EXTENSION OF THE NAVIER-STOKES EQUATIONS TO TRANSITION REGIME USING INFORMATION PRESERVATION METHOD

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ABSTRACT
The main goal of the current study is to extend the range of application of the Navier Stokes equations beyond the slip-flow regime by using information preservation (IP) method. In addition to a correct velocity profile, the continuum-based equations should predict accurate mass flow rate and axial pressure distribution. The second-order slip velocity model based on the kinetic theory provides accurate velocity profiles while it gives erroneous mass flow rate. In this study, we use shear stress distribution obtained from an extended IP code to develop analytical formula for dynamic viscosity in the mid-transition regime and use it to modify Navier Stokes equations. Using the new viscosity coefficient, analytical expression is derived for mass flow rate in the range of 0.1<Kn<0.5. Comparing with experimental data and analytical solutions, the new model accurately predicts mass flowrate for a much wider range of Knudsen number. Meanwhile, we observe that axial pressure distribution is of the highest sensitivity to the viscosity coefficient and unpredicted by new model at high Knudsen numbers.

KEY WORDS: IP method, transition regime, mass flow rate scaling, Navier Stokes equations, shear stress.

1. VISCOSITY MODEL AND VALIDATION
One of the most accurate second order slip boundary conditions is the one derived from kinetic theory [1], given by

$$U_s - U_w = \frac{2-s_v}{s_v}\left[Kn\frac{\partial u}{\partial y} + \frac{Kn^2}{2}\frac{\partial^2 u}{\partial y^2}\right] \tag{1}$$

This slip boundary condition gives accurate wall slip value up to Kn<0.5 but is very poor in predicting mass flow rate, since N-S equations becomes invalid as Kn increases. To remedy this problem, the rarefaction effects must be considered on dynamic viscosity. Here, we apply IP scheme to calculate dynamic viscosity variation with Knudsen number. IP Shear stress is used to obtain viscosity coefficient as follows

$$m_{effective}(kn) = \frac{t_{w,IP}(x)}{\partial V_t / \partial n} \tag{2-a}$$

$$t_{w,IP} = \sum_{j=1}^{N_s} m(V_{t,j}^{in} - V_{t,j}^{re})/(t_s A) \tag{2-b}$$

where $t_{w,IP}$ is the IP shear stress. For small Knudsen flow, the linear dependence of stress-strain is held but as Knudsen increases; higher order terms show up in shear stress formula:

$$t_{w,IP} = m_{NS}\frac{\partial V_t}{\partial n} + t_w^{(B)} + t_w^{(AB)} + O(Kn^4) \tag{3}$$

Superscripts NS, B, and AB stands for Navier-Stokes, Burnett and augmented Burnett equations; respectively. We approximate Eq. (3) to obtain an effective viscosity coefficient for Navier-Stokes equations:

$$t_{w,IP}(x) \approx m_{effective,NS}(x)\frac{\partial V_t}{\partial n} + O(Kn^3) \tag{4}$$

The value of $t_w^{(B)}$ is negligible for low Mach number, isothermal flows in long channels [1]. Holding other terms such as $t_w^{(AB)}$ in Eq. (4) is not useful for our current purpose in that the resulting viscosity coefficient will be suitable for augmented Burnett equations not N-S equations. The validity of this approximation decreases as *Kn* increases. In fact, the computed viscosity coefficient from Eq. (4) transforms the IP shear stress to N-S equations and permits these equations to be extended beyond the limit of slip flow regime.

We obtain the IP viscosity predictions for the range of 0.1<Kn<0.5 as follows

$$\left(\frac{m(kn)}{m_0}\right)_{IP} = \frac{1+0.89kn+4.70kn^2}{1+0.75kn+19.98kn^2} \tag{5}$$

Figure 1 compares the above model with different analytical models reported in literature. Colin [2] model is accurate up to kn<0.3 and the current formula closely follows it for this region.

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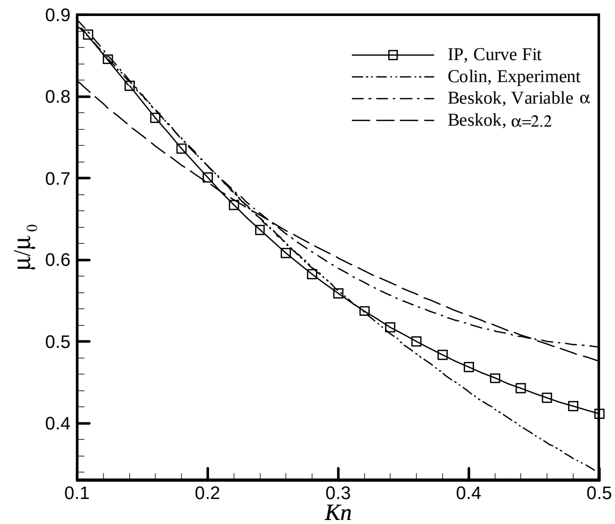


Figure 1. Variation of viscosity coefficient with Kn from different methods, 0.1<Kn<0.5.

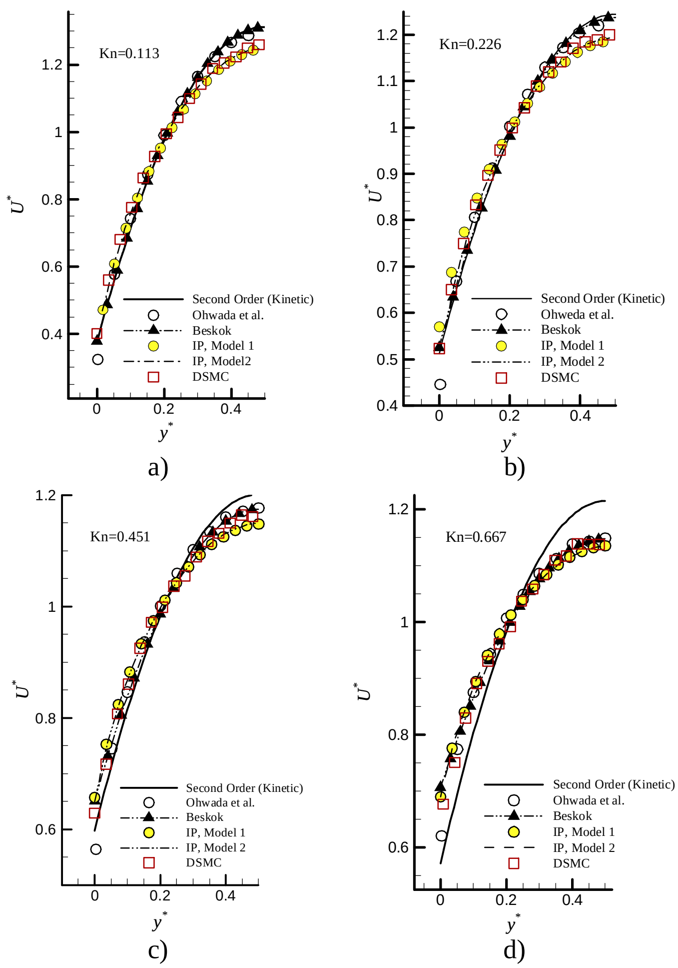


Figure 2. Comparison of velocity profiles from different molecular and continuum based models at different Knudsen numbers.

To further evaluate the accuracy of our IP solution, a comparison is made between different velocity profiles obtained from current DSMC and IP solution (with two collision models, see Ref. [3]), Ohwada et al. linearized Boltzmann (LB), Beskok analytical model [1], and developed numerical Navier-Stokes solver using second order kinetic slip velocity model (Eq. 1) for different Knudsen numbers. The developed N-S solver uses a finite-volume-based-finite-element method. It has extensively been validated with DSMC, Lattice Boltzmann solution and different experimental data and analytical solutions for micro-nano flows and heat transfer [4-5]. DSMC solution is considered as the closest solution to Boltzmann equation. For the simulated cases, LB solution under-predicts slip velocity comparing with DSMC solution but it carefully follows velocity profile curvature. Second order kinetic model (Eq. 1) is accurate for slip but it degrades as Kn increase. It is expected because this model is derived based on a maximum thickness of one mean free path for Knudsen layer, which is valid for small Kn. As Knudsen increases, Beskok analytical model over-predicts slip velocity (Fig. 4-c) but it correctly predicts maximum velocity. Our simulations show that combination of our viscosity model with second order kinetic model accurately predicts mass flow rate for mid-range transition regime (0.1<Kn<0.5) for different pressure ratios, accommodation coefficients and outlet Knudsen numbers as compared with experimental data. Since second order kinetic model (Eq. 1) is accurate within this Knudsen range, our viscosity model is physically accurate.

2. ANALYTICAL DEVELOPMENT FOR MASS FLOW RATE

We combine IP analytical expressions for viscosity coefficient (Eq. 5) with suggested velocity slip (Eq. 1) to analytically calculate mass flowrate. The resulting expression obtained for normalized mass flow rate, i.e., slip coefficient, S, is:

$$S = \frac{\dot{m}}{\dot{m}_c} = 1 + a_1 \frac{kn_o}{\Pi + 1} + \frac{kn_o^2}{(1-\Pi^2)}(51.0\ln(\Pi) + 34.07\ln(a_2))$$

$$a_1 = 11.72 + \frac{89.9 \times 0.47}{1 + (0.21 + \frac{0.47}{kn_o})(0.21 + \frac{0.47\Pi}{kn_o})}, \qquad (6)$$

$$a_2 = \frac{1 + 0.89kn_o + 4.7kn_o^2}{\Pi^2 + 0.89kn_o\Pi + 4.7kn_o^2}$$

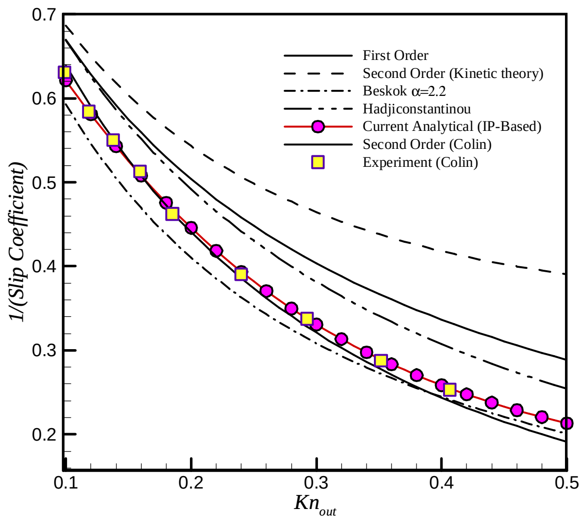
where Π is the pressure ratio. Note that coefficients of Knudsen in IP viscosity model reappear in the slip coefficient. For a general second order boundary condition, slip coefficient is given by:

$$S = 1 + 12C_1 \frac{Kn_o}{\Pi + 1} + 12C_2 \frac{Kn_o^2}{\Pi^2 - 1}\ln(\Pi) \qquad (7)$$

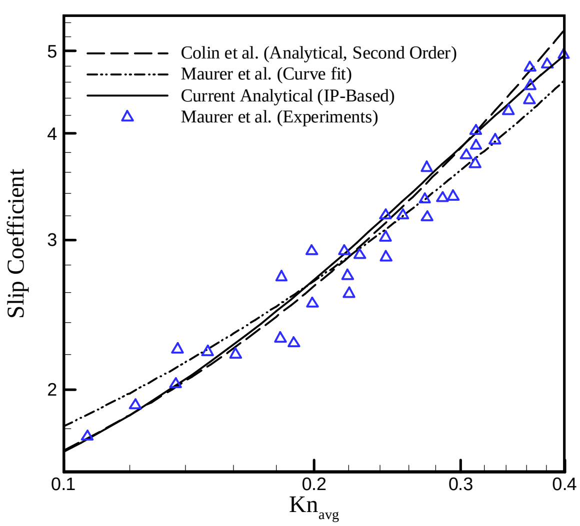
where C_1 and C_2 are constants. Although Eq. (7) seems quite simple, it is derived from a velocity profile which is either correct for mass flow rate or velocity shape prediction. It includes no viscosity correction.

2.1 VALIDATION

Having developed analytical expressions for mass flow rate, we investigate the accuracy of current viscosity formula. In order to compare our results with experimental data, slip variation for condition of $\Pi = 1.8$ is studied, where experimental data of Colin are available [2]. Other solutions from different analytical expressions; i.e., Maxwell first order slip B.C., second order slip B.C.s (kinetic theory, Colin and Hadjiconstantinou formula), and Beskok model, are also included for comparison. According to Fig. 3-a, IP-model excellently follows experimental measurement while Colin formula departs from it as soon as *Kn* increases above 0.3. This figure also shows low sensitivity of IP viscosity coefficient to accommodation coefficient. As mentioned earlier, IP simulations are performed for full momentum accommodation. Figure 3-b compares slip coefficient variation with average Knudsen from IP-model, Colin formula, Maurer et al. [6] experimental data and their empirical formula. As observed, the current model also agrees with experimental data of Maurer et al.. We can conclude that correct mass flowrate prediction of IP and suitable accuracy of second order kinetic B.C are the key aspects resulting in an accurate slip formula which predicts flowrate perfectly within the 0.1<Kn<0.5 region.



a) Π =1.8, s =0.93



b) Π =2

Figure 3. Variation of slip coefficient with Knudsen, comparison of different analytical models at two pressure ratios, experimental data [2, 6] are included.

2.2 EFFECT OF PRESSURE RATIO

Analytical expressions derived for mass flow rate depends on both Knudsen and pressure ratio. Figure 4 shows the variation of slip coefficient with pressure ratio for simulation with Kn_o=0.47. Experimental data are taken from Ref. [2] and suitably normalized. IP-model agrees well with experimental data and Beskok model. It is a key point about the current analytical expression which was originally derived for a constant pressure ratio of 2. It is seen that it is extended to other pressure ratios without loosing accuracy.

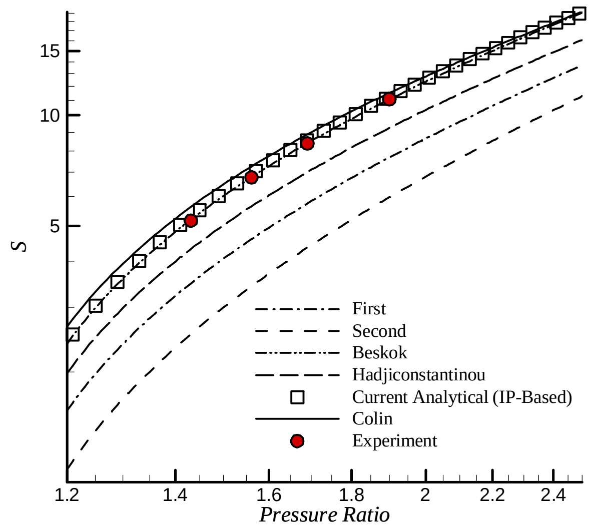


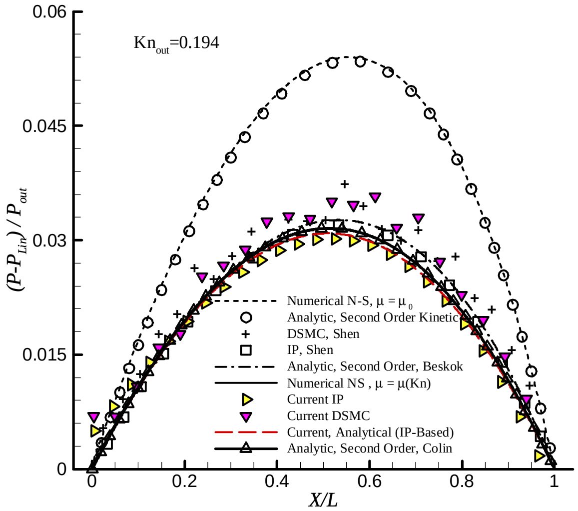
Figure 4. Comparison of different analytical models for slip coefficient variation with pressure ratio, Kn_o=0.47, experimental data from [2].

3. AXIAL PRESSURE DISTRIBUTION

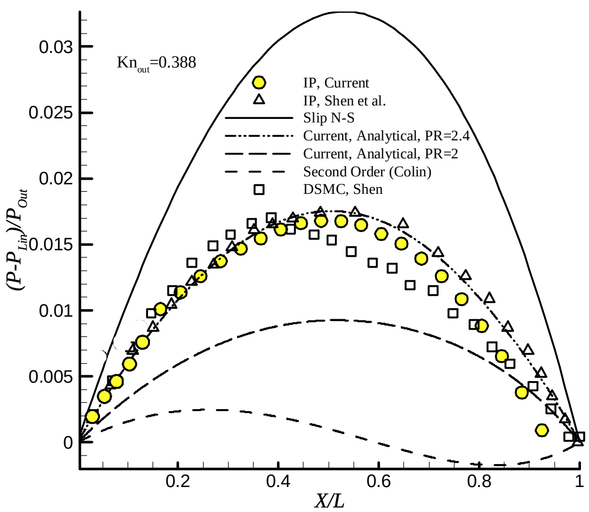
Another important property that must be carefully studied is the axial pressure distribution. The axial pressure distribution can be found by equating the conservation of the mass flow rate through the channel. Figure 5-a shows deviation of axial pressure from the linear incompressible distribution for different analytical models and numerical schemes. The maximum deviation at the mid-channel is around 0.03, as concluded from DSMC-IP solutions (DSMC solution shows similar trend with IP but exhibits relatively large scatter). It is observed that second order kinetic model (used either analytically or in our N-S solver) fully over-predicts pressure deviations while using current IP viscosity model (either in analytical form or in numerical solver), Colin formula, Beskok second order (using $\overline{a} = 6$, see Ref. [1]) results in accurate solutions compared with current DSMC/IP solutions and DSMC/IP solutions of Shen [7]. According to Fig. 5, it is concluded that incorrect dynamic viscosity not only deteriorates mass flow rate calculations, but also makes pressure distribution erroneous as well, i.e., second order kinetic solution without viscosity modification.

Finally, Fig. 5-b shows deviation of the centerline pressure for a relatively higher Knudsen of 0.388. The numerical DSMC-IP simulation is done for pressure ratio of 2. Analytical IP-Based solution falls below real pressure deviations even tough it correctly predicts mass flow rate for Kn<0.5 flows. In fact, mass flowrate depend on pressure ratio, i.e., Eq. 6, and not on local variations of pressure. Correct mass flowrate is obtained as a combination of velocity slip and suggested viscosity coefficient. As 1-D N-S equation suggests, pressure gradient depends on the longitudinal variation of shear stress:

$$\frac{dp}{dx} = \frac{\partial}{\partial y}(t_{xy}) \tag{8}$$



a) Kn_o=0.194



b) Kn_o=0.388

Figure 5. Deviation of centerline pressure from linear distribution, comparison of current IP solution with DSMC and IP solution of Shen [7] and different analytical solutions.

The shear variation in normal direction is not considered in our derivation. As soon as the thickness of Knudsen layer increases, the error due to this approximation increases and therefore, pressure gradient is calculated incorrectly. Additionally, this figure shows that Colin formula is fully erroneous in pressure estimation for higher Knudsen regimes in that it gives a wavy pattern. This confirms that obtaining correct mass flowrate via viscosity correction does not guaranty correct pressure gradient. For this case, current model needs a pressure ratio of 2.4 to match correct variation. The deviation between current IP and DSMC is also attributed to the limitations in phenomenological collision model of Sun and Boyd [3], i.e., model 2, which is applied here. As a conclusion, we note that correct pressure calculation needs considering accurate variation of shear stress.

4. CONCLUSION

In order to extend the basic N-S equations beyond slip flow regime, the current study develops and validates an analytical expression for the variation of viscosity coefficient with Knudsen number using the results of IP simulation. There are second order accurate slip boundary conditions such as kinetic-theory based and Beskok models which accurately predict velocity profile inside and outside of the Knudsen layer for high Knudsen flows. To achieve correct mass flow rate in addition, it required that N-S equations consider rarefaction effects. Using IP shear stress, we modified dynamic viscosity so that N-S equations capture correct variation of flowrate. Based on the new viscosity model, we developed analytical expressions for mass flow rate and axial pressure. For the derivation range, 0.1<Kn<0.5, the current IP-based model accurately predicts mass flowrate while compared with experimental data. Although correct mass flowrate is predicted at higher Knudsen flows, axial pressure remains under-predicted by the current model in that local variations of shear stress are not included in the derivation. Assuming linear relation between shear stress and velocity gradient and limited accuracy of kinetic-based slip model up to Kn<0.5 prevents extending the developed viscosity model to higher Knudsen values. Finding alternative approaches to extend the current model to higher Knudsens is the topic of our ongoing research.

REFERENCES

[1] Karniadakis, G. E., Beskok, A., and Aluru, N., 2005, Micro flows and Nano flows: Fundamentals and Simulation, Springer-Verlag, New York.

[2] Colin S, 2005, Rarefaction and compressibility effects on steady and transient gas flow in microchannels. Microfluid Nanofluidics 1(3) 268–279.

[3] Sun, Q., and Boyd, I.D., 2002, A direct simulation method for subsonic microscale gas flows, J. Computational Physics, 179 400-425.

[4] Darbandi, M., and Vakilipour, S., 2007, Developing Consistent Inlet Boundary Conditions to Study the Entrance Zone in Microchannels., Journal of Thermophysics and Heat Transfer, Vol. **21** (3) , pp. 596-607.

[5] Vakilipour, S., Darbandi, M., 2008, Advancement in numerical study of gas flow and heat transfer in microchannels, Accepted in Journal of Thermophysics and Heat Transfer.

[6] Maurer, J., Tabelin, P., Joseph, P., Willaime, H., 2003, Second-order slip laws in microchannels for helium and nitrogen. Phys. Fluids 15, 2613–2621.

[7] Shen, C., 2005, Rarefied Gas Dynamics; Fundamentals, Simulations and Micro Flows, Springer-Verlag Berlin Heidelberg.