
12,14	-44900	-44698	-44900	-44699	0	1
13,15	40900	40672	40900	40674	0	-2
2,3	150	151	0	-60	150	211
4,5	628	629	0	4	628	625
6,7	1250	1301	0	96	1250	1205
8,9	641	643	0	29	642	614
10,11	0	-46	0	-1	0	-45
12,13	0	-4	0	-8	0	4
2,5	-211	-212	120	120	-331	-332
3,4	-683	-682	-351	-351	-332	-331
4,7	-415	-417	224	227	-639	-644
5,6	-1180	-1179	539	-535	-641	-644
6,9	-656	-720	351	295	-1007	-1015
7,8	-1717	-1659	-709	-665	-1008	-1014
8,11	623	577	554	509	69	68
9,10	-772	-726	-840	-795	68	69
10,13	727	690	733	696	-6	-6
11,12	-1045	-1008	-1040	-1003	-5	-5
12,15	945	915	945	915	0	0
13,14	-1251	-1221	-1250	-1221	-1	0

Table 5 Comparison of methods

Celernomic Analysis	General Nonlinear Analysis	Proposed Meth. (Complementary)
9 Step	55 Cycles	8 Cycles

presented in Table (4). The required steps and iterations for different methods are compared in Table (5).

6. Conclusion

In the present study, a new method was presented for elasto-plastic analysis of trusses. Comparing the method with other ones shows that:

- The proposed method is useful for the elasto-plastic analysis of the structures including truss members.

- In the proposed method, the external load vector is modified instead of stiffness matrix. This vector is achieved by mathematical programming and applied to structure in one step, so, the iterative structural analysis is eliminated.

- Changing the elasto-plastic analysis to a linear programming problem guarantees the convergence.

- The results of a linear programming problem are exact, so the results of the proposed method are exact too.

- The usage of complementary programming instead of quadratic programming reduces the number of variables and improves rapidity remarkably.

- The required variables are fewer than those required for other mathematical methods. So the required memory is less, and rapidity is better than other mathematical ones.

- The proposed method can be used for path dependent and independent problems.

- This technique can also be developed for other nonlinear problems.

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