Phys. Scr. **79** (2009) 065009 (7pp) doi:10.1088/0031-8949/79/06/065009

Application of the homotopy method for analytical solution of non-Newtonian channel flows

Ehsan Roohi[1,2], Shahab Kharazmi[3] and Yaghoub Farjami[4]

[1] Department of Aerospace Engineering, Sharif University of Technology, PO Box 11365-8639, Azadi Avenue, Tehran, Iran
[2] Aerospace Research Institute, Ministry of Science, Research, and Technology, PO Box 14665-834, Tehran, Iran
[3] Department of Mechanical Engineering, Sharif University of Technology, PO Box 11365-8639, Azadi Avenue, Tehran, Iran
[4] Department of Computer Engineering, University of Qom, Qom, Iran

E-mail: roohi@sharif.edu

Received 1 January 2009
Accepted for publication 29 April 2009
Published 27 May 2009
Online at stacks.iop.org/PhysScr/79/065009

Abstract
This paper presents the homotopy series solution of the Navier–Stokes and energy equations for non-Newtonian flows. Three different problems, Couette flow, Poiseuille flow and Couette–Poiseuille flow have been investigated. For all three cases, the nonlinear momentum and energy equations have been solved using the homotopy method and analytical approximations for the velocity and the temperature distribution have been obtained. The current results agree well with those obtained by the homotopy perturbation method derived by Siddiqui *et al* (2008 *Chaos Solitons Fractals* **36** 182–92). In addition to providing analytical solutions, this paper draws attention to interesting physical phenomena observed in non-Newtonian channel flows. For example, it is observed that the velocity profile of non-Newtonian Couette flow is indistinctive from the velocity profile of the Newtonian one. Additionally, we observe flow separation in non-Newtonian Couette–Poiseuille flow even though the pressure gradient is negative (favorable). We provide physical reasoning for these unique phenomena.

PACS number: 02.30.Hq

1. Introduction

Although most of the common fluids in the real world exhibit Newtonian behavior, there are important classes of fluids that are classified as non-Newtonian. Non-Newtonian fluids are those whose constitutive equation, the equation that relates the stress and strain, is not a simple linear relation. Blood and coal-based slurries are sample examples of non-Newtonian fluids. In the current study, we seek analytical velocity and temperature profiles of a third grade fluid between two parallel plates with different temperatures and velocities. According to the relative motion of plates, three different problems are considered, the Couette flow, the Poiseuille flow and the Couette–Poiseuille flow. As is known, the governing partial differential equations (PDEs) for velocity and temperature fields are nonlinear and no exact solutions are available. Consequently, asymptotic methods are usually applied to find analytical solution for these equations. A brief review of these methods is given in [2]. Of various asymptotic methods, the one due to Liao is the homotopy method [2–6]. The homotopy method is an extension of the traditional perturbation method coupled with the homotopy concept as used in the topology. The homotopy method has been applied to obtain analytical solutions for a wide class of stochastic and deterministic problems in science and engineering involving algebraic, differential, integro-differential and integral differential equations [7, 8].

Siddiqui *et al* [1, 9–13] have recently applied the homotopy perturbation (HP) method, which is a specific form of the basic homotopy method, to analyze flow problems of non-Newtonian fluid mechanics. In the present paper, we apply the more general homotopy method to study the non-Newtonian channel flow and compare our results with that derived in [1] by using the HP method. Besides providing analytical expressions for velocity and temperature, we discuss some unique physical phenomenon related to non-Newtonian fluids.

2. Governing equations

The governing equations for the conservation of mass, momentum and energy for an incompressible fluid in tensor notation are given by

$$\begin{aligned} u_{i,j} &= 0, \\ \rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right) &= \rho f+\tau_{ij,j}, \\ \rho c_p\left(\frac{\partial \theta}{\partial t}+u_j\frac{\partial \theta}{\partial x_j}\right) &= \kappa\theta_{,jj}+\tau_{ij}l_{i,j}, \end{aligned} \tag{1}$$

where u is the velocity, f is the body force, τ is the stress tensor, θ is the temperature, ρ is the constant fluid density, κ is the thermal conductivity, c_p is the constant pressure specific heat and l is the gradient of u. The constitutive equation for a third grade fluid is

$$\begin{aligned} \tau_{ij} = &-p\delta_{i,j}+\mu S_{1ij}+\alpha_1 S_{2ij}+\alpha_2 S_{1ij}^2+\beta_1 S_{3ij} \\ &+\beta_2(S_{1ij}S_{2ij}+S_{2ij}S_{1ij})+\beta_3(tr S_{2ij})S_{1ij}, \end{aligned} \tag{2}$$

where p is the fluid pressure, μ is the coefficient of viscosity, $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are material constants and $S_{1ij}, S_{2ij}, S_{3ij}$ are line kinematics tensors defined by

$$S_{1ij} = (l_{j,i}+l_{i,j}), \tag{3}$$

$$S_{nij} = \frac{DS_{(n-1)ij}}{Dt}+S_{(n-1)ij}l_{i.j}+l_{j,i}S_{(n-1)ij}, \quad n = 2, 3, 4 \ldots. \tag{4}$$

2.1. Couette flow

Shear-driven flows are encountered in micromotors, comb mechanisms and microbearings. Consider the steady-state flow of a third grade fluid between two long parallel plates distance $2h$ apart. The lower plate is stationary and the upper plate is moving with a constant speed U. The temperature of the lower plate is θ_0 and that of the upper plate is θ_1. The lower and upper plates are located in the planes $y = -h$ and $y = \text{h}$. The pressure gradient is zero and the velocity and temperature fields are assumed to be of the form

$$u = u(y), \quad \theta = \theta(y), \tag{5}$$

The equation of continuity is satisfied and the momentum and energy equations become ($\beta_1 = \beta_3$)

$$\begin{aligned} &\mu\frac{d^2u}{dy^2}+6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = 0, \\ &\kappa\frac{d^2\theta}{dy^2}+\mu\left(\frac{du}{dy}\right)^2+2(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^4 = 0. \end{aligned} \tag{6}$$

As a result, the problem reduces for solving the equations (6) subject to the conditions of no slip and no temperature jump at both of the plates

$$\begin{aligned} u(-h) = 0, \quad u(h) = U, \\ \theta(-h) = \theta_0, \quad \theta(h) = \theta_1. \end{aligned} \tag{7}$$

We consider h as the characteristic length, U as the characteristic velocity, and θ_0 and θ_1 as characteristic temperatures and rewrite the above equations in dimensionless form by using

$$y^* = \frac{y}{h}, \quad u^* = \frac{u}{U}, \quad \theta^* = \frac{\theta-\theta_0}{\theta_1-\theta_0}. \tag{8}$$

In a non-dimensional form, after dropping the asterisks, equations (6) become

$$\begin{aligned} &\frac{d^2u}{dy^2}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} = 0, \\ &\frac{d^2\theta}{dy^2}+\lambda\left(\frac{du}{dy}\right)^2+2\beta\lambda\left(\frac{du}{dy}\right)^4 = 0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \beta &= \left(\frac{\beta_2+\beta_3}{\mu}\right)\left(\frac{U}{h}\right)^2, \\ \lambda &= \frac{\mu \text{U}^2}{\kappa(\theta_1-\theta_0)} = \frac{\mu c_p}{\kappa}\times\frac{U^2}{c_p(\theta_1-\theta_0)} = Pr Ec, \end{aligned} \tag{10}$$

where $Pr\,Ec$ is the Brinkman number that is the product of the Prandtl number Pr and Eckert number Ec. The corresponding boundary conditions are

$$\begin{aligned} u(-1) = 0, \quad u(1) = 1, \\ \theta(-1) = 0, \quad \theta(1) = 1. \end{aligned} \tag{11}$$

2.2. Poiseuille flow

Now, the same problem is considered when both plates are stationary and the fluid motion is induced by a constant pressure gradient, other conditions on the velocity and the temperature fields remain unchanged. In this case, the momentum and energy equations yield

$$\begin{aligned} \mu\frac{d^2u}{dy^2}+6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} &= \frac{\partial\hat{p}}{\partial x}, \\ \frac{\partial\hat{p}}{\partial y} = \frac{\partial\hat{p}}{\partial z} &= 0, \\ \kappa\frac{d^2\theta}{dy^2}+\mu\left(\frac{du}{dy}\right)^2+2(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^4 &= 0. \end{aligned} \tag{12}$$

where $\hat{p}$ denotes the generalized pressure given by

$$\hat{p} = p-(2\alpha_1+\alpha_2)\left(\frac{du}{dy}\right)^2. \tag{13}$$

We find from (12) that

$$\frac{\partial\hat{p}}{\partial x} = \text{constant} = \frac{d\hat{p}}{dx}. \tag{14}$$

Thus, the problem reduces for solving the following differential equations

$$\mu\frac{d^2u}{dy^2}+6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=\frac{d\hat{p}}{dx},$$
$$\kappa\frac{d^2\theta}{dy^2}+\mu\left(\frac{du}{dy}\right)^2+2(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^4=0, \qquad (15)$$

$$u(-h)=0,\quad u(h)=0,$$
$$\theta(-h)=\theta_0,\quad \theta(h)=\theta_1. \qquad (16)$$

The non-dimensional form is

$$\frac{d^2u}{dy^2}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=-B,$$
$$\frac{d^2\theta}{dy^2}+\lambda\left(\frac{du}{dy}\right)^2+2\beta\lambda\left(\frac{du}{dy}\right)^4=0, \qquad (17)$$

$$B=\frac{-h^2}{U\mu}\frac{d\hat{p}}{dx},$$

$$u(-1)=0,\quad u(1)=0,$$
$$\theta(-1)=0,\quad \theta(1)=1. \qquad (18)$$

2.3. Couette–Poiseuille flow

For the Couette–Poiseuille flow, we assume that the fluid motion is produced by both the motion of the upper plate with constant velocity U and by a constant pressure gradient. All other conditions on the temperature and the velocity remain unchanged. Thus, the momentum and energy equations take the form

$$\mu\frac{d^2u}{dy^2}+6(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=\frac{\partial\hat{p}}{\partial x},$$
$$\frac{\partial\hat{p}}{\partial y}=\frac{\partial\hat{p}}{\partial z}=0, \qquad (19)$$
$$\kappa\frac{d^2\theta}{dy^2}+\mu\left(\frac{du}{dy}\right)^2+2(\beta_2+\beta_3)\left(\frac{du}{dy}\right)^4=0,$$

with the boundary conditions

$$u(-h)=0,\quad u(h)=U,$$
$$\theta(-h)=\theta_0,\quad \theta(h)=\theta_1. \qquad (20)$$

In non-dimensional forms, we have

$$\frac{d^2u}{dy^2}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}=-B,$$
$$\frac{d^2\theta}{dy^2}+\lambda\left(\frac{du}{dy}\right)^2+2\beta\lambda\left(\frac{du}{dy}\right)^4=0, \qquad (21)$$

$$u(-1)=0,\quad u(1)=1,$$
$$\theta(-1)=0,\quad \theta(1)=1. \qquad (22)$$

In the subsequent section, we use the homotopy method to solve the three boundary-value problems described in this section.

3. Homotopy method

By means of generalizing the traditional concept of homotopy, Liao constructs the so-called zero-order deformation equation [2, 3]

$$(1-p)L(\phi(r,t,p)-f_0(r,t))=hH(r,t)pN[\phi(r,t,p)], \qquad (23)$$

where $p\in[0-1]$ is the embedding parameter, h is a nonzero auxiliary parameter, H is an auxiliary function, L is an auxiliary linear operator, $f_0(r,t)$ is an initial guess of $f(r,t)$ and $\phi(r,t,p)$ is an unknown function, respectively. It should be emphasized that one has great freedom to choose the initial guess, the auxiliary linear operator, the auxiliary parameter and the auxiliary function H. Obviously, when $p=0$ and $p=1$, it holds that

$$\phi(r,t,0)=f_0(r,t),\quad \phi(r,t,1)=f(r,t). \qquad (24)$$

Hence, as p increases from 0 to 1, $\phi(r,t,p)$ varies, or deforms, from the initial guess $f_0(r,t)$ to the solution $f(r,t)$. Expanding $\phi(r,t,p)$ in the Taylor series with respect to the embedding parameter p, one has

$$\phi(r,t,p)=f_0(r,t)+\sum_{k=1}^{\infty}f_k(r,t)p^k, \qquad (25)$$

where

$$f_k(r,t)=\frac{1}{k!}\frac{\partial^k\phi(r,t,p)}{\partial p^k}\bigg|_{p=0}. \qquad (26)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function are so properly chosen that the above series converges at $p=1$, one has

$$f(r,t)=f_0(r,t)+\sum_{k=1}^{\infty}f_k(r,t). \qquad (27)$$

Differentiating equation (23) m times with respect to the embedding parameter p and then setting $p=0$ and finally dividing by $m!$, we have the so-called m^{th}-order deformation equation

$$L(\phi_n(r,t,p)-\chi_n\phi_{n-1}(r,t,p))=hH(r,t)R_N[\phi_{n-1}(r,t,p)] \qquad (28)$$

$$\chi_m=\begin{cases}0, & m\leqslant 1\\ 1, & \text{otherwise.}\end{cases}$$

Subject to the initial condition

$$\phi_m(r,0,p)=0, \qquad (29)$$

where

$$R_n(\phi_{n-1}(r,t,p))=\frac{1}{(n-1)!}\frac{\partial^{n-1}N[\phi(r,t,p)]}{\partial p^{n-1}}\bigg|_{p=0}. \qquad (30)$$

After assuming an initial guess, we can obtain subsequent terms in homotopy series solution using equations (28)–(30).

4. Solution and discussion

4.1. Couette flow

The linear part of the momentum equation in equation (9), d^2u/dy^2, is chosen as the linear operator L in equation (23). The non-linear operator N is the same as equation (9). Let us take $H = 1$ and assume a zero-order solution of the form of equation (31) which satisfies the boundary conditions,

$$u_0(y) = \tfrac{1}{2}(1 + y). \quad (31)$$

If we apply equation (28) to the above initial approximation, integrating twice with respect to y and applying the homogenous boundary conditions, we obtain the first-order solution

$$u_1(y) = 0. \quad (32)$$

Similarly the solution to higher-order boundary value problem is $u_2(y) = 0$. Hence,

$$u(y, p) = u_0(y) = \tfrac{1}{2}(1 + y). \quad (33)$$

Substituting $u(y) = (1 + y)/2$ into the energy equation and integrating the resulting equation with respect to y, we obtain the solution for the temperature field

$$\theta(y) = \frac{\lambda}{16}(2 + \beta)(1 - y^2) + \frac{1}{2}(1 + y). \quad (34)$$

The solution $u(y)$ in (33) provides a linear velocity profile that is the same as the solution of a Newtonian fluid and is independent of the non-Newtonian parameter β. This feature is attributed to a constant velocity boundary condition and independent of flow properties from the pressure gradient in the Couette flow. In fact, since pressure is constant along the flow, the steady-state flow is only affected by shear stress and the velocity boundary condition. The latter is a first-order linear function in terms of velocity (see equation (7)). When we decompose the momentum governing equation into linear and non-linear terms, see equation (11), the linear boundary condition satisfies the linear part of the equation and the non-linear part, which is due to non-Newtonian effects, remains trivial. Consequently, the steady state velocity profile remains unaffected by nonlinearity or non-Newtonian forces. From a physical point of view, we can interpret that the nonlinear (non-Newtonian) modes in flow are not excited by a constant wall velocity boundary condition. Therefore, the velocity profile is the same for both Newtonian and non-Newtonian flows. However, dependence of temperature on the parameter β is evident, see equations (9), (11) and (34).

Figure 1 shows the effect of the variation of β when λ is fixed. It is observed that the fluid temperature increases with increase in the value of β. In order to compare homotopy solutions with numerical results, we solved the nonlinear ODE system given by equations (9) and (11) with MAPLE. As observed in this figure, numerical solution agrees quite well with the homotopy solution. In figure 2, the effect of the change of the number λ for $\beta = 1$ is presented. It is observed that the behavior of the temperature profile is similar to that in figure 1 except for somewhat larger temperature rise in this case.

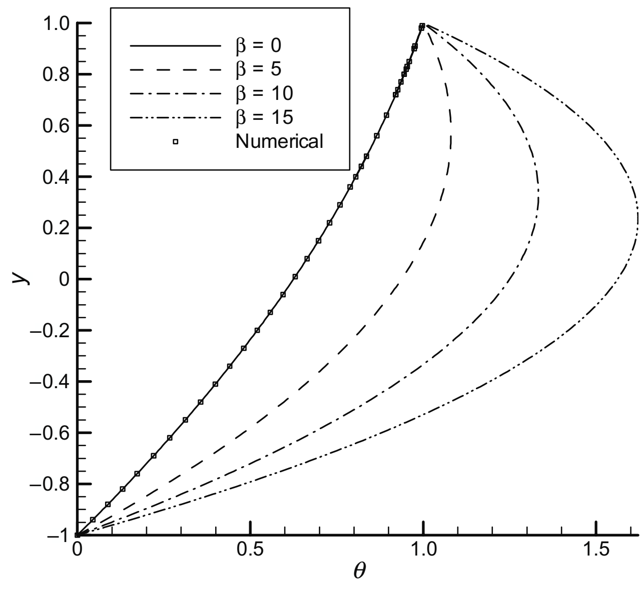

Figure 1. Effect of the parameter β (when $\lambda = 1$) on the Couette flow temperature profile.

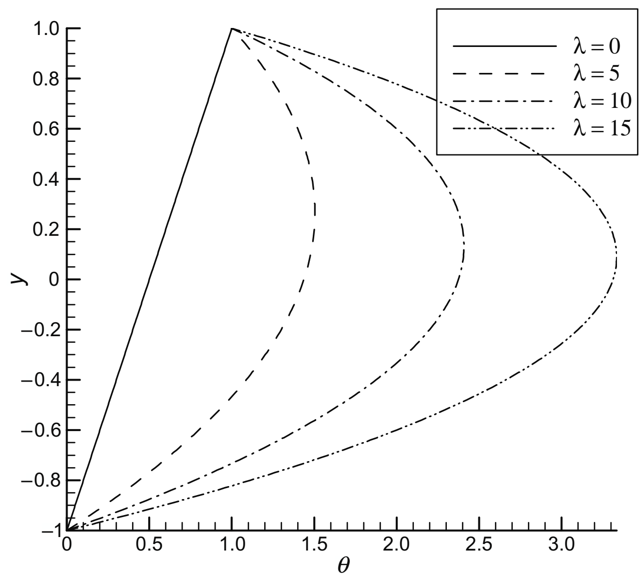

Figure 2. Effect of the parameter λ (when $\beta = 1$) on the Couette flow temperature profile.

4.2. Poiseuille flow

Again, the linear part of the momentum equation in equation (12), d^2u/dy^2, is chosen as the linear operator L in equation (23). The nonlinear operator N is the same as equation (12). We take $H = 1$ and assume a zero-order solution of the form of equation (35) which satisfies the boundary conditions,

$$u_0(y) = \frac{B}{2}(1 - y^2). \quad (35)$$

By applying equation (28) to the above initial approximation, we obtain other terms

$$u_1(y) = -\tfrac{1}{2}h\beta B^3(1 - y^4). \quad (36)$$

$$u_2(y) = -2h^2\beta^2 By^6 - 0.5(h\beta B^3 + h^2\beta B^3)y^4 + 0.5h\beta B^3 + h(2h\beta^2 B^5 + 0.5h\beta B^3) \quad (37)$$

$$\vdots$$

Other terms of the series, which depend on h, are complicated and we do not present them here. If we set $h=-1$, we get the same results as the HP method calculated by Siddiqui *et al* [1]

$$h=-1 \Rightarrow u_2(y) = -2\beta B^5(1-y^6). \quad (38)$$

$$h=-1 \Rightarrow u_3(y) = -12\beta B^7 y^8 - 4\beta^2 B^5 y^6 + 4\beta B^5 + 12\beta^3 B \quad (39)$$

$$\vdots$$

Adding the first two terms to the zero-order solution, we get an analytical expression for the velocity field

$$u(y) = \tfrac{1}{2}(1-y^2) - \tfrac{1}{2}\beta B^3(1-y^4) + 2\beta^2 B^5(1-y^6). \quad (40)$$

We considered only the above terms in order to compare our result with [1]. Equation (40) is exactly the same as the result of the HP method of Siddiqui *et al* [1]. Figures 3 and 4 present the velocity profile $u(y)$, given by equation (40), for various values of β when B is fixed at one. It is observed from this figure that the velocity increases with increasing β. Similar behavior of the solution is shown in figure 4, which shows the $u(y)$ curve for various values of B when the value of β is taken to be one. Substituting $u(y)$ from (40) into (17) and solving the ODE equation, we obtain

$$\theta(y) = \tfrac{1}{2} + \tfrac{1}{2}y + \tfrac{1}{12}\lambda B^2(1-y^4) - \tfrac{1}{15}\lambda\beta B^4(1-y^6) + \tfrac{3}{14}\lambda\beta^2 B^6(1-y^8) - \tfrac{124}{33}\lambda\beta^4 B^{10}(1-y^{12}) + 16\lambda\beta^5 B^{12}(1-y^{14}) - 32\lambda\beta^6 B^{14}(1-y^{16}) + \tfrac{1152}{17}\lambda\beta^7 B^{16}(1-y^{18}) - \tfrac{6912}{95}\lambda\beta^8 B^{18}(1-y^{20}) + \tfrac{6912}{95}\lambda\beta^9 B^{20}(1-y^{22}). \quad (41)$$

The expression for temperature $\theta(y)$, given by equation (41), is plotted in figure 5 for different values of β when $\lambda = 1$, $B = 1$. This graphical study shows that the temperature rapidly rises with increasing values of the parameters β.

4.3. Couette–Poiseuille flow

Again, the linear part of the momentum equation in equation (21) is chosen as the linear operator L in equation (23). The nonlinear operator N is the same as equation (21). We take $H=1$ and the initial guess $u_0(y)$, which satisfies the boundary conditions, is chosen to be

$$u_0(y) = \frac{1}{2}(1+y) + \frac{B}{2}(1-y^2). \quad (42)$$

Applying the homotopy method, we have

$$u_1(y) = -\frac{1}{32}\frac{h\beta(2By-1)^4}{B} - \frac{1}{4}h\beta(4B^2+1)y + \frac{1}{32}\frac{h\beta(16B^4+24B^2+1)}{B}. \quad (43)$$

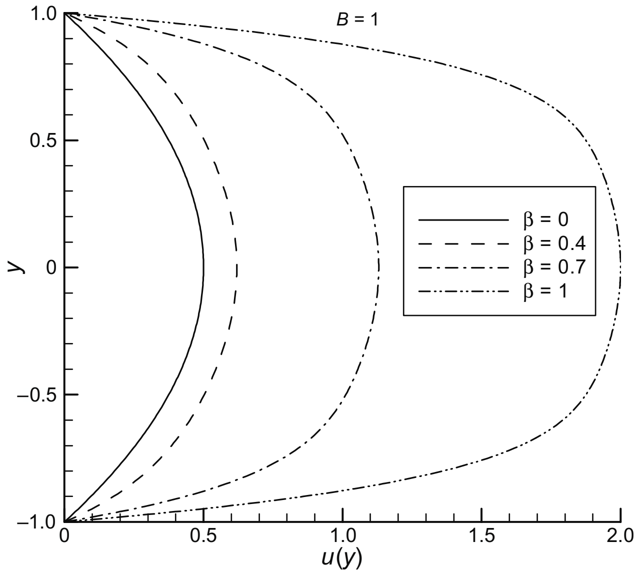

Figure 3. Effect of the parameter β (when $B=1$) on the Poiseuille flow velocity profile.

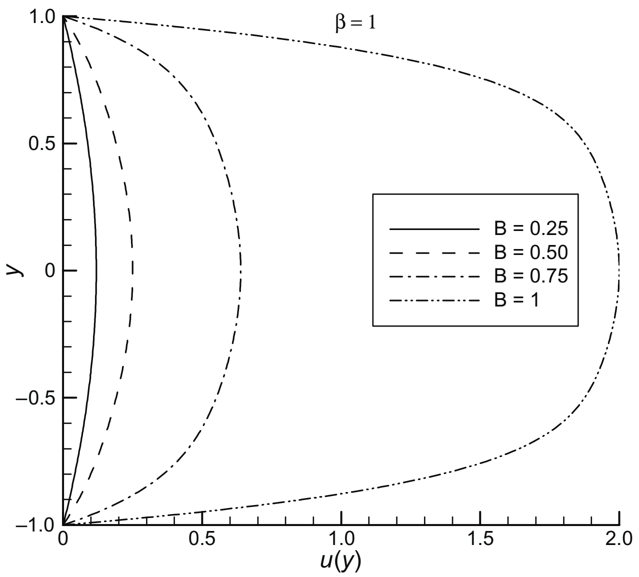

Figure 4. Effect of the parameter B (when $\beta=1$) on the Poiseuille flow velocity profile.

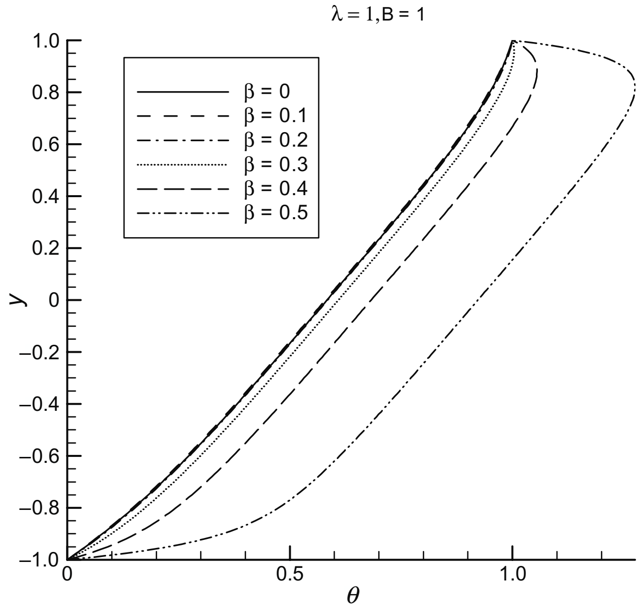

Figure 5. Effect of the parameter β (when $B = 1$, $\lambda = 1$) on the Poiseuille flow temperature profile.

$$u_2(y) = -2\beta^2 B^5 y^6 + 6\beta^2 B^4 y^5 - \frac{15}{2}\beta^2 B^3 y^4 + \left(-2\beta^2 B^4 + \frac{9}{2}\beta^2 B^2\right) y^3 + \left(-\frac{9}{8}\beta^2 B + 3\beta^2 B^3\right) y^2 + \left(\frac{1}{4}\beta(4B^2+1) - 4\beta^2 B^4 - \frac{9}{2}\beta^2 B^2 - \beta B^2 - \frac{1}{4}\beta\right) y + \frac{1}{32}\frac{\beta}{B} + 2\beta^2 B^3 - \frac{1}{32}\frac{\beta(16B^4+24B^2+1)}{B} + \frac{1}{2}\beta B^3 + \frac{3}{4}\beta B + \frac{9}{8}\beta^2 B + \frac{9}{2}\beta^2 B^3. \quad (44)$$

$$\vdots$$

Again, the other terms are complicated and we avoid writing them here. Setting $h = -1$, we have

$$u_1(y) = \tfrac{1}{4}\beta[3B(y^2-1) - 4B^2(y^3-y) + 2B^3(y^4-1)]. \quad (45)$$

$$\vdots$$

Similar to [1], we add up the zero- and first-order terms

$$u(y) = \frac{1}{2}(1+y) + \frac{B}{2}(1-y^2) + \frac{\beta}{4}[3B(y^2-1) + 4B^2(y-y^3) + 2B^3(y^4-1)]. \quad (46)$$

The velocity profile (46) is plotted in figures 6 and 7 for various values of β and B when one of them is chosen to be one. Interesting cases are the curves for $\beta = 0.3$ and $B = 0.4$. For $\beta > 0.3$ and $B > 0.4$, there is a back flow at the lower plate. This back flow becomes more dominant as

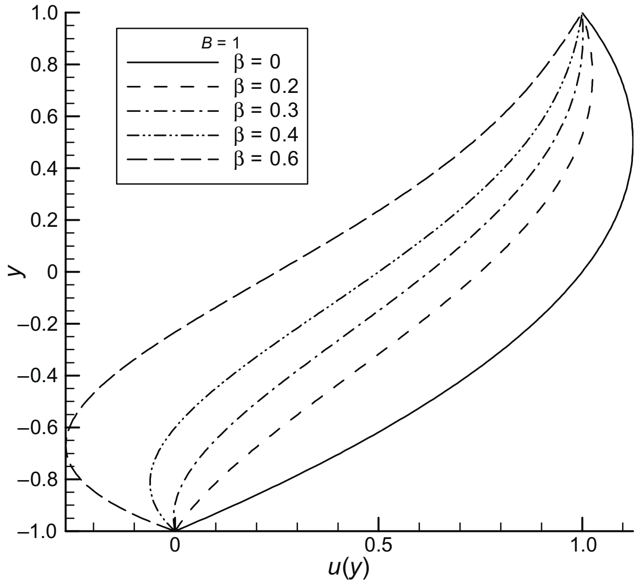

Figure 6. Effect of the parameter β (when $B = 1$) on the Couette–Poiseuille flow velocity profile.

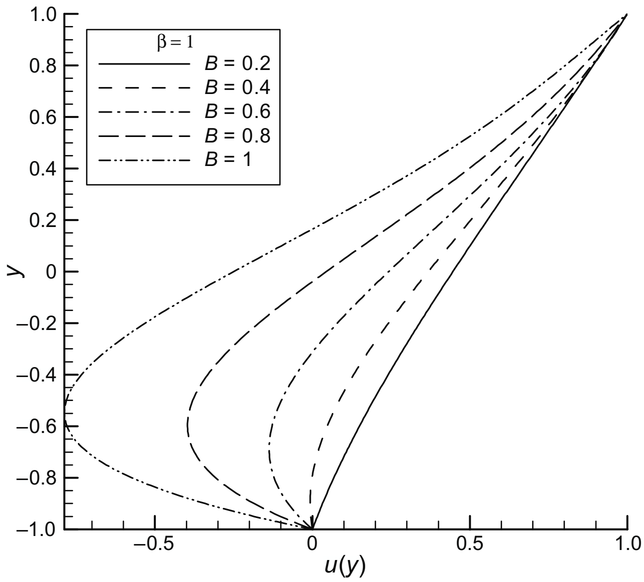

Figure 7. Effect of the parameter B (when $\beta = 1$) on the Couette–Poiseuille flow velocity profile.

the values of β and B increase. In fact, the occurrence of back flow is an indication of flow separation. For Newtonian Couette–Poiseuille flow, back flow and separation are prone to occur whenever an adverse pressure gradient ($\mathrm{d}p/\mathrm{d}x > 0$) is applied along the flow, for example, see figure 3.8 in [14]. In the case of Newtonian Couette–Poiseuille flow, the flow solution is a linear superposition of corresponding Couette and Poiseuille flows. The solution corresponding to the Couette flow resists an adverse pressure gradient of the Poiseuille flow until $B = -0.5$. Once B falls below this value, shear stress at the walls becomes negative and separation and back flow occurs. However, an interesting feature is observed here for the non-Newtonian Couette–Poiseuille flow. Even the pressure gradient is favorable ($\mathrm{d}p/\mathrm{d}x < 0$), i.e. $B = 0.4$, the flow starts to separate. This behavior can only be attributed to the interaction of nonlinear modes in the stress–strain relation

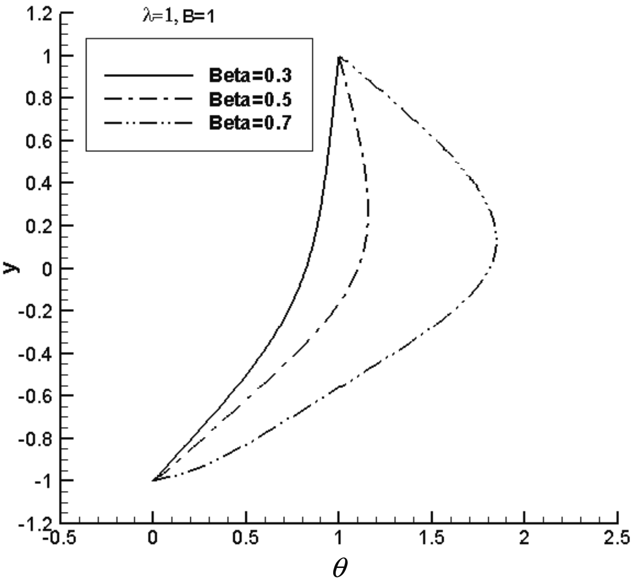

Figure 8. Effect of the parameter β (when $B = 1$, $\lambda = 1$) on the Couette–Poiseuille flow temperature profile.

for non-Newtonian fluids with the pressure gradient, see the third term on the left-hand side of equation (46). Therefore, an important conclusion can be drawn here. The classical criterion for flow separation which assumes flow separation is unlikely to happen under a favorable pressure gradient, is restricted to Newtonian fluids. As observed here, we had shown that flow separation occurs under a favorable pressure gradient for a third grade fluid.

Using the velocity field, we solve the boundary value problem (21) and (22) to obtain the temperature distribution. Thus, we find

$$\theta(y) = A_0 + A_0 y + A_0 y^2 + \cdots + A_0 y^{13} + A_0 y^{14}. \qquad (47)$$

The coefficients A_i are the same as what is calculated in [1] by the HP method. In figure 8, the temperature, given by equation (48), is plotted for different values of β when B and λ are fixed at one. The temperature rise with the increase of β is again observed.

5. Conclusion

Analytical solutions for the velocity and the temperature profile have been found for a non-Newtonian third grade fluid between two parallel infinite plates, which are at different temperatures and have different velocities. Considering three common cases, the Couette flow, the Poiseuille flow and the Couette–Poiseuille flow, Liao's homotopy method has been used to obtain analytical series solutions. It was observed that the fluid velocity depends upon the parameter β except in the case of the Couette flow. In addition, the velocity increases with increasing value of β or B. In the Couette flow, nonlinear modes in the stress–strain relation are not excited by constant velocity of the upper plate. In the case of the Couette–Poiseuille flow, the back flow behavior is observed in some specific cases according to the values of β and B. In other words, we have shown that flow separation may occur under a favorable (negative) pressure gradient for a non-Newtonian fluid. This phenomenon is unlikely for Newtonian fluids. In all the three cases, the fluid temperature depends on β and B. The obtained analytical solutions agree well with the solutions calculated by He's HP method [1].

Acknowledgment

ER thanks the anonymous referee for fruitful comments that helped in improving the quality of the paper.

References

[1] Siddiqui A M, Zeb A, Ghori Q K and Benharbit A M 2008 Homotopy perturbation method for heat transfer flow of a third grade fluid between parallel plates *Chaos Solitons Fractals* **36** 182–92

[2] Liao S J 2003 *Beyond Perturbation: Introduction to Homotopy Analysis Method* (Boca Raton, FL: CRC Press)

[3] Liao S J 1992 The proposed homotopy analysis technique for the solution of nonlinear problems *PhD Thesis* Shanghai Jiao Tong University

[4] Liao S J 1995 An approximate solution technique not depending on small parameters: a special example *Int. J. Non-Linear Mech.* **30** 371–80

[5] Liao S J 1997 A kind of approximate solution technique which does not depend upon small parameters—II: an application in fluid mechanics *Int. J. Non-Linear Mech.* **32** 815–22

[6] Liao S J 2004 On the homotopy analysis method for nonlinear problems *Appl. Math. Comput.* **147** 499–513

[7] Liao S J 2006 An analytic solution of unsteady boundary-layer flows caused by an impulsively stretching plate *Commun. Nonlinear Sci. Numer. Simul.* **11** 326–39

[8] Roohi E, Rasi Marzabadi F and Farjami Y 2008 Application of the homotopy perturbation method to linear and nonlinear fourth-order boundary value problems *Phys. Scr.* **77** 055004

[9] Siddiqui A M, Ahmed M and Ghori Q K 2006 Couette and Poiseuille flows for non-Newtonian fluids *Int. J. Nonlinear Sci. Numer. Simul.* **7** 15

[10] Siddiqui A M, Ahmed M and Ghori Q K 2007 Thin film flow of non-Newtonian fluids on a moving belt *Chaos Solitons Fractals* **33** 1006–16

[11] Siddiqui A M, Mahmood R and Ghori Q K 2006 Homotopy perturbation method for thin film flow of a fourth grade fluid down a vertical cylinder *Phys. Lett.* A **352** 404–10

[12] Siddiqui A M, Mahmood R and Ghori Q K 2006 Thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method *Int. J. Nonlinear Sci. Numer. Simul.* **7** 7

[13] Siddiqui A M, Mahmood R and Ghori Q K 2006 Homotopy perturbation method for thin film flow of a third grade fluid down an inclined plane *Chaos Solitons Fractals* **35** 140–7

[14] White M F 1991 *Viscous Fluid Flow* 2nd edn (New York: McGraw-Hill)