COMPARATIVE ANALYSIS USING THE NUMERICAL METHODS OF FINITE ELEMENT, BOUNDARY ELEMENT, ELEMENT FREE GALERKIN AND FINITE VOLUME IN THE FIELD OF SOLID MECHANICS

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Abstract. In this paper four eminent numerical techniques of Finite Element Method (FEM), Boundary Element Method (BEM), Element Free Galerkin Method (EFGM) and Finite Volume Method (FVM) are evaluated comparatively. The bench marking plane strain problem of cantilever beam under end load is used as the basis for the analysis. Using MATLAB package a computer code has been written and examined for each routine. Triangular elements with nodes on their corners are used in FEM, linear elements are used in BEM, the polynomial of degree one is used for the approximation of the field function around each node in EFGM and constant strain triangles are used in the formation of the volumes in FVM. The effect of mesh refinement upon the accuracy of the results and the period of execution are evaluated and used as a criterion for the assessment of the routines.

1 INTRODUCTION. The Finite Element Method (FEM) seems to be dominant to different computing disciplines such as the Computational Structure Mechanics (CSM), but this hasn't caused the researchers to stop the struggles for finding better numerical approaches to solve problems more efficiently. Introducing fairly newer numerical methods such as Element Free Galerkin (EFG), Finite Volume (FV) for solid mechanics, Boundary Element (BE), Petrov-Galerkin and so on are approaches which have formulated to conquer the FEM limitations. One of the FEM deficiencies is that the method requires obtaining all of the nodal values even if we require only one nodal value in a domain. Also in some cases such as the large deformation or crack growth problems the dependence of the method on a mesh leads to some complexities. In such cases mesh pattern may lead to skewed or compressed elements. So we need methods which could deal well with the extended regions, contrasting the FEM that yields large errors in this context. So first of all let's have a brief survey on the history of the abovementioned methods.

In the beginning of the twentieth century Fredholm (1) conducted a rigorous investigation into the classical kinds of integral equations. Most of the earlier researches were focused on the formulations of the potential theory and elasticity by means of the integral equations. The endeavors were based on the analytical procedures which reduced the application of integral equations to simple and trivial problems (2).

In 1967, Rizzo (3) used the direct boundary element method to solve two-dimensional elastostatic problems. Cruse (4), extended Rizzo's work to three-dimensional problems of Elastostatics. Triangular elements were used to model the curved surface, thus converting the curved boundary into a piecewise flat surface.

Within a relatively short span of time a number of important contributions in various fields were made by Cruse. Elastodynamic problems (5), Elastoplasticity (6), Fracture Mechanics (7) were successfully attempted by his team of researchers who paved the way to further accelerate the upraising of the boundary element method.

The element free Galerkin method (EFGM) is a relatively recent numerical method. At the beginning, Belytschko et al. (8) proposed this tactic. Since then it has been applied to various fields such as the Fracture Mechanics (9), thin plates and shells (10), transient coupled problems (11) and material interfaces modeling (12). Recently more tendencies aroused to upgrade the type of the enforcement of essential boundary conditions in the EFG Method (13).

Finite Volume (FV) is a fairly new numerical approach in the field of Computational Solid Mechanics (CSM). In this approach, instead of utilizing the energy minimization as in the FEM, the balance law is directly applied to the discretized volumes forming a domain. This approach has enabled successful analysis of the complex problems involving moving fluids and structure interactions (1-4). The early developed FV codes were based on the structural analysis procedures capable of predicting displacements, strains and stresses in two and three dimensional linear elastic loaded structures under small deformations (15). More recently these procedures have been focused on the nonlinear behavior of materials (16) as well as the large deformation of solids (17).

In this paper, we present some comparisons between four numerical methods of Finite Element (FE), Boundary Element (BE), Finite Volume (FV) in the CSM field and Element Free Galerkin (EFG). To do this, the bench marking plane strain problem of cantilever beam under end load which it's analytical solution is available is used as the basis for the analysis. Using MATLAB package a computer code is written and examined for each method. First order polynomials are employed for the interpolation of the field and the physical variables. For this purpose, triangular elements with nodes on their corners are used in FEM, linear elements are used in BEM, the polynomial of degree one is used for the approximation of the field function around each node in EFGM and constant strain triangles (CST) are used in the formation of the volumes in the FVM. The effect of mesh refinement or numbers of nodes upon the accuracy of the results is studied by means of the evaluation of the displacements or energy norms. The period of execution for each one of the computer codes is measured and used as a criterion for the assessment of the routines. Based on the potential of each method in the solution of the benchmarking problem, some suggestions regarding the choice of proper method in some instances in the field of solid mechanics are recommended. In the next we will bring the formulation of FVM because of being relatively newer than other mentioned methods.

2 FORMULATION OF THE FVM IN THE CSM FIELD. A two dimensional structure can be represented by a mesh of triangular elements connected alongside at their nodes. Finite control volumes are constructed around each node by connecting the centroids of the triangular elements to the midpoints of their sides. A typical volume constructed around a node P, surrounded by N elements is shown in Figure 1.

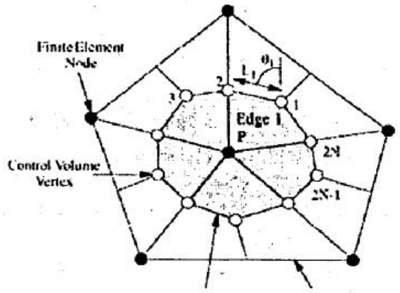

Figure 1. Volume made up of triangular cells

In general, the vertices of the volume are numbered from 1 to 2N, commencing at one of the element centroids and continuing around anticlockwise via the element edge midpoints and centroids one after the other.

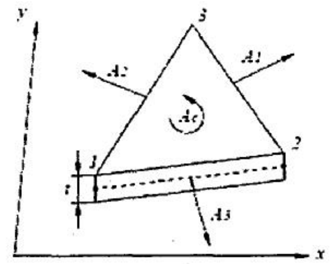

Figure 2. Geometry values of a triangular cell.

According to figure 2, considering one triangular cell constructing the control volume we can write for the strain components

$$\boldsymbol{\varepsilon}_c = \mathbf{B}\,\mathbf{u}_c \tag{1}$$

where **B** is the differential operation matrix and $\mathbf{u}_c$ collects the displacement components u_i and v_i. Consequently the strain components can be written as,

$$\varepsilon_x = -\frac{1}{2A_c t}\sum_{i=1}^{3} A_{ix} u_i \tag{2}$$

$$\varepsilon_y = -\frac{1}{2A_c t}\sum_{i=1}^{3} A_{iy} v_i \tag{3}$$

$$\gamma_{xy} = -\frac{1}{2A_c t}\sum_{i=1}^{3} (A_{ix} v_i + A_{iy} u_i) \tag{4}$$

where t is the thickness of the sample and A_c is the area of the cell. Besides the meaning of A_{ix} and A_{iy} can be found from Figure 2 where for example A_{ix} is the x component of A_i and A_i is in turn the area of the side cross section opposite to the vertex i.

The constitutive equation may be written as

$$\boldsymbol{\sigma}_c = \mathbf{D}_c\,\boldsymbol{\varepsilon}_c = \mathbf{D}_c\,\mathbf{B}\,\mathbf{u}_c \tag{5}$$

where $\boldsymbol{\sigma}_c$ collects the stress components, and the matrix $\mathbf{D}_c$ represents Hooke's law for the isotropic homogeneous material.

In order to write equilibrium equations it is necessary to express the forces acting through each side of the control volume surrounding the considered node. As stress components are uniform within each cell, the surface force such as T_1 in Figure 3 will be given by

$$\begin{bmatrix} T_{1x} \\ T_{1y} \end{bmatrix}_c = \frac{1}{2}\begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \end{bmatrix}_c \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_c \tag{6}$$

and in a similar manner for the three nodes of the cell applied forces can be obtained by multiplying area and stress matrix as,

$$\begin{bmatrix} T_{1x} \\ T_{1xy} \\ T_{2x} \\ T_{2y} \\ T_{3x} \\ T_{3y} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \\ A_{2x} & 0 & A_{2y} \\ 0 & A_{2y} & A_{2x} \\ A_{3x} & 0 & A_{3y} \\ 0 & A_{3y} & A_{3x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \tag{7}$$

Using Equation (5), Equation (6) can be also written as

$$\mathbf{T}_c = -t\,A_c\,\mathbf{B}\,\mathbf{D}_c\,\mathbf{B}\,\mathbf{u}_c \tag{8}$$

It is now possible to write equilibrium condition for each cell. We use the following equation,

$$\mathbf{T}_p = \sum_c \mathbf{T}_p^c \tag{9}$$

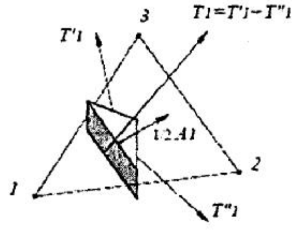

Figure 3. Forces corresponding to the node 1 of the cell

where $\mathbf{T}_P^c$ is the resultant of the forces that act on the two sides of the volume surrounding node P belonging to the cell c and $\mathbf{T}_P$ is the total force acting on all sides of volume surrounding node P. Similarly for external forces we have,

$$\mathbf{E}_P = \sum_c \mathbf{E}_P^c \tag{10}$$

where $\mathbf{E}_P^c$ is the external force that acts on the portion of the cell c which belongs to the volume surrounding node P, and $\mathbf{E}_P$ is the resultant of the external force acting on that volume.
Equilibrium can then be written for the volume surrounding node P as

$$\mathbf{T}_h + \mathbf{E}_h = \mathbf{0} \tag{11}$$

Equation (11) is a set of 2n linear equations in the 2n unknown u_i and v_i (i=1,...,n) which can be solved with the usual methods.

3 NUMERICAL RESULTS. In this section the results obtained by applying the four explicated numerical methods to a benchmarking problem are introduced. Having in hand the exact solution of the problem, in each case different assessments of errors are calculated. Also the period of execution for each one of the computer codes is measured and used as a criterion for the comparison of the methods.
Consider a cantilever beam of length L, height H and thickness t subjected to end load P. This beam is shown in Figure 4. Here the beam geometrical dimensions and the amount of load are taken to be,

$$L = 2\text{ m},\quad H=0.5\text{ m},\quad t = 0.05\text{ m},\quad P = 1\times10^5\text{ N}$$

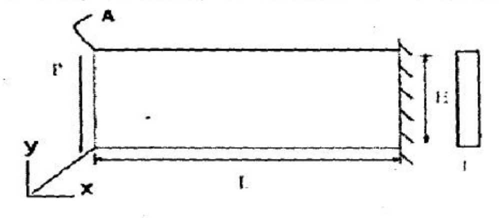

Figure 4. Cantilever beam with the end loading

Figures 5 to 8 show the manner which is used to discretise the domain to n-divisions along the length and m-divisions along the width. In each case the method that nodes are numbered is indicated on the picture.

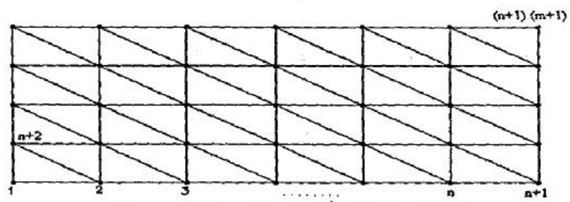

Figure 5. Discretised domain in the FEM

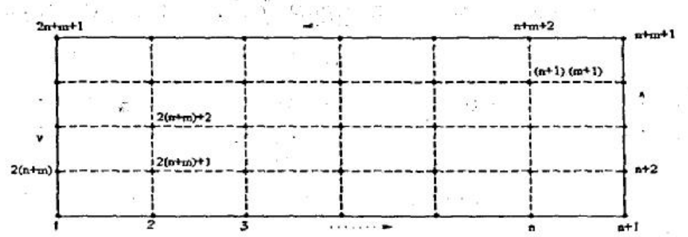

Figure 6. Discretised domain in the BEM

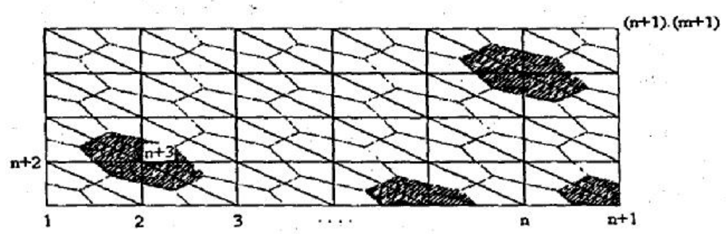

Figure 7. Discretised domain in the FVM

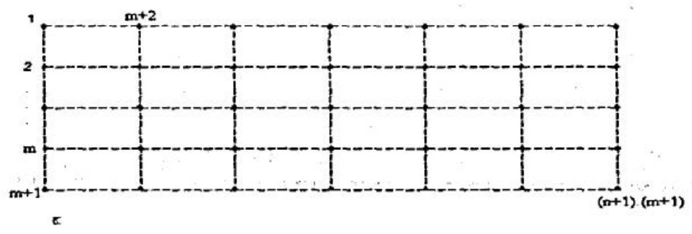

Figure 8. Discretised domain in the EFGM

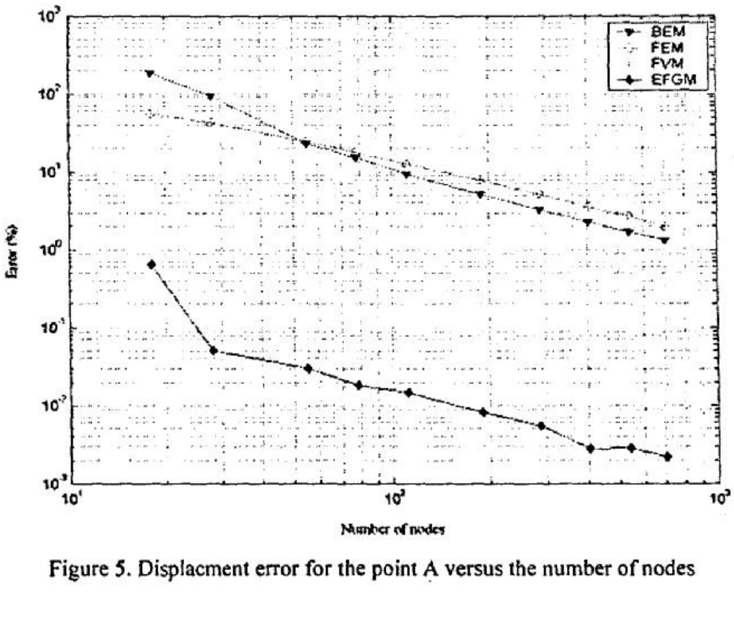

Figure 5. Displacment error for the point A versus the number of nodes

3-1 Node numbers effect. A closed form solution to this problem has been given by Timoshenko and Goodier (18). The percentage of the error in the displacement of point A (see Figure 4) as per different numerical techniques is depicted in Figure 9.

In Figure 9, the abscissa is the number of nodes and the ordinate shows the percentage of error calculated in different techniques. Based on Figure 9 the method of Element Free Galerkin (EFG) shows the most accurate results and its error is less than %1 for all number of divisions. By using coarse meshes the error in the BEM solution will grow up to a level greater than that in the FEM, but as Figure 9 reveals by refining the mesh, the answers of BEM draws near to the exact solution of the problem. So at least in this case, on the boundary nodes such as the point A, BEM solution yields more accurate answers than FEM.
As can be seen in Figure for the defined problem the displacement error of the FVM and FEM solutions are coincident.

3-2 The effect of meshing on the energy norm. The energy norm for a structural numerical analysis is often calculated as

$$\text{energy norm} = \left\{ \frac{1}{2} \int_{\Omega} (\varepsilon^{NUM} - \varepsilon^{EXACT})^T \mathbf{D} (\varepsilon^{NUM} - \varepsilon^{EXACT}) \, d\Omega \right\}^{1/2}$$

in which ε^{EXACT} is the strain calculated from an exact solution, ε^{NUM} is the strain obtained from the numerical approach and $d\Omega$ is the partial volume of the domain. The energy norm versus the number of nodes obtained in this study is depicted in Figure10. The figure shows that norm of energy in the EFGM solution is several times less than that in three other methods. Figure 10 also points out that compared to other methods the more is the number of the nodes the less will be the norm of energy in the EFGM solution. That is the EFGM line is steeper than the curve of other methods. Figure 10 also shows that by using coarse meshes the norm of energy in FEM solution is less than BEM energy norm. The situation will be reversed if fine meshes are used, so the BEM is more mesh-sensible than the FEM. This study suggests that typically by using fine meshes, the result of BEM is more accurate than that of the FEM solution. Figure also shows that the energy norm in FVM is similar to the FEM norm.
The norm of energy values have also listed in table 1.

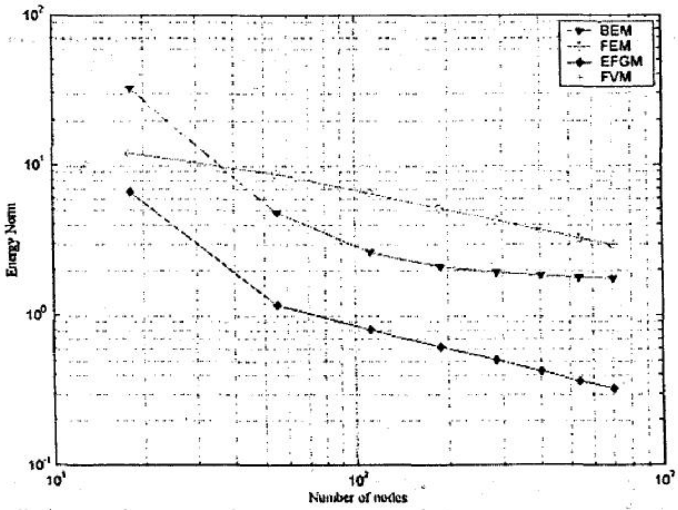

Figure 6. Energy norm

3-3 Execution time. Using several divisions of domain, the execution period elapsed in each one of the algorithms have been listed in Table 2 and also depicted in Figure11. Figure 11 discloses that EFGM is a weighty and prolonged method. The Figure also shows that the BEM takes less time than the FEM and FVM, and the FEM is slower than the FVM.

Table 1. The values of energy norms

divisions	*5×2*	*10×4*	*15×6*	*30×8*	*25×10*	*30×12*	*35×14*	*40×16*
FEM	11.9994	8.7863	6.5265	5.1719	4.3046	3.7129	3.2886	2.9726
BEM	32.6533S	4.8115	2.6487	2.1174	1.9283	1.8417	1.7946	1.7659
FVM	11.9994	8.7863	6.5265	5.1719	4.3046	3.7129	3.2886	2.9726
EFGM	6.6553	1.1801	0.8038	0.6208	0.5063	0.4276	0.3700	0.3262

Table 2. The period of calculation for the nodal displacements in seconds

number of nodes	*189*	*835*	*828*	*1271*
BEM	6.13	18.48	45.11	83.70
FEM	1.15	7.35	38.16	482.96
EFGM	71.84	292.30	814.7	1651
FVM	1.30	7.73	41.29	157.10

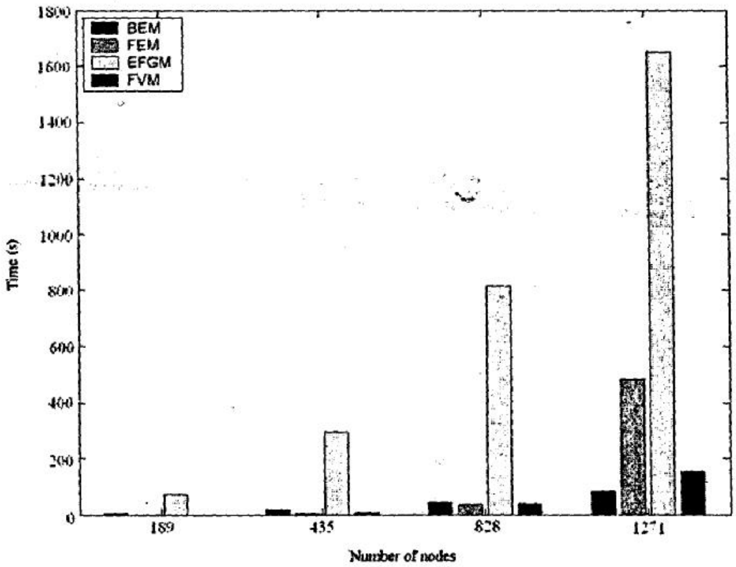

Figure 7. Expended time for the calculation of nodal displacements

4 CONCLUSIONS. In this paper in order to compare some techniques of computational mechanics, the local error in the displacement of a check point and the global norm of energy on the entirety of a cantilever beam under end loading as a benchmarking problem has been derived. Using the four methods of BEM, EFGM, FVM and FEM with the linear interpolation functions or basis some measures of errors are obtained and represented. The results show that both local and global norms of error for the EFGM were several times less than other three methods and that by using fine meshes the BEM is more accurate than the FEM. Also, the time of execution for each one of the computer codes has been measured and compared. While using the CST leads to a similar stiffness matrix and consequently a comparable level of the accuracy in the FVM and FEM, it is interesting that the execution time for the FVM is less than that of the FEM for sufficiently fine mesh. The study also reveals that EFGM is the most time consuming method and also that the BEM takes less time than FVM or FEM.

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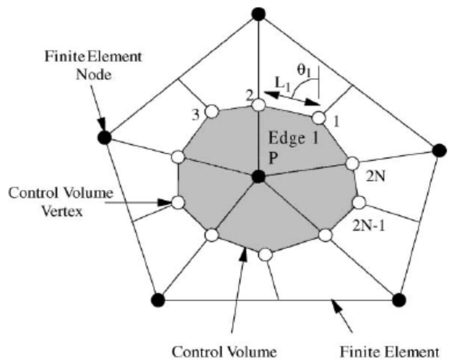

Figure ١. Volume made up of triangular cells

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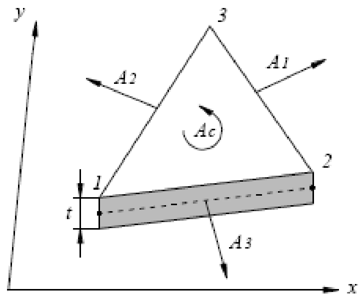

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$$\varepsilon_y = -\frac{1}{2A_c t}\sum_{i=1}^{3} A_{iy} v_i \qquad (٣)$$

$$\gamma_{xy} = -\frac{1}{2A_c t}\sum_{i=1}^{3} (A_{ix} v_i + A_{iy} u_i) \qquad (٤)$$

where t is the thickness of the sample and A_c is the area of the cell. Besides the meaning of A_{ix} and A_{iy} can be found from Figure ٢ where for example A_{ix} is the x component of A_i and A_i is in turn the area of the side cross section opposite to the vertex i.
The constitutive equation may be written as

$$\boldsymbol{\sigma}_c = \mathbf{D}_c\,\boldsymbol{\varepsilon}_c = \mathbf{D}_c\,\mathbf{B}\,\mathbf{u}_c \qquad (٥)$$

where $\boldsymbol{\sigma}_c$ collects the stress components, and the matrix $\mathbf{D}_c$ represents Hooke's law for the isotropic homogeneous material.
In order to write equilibrium equations it is necessary to express the forces acting through each side of the control volume surrounding the considered node. As stress components are uniform within each cell, the surface force such as $\mathbf{T}_1$ in Figure ٣ will be given by

$$\begin{bmatrix} T_{1x} \\ T_{1y} \end{bmatrix}_c = \frac{1}{2}\begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \end{bmatrix}_c \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_c \qquad (٦)$$

and in a similar manner for the three nodes of the cell applied forces can be obtained by multiplying area and stress matrix as,

$$\begin{bmatrix} T_{1x} \\ T_{1xy} \\ T_{2x} \\ T_{2y} \\ T_{3x} \\ T_{3y} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_{1x} & 0 & A_{1y} \\ 0 & A_{1y} & A_{1x} \\ A_{2x} & 0 & A_{2y} \\ 0 & A_{2y} & A_{2x} \\ A_{3x} & 0 & A_{3y} \\ 0 & A_{3y} & A_{3x} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad (٧)$$

Using Equation (٥), Equation (٦) can be also written as

$$\mathbf{T}_c = -t\,A_c\,\mathbf{B}\,\mathbf{D}_c\,\mathbf{B}\,\mathbf{u}_c \qquad (٨)$$

It is now possible to write equilibrium condition for each cell. We use the following equation,

$$\mathbf{T}_p = \sum_c \mathbf{T}_p^c \qquad (٩)$$

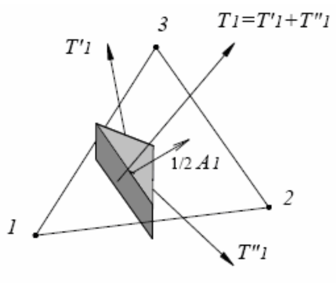

Figure ۳. Forces corresponding to the node ۱ of the cell

where $\mathbf{T}_p^c$ is the resultant of the forces that act on the two sides of the volume surrounding node P belonging to the cell c and $\mathbf{T}_P$ is the total force acting on all sides of volume surrounding node P. Similarly for external forces we have,

$$\mathbf{E}_P = \sum_c \mathbf{E}_P^c \qquad (۱۰)$$

where $\mathbf{E}_P^c$ is the external force that acts on the portion of the cell c which belongs to the volume surrounding node P, and $\mathbf{E}_P$ is the resultant of the external force acting on that volume.
Equilibrium can then be written for the volume surrounding node P as

$$\mathbf{T}_h + \mathbf{E}_h = \mathbf{0} \qquad (۱۱)$$

Equation (۱۱) is a set of ۲n linear equations in the ۲n unknown u_i and v_i (i=۱,…,n) which can be solved with the usual methods.

۳ NUMERICAL RESULTS. In this section the results obtained by applying the four explicated numerical methods to a benchmarking problem are introduced. Having in hand the exact solution of the problem, in each case different assessments of errors are calculated. Also the period of execution for each one of the computer codes is measured and used as a criterion for the comparison of the methods.
Consider a cantilever beam of length L, height H and thickness t subjected to end load P. This beam is shown in Figure ٤. Here the beam geometrical dimensions and the amount of load are taken to be,

L = ۲ m, H=۰٫۵ m, t = ۰٫۰۵ m, P = ۱×۱۰^۵ N

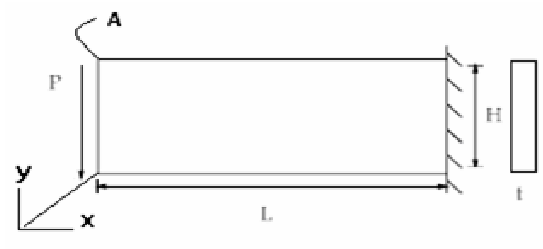

Figure ٤. Cantilever beam with the end loading

Figures ٥ to ٨ show the manner which is used to discretise the domain to n-divisions along the length and m-divisions along the width. In each case the method that nodes are numbered is indicated on the picture.

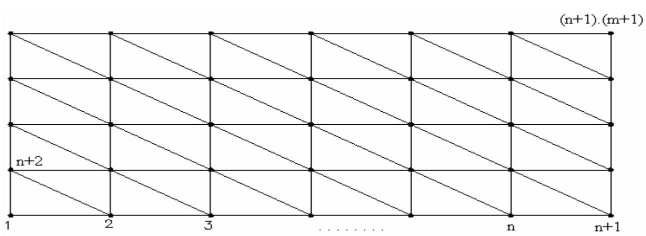

Figure ٥. Discretised domain in the FEM

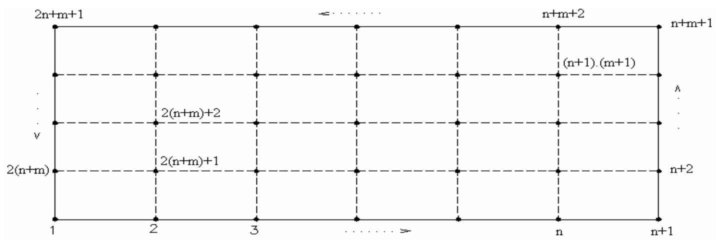

Figure ٦. Discretised domain in the BEM

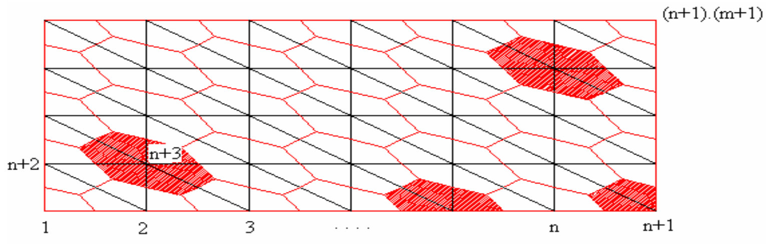

Figure ٧. Discretised domain in the FVM

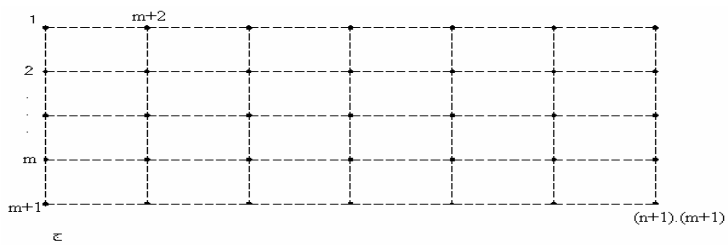

Figure ٨. Discretised domain in the EFGM

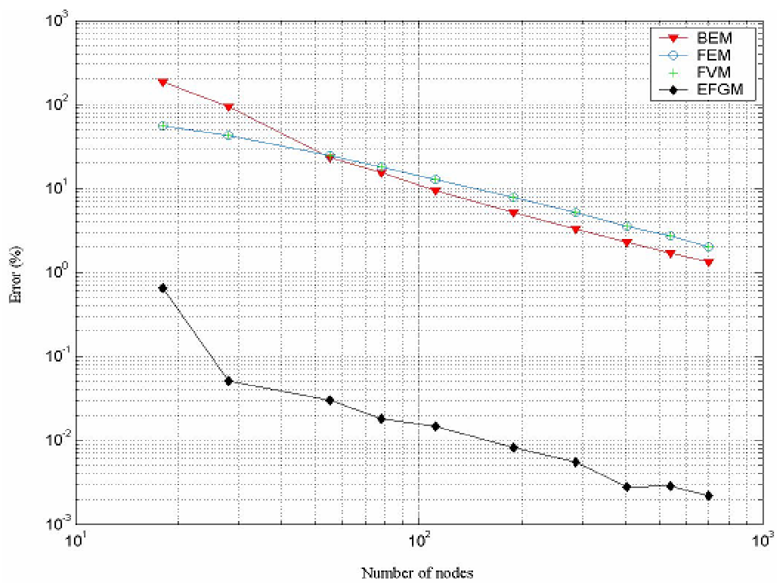

Figure ٥. Displacment error for the point A versus the number of nodes

٣-١ Node numbers effect. A closed form solution to this problem has been given by Timoshenko and Goodier (١٨). The percentage of the error in the displacement of point A (see Figure ٤) as per different numerical techniques is depicted in Figure ٩.

In Figure ٩, the abscissa is the number of nodes and the ordinate shows the percentage of error calculated in different techniques. Based on Figure ٩ the method of Element Free Galerkin (EFG) shows the most accurate results and its error is less than ٪١ for all number of divisions.

By using coarse meshes the error in the BEM solution will grow up to a level greater than that in the FEM, but as Figure ٩ reveals by refining the mesh, the answers of BEM draws near to the exact solution of the problem. So at least in this case, on the boundary nodes such as the point A, BEM solution yields more accurate answers than FEM.

As can be seen in Figure for the defined problem the displacement error of the FVM and FEM solutions are coincident.

٣-٢ The effect of meshing on the energy norm. The energy norm for a structural numerical analysis is often calculated as

$$\text{energy norm} = \left\{ \frac{1}{2} \int_{\Omega} (\boldsymbol{\varepsilon}^{\text{NUM}} - \boldsymbol{\varepsilon}^{\text{EXACT}})^{\text{T}} \, \mathbf{D} (\boldsymbol{\varepsilon}^{\text{NUM}} - \boldsymbol{\varepsilon}^{\text{EXACT}}) \, d\Omega \right\}^{1/2}$$

in which $\boldsymbol{\varepsilon}^{\text{EXACT}}$ is the strain calculated from an exact solution, $\boldsymbol{\varepsilon}^{\text{NUM}}$ is the strain obtained from the numerical approach and $d\Omega$ is the partial volume of the domain. The energy norm versus the number of nodes obtained in this study is depicted in Figure١٠. The figure shows that norm of energy in the EFGM solution is several times less than that in three other methods. Figure ١٠ also points out that compared to other methods the more is the number of the nodes the less will be the norm of energy in the EFGM solution. That is the EFGM line is steeper than the curve of other methods. Figure ١٠ also shows that by using coarse meshes the norm of energy in FEM solution is less than BEM energy norm. The situation will be reversed if fine meshes are used, so the BEM is more mesh-sensible than the FEM. This study suggests that typically by using fine meshes, the result of BEM is more accurate than that of the FEM solution. Figure also shows that the energy norm in FVM is similar to the FEM norm.

The norm of energy values have also listed in table ١.

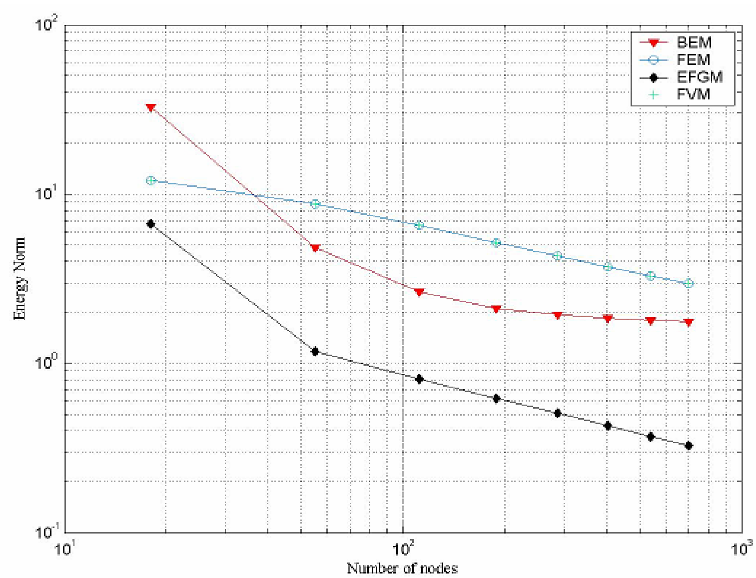

Figure ٦. Energy norm

٣-٣ Execution time. Using several divisions of domain, the execution period elapsed in each one of the algorithms have been listed in Table ٢ and also depicted in Figure١١. Figure ١١ discloses that EFGM is a weighty and prolonged method. The Figure also shows that the BEM takes less time than the FEM and FVM, and the FEM is slower than the FVM.

Table ١. The values of energy norms

divisions	٥× ٢	١٠× ٤	١٥× ٦	٣٠× ٨	٢٥× ١٠	٣٠× ١٢	٣٥× ١٤	٤٠× ١٦
FEM	١١,٩٩٩٤	٨,٧٨٦٣	٦,٥٢٦٥	٥,١٧١٩	٤,٣٠٤٦	٣,٧١٢٩	٣,٢٨٨٦	٢,٩٧٢٦
BEM	٣٢,٦٥٣٣S	٤,٨١١٥	٢,٦٤٨٧	٢,١١٧٤	١,٩٢٨٣	١,٨٤١٧	١,٧٩٤٦	١,٧٦٥٩
FVM	١١,٩٩٩٤	٨,٧٨٦٣	٦,٥٢٦٥	٥,١٧١٩	٤,٣٠٤٦	٣,٧١٢٩	٣,٢٨٨٦	٢,٩٧٢٦
EFGM	٦,٦٥٥٣	١,١٨٠١	٠,٨٠٣٨	٠,٦٢٠٨	٠,٥٠٦٣	٠,٤٢٧٦	٠,٣٧٠٠	٠,٣٢٦٢

Table ٢. The period of calculation for the nodal displacements in seconds

number of nodes	١٨٩	٨٣٥	٨٢٨	١٢٧١
BEM	٦,١٣	١٨,٤٨	٤٥,١١	٨٣,٧٠
FEM	١,١٥	٧,٣٥	٣٨,١٦	٤٨٢,٩٦
EFGM	٧١,٨٤	٢٩٢,٣٠	٨١٤,٧	١٦٥١
FVM	١,٣٠	٧,٧٣	٤١,٢٩	١٥٧,١٠

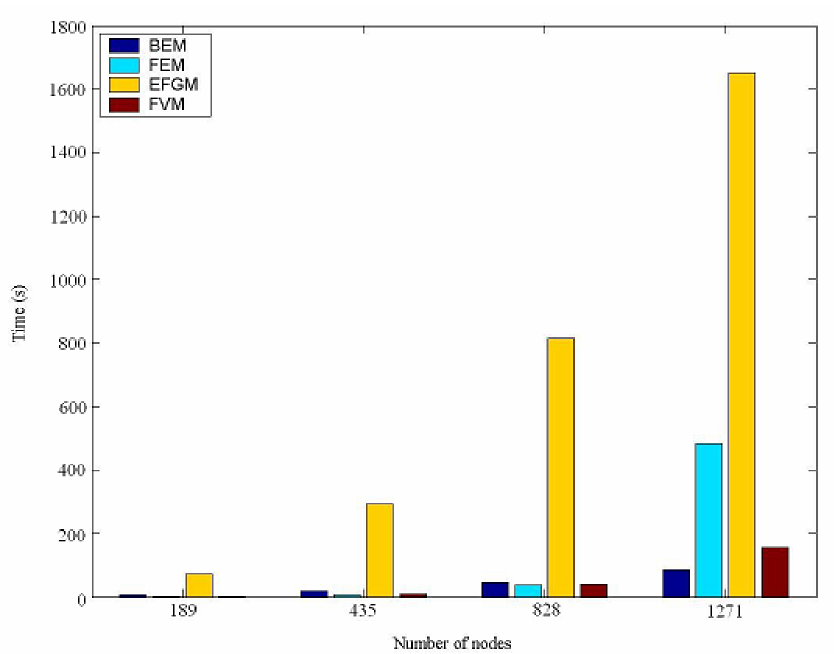

Figure ٧. Expended time for the calculation of nodal displacements

٤ CONCLUSIONS. In this paper in order to compare some techniques of computational mechanics, the local error in the displacement of a check point and the global norm of energy on the entirety of a cantilever beam under end loading as a benchmarking problem has been derived. Using the four methods of BEM, EFGM, FVM and FEM with the linear interpolation functions or basis some measures of errors are obtained and represented. The results show that both local and global norms of error for the EFGM were several times less than other three methods and that by using fine meshes the BEM is more accurate than the FEM. Also, the time of execution for each one of the computer codes has been measured and compared. While using the CST leads to a similar stiffness matrix and consequently a comparable level of the accuracy in the FVM and FEM, it is interesting that the execution time for the FVM is less than that of the FEM for sufficiently fine mesh. The study also reveals that EFGM is the most time consuming method and also that the BEM takes less time than FVM or FEM.

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