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Inference for Weibull distribution based on progressively Type-II hybrid censored data

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ABSTRACT

Progressive Type-II hybrid censoring is a mixture of progressive Type-II and hybrid censoring schemes. In this paper, we discuss the statistical inference on Weibull parameters when the observed data are progressively Type-II hybrid censored. We derive the maximum likelihood estimators (MLEs) and the approximate maximum likelihood estimators (AMLEs) of the Weibull parameters. We then use the asymptotic distributions of the maximum likelihood estimators to construct approximate confidence intervals. Bayes estimates and the corresponding highest posterior density credible intervals of the unknown parameters are obtained under suitable priors on the unknown parameters and also by using the Gibbs sampling procedure. Monte Carlo simulations are then performed for comparing the confidence intervals based on all those different methods. Finally, one data set is analyzed for illustrative purposes.

1. Introduction

Censoring occurs commonly in reliability and survival analysis. The most common censoring schemes are Type-I censoring, where the life-testing experiment stops at a predetermined time T; and Type-II censoring, where the life-testing experiment stops when predetermined number (r) are observed to have failed. The mixture of Type-I and Type-II censoring schemes is known as the hybrid censoring scheme which was first introduced by Epstein (1954). But recently it becomes quite popular in the reliability and life-testing experiments. See for example the work of Chen and Bhattacharya (1988), Childs et al. (2003), Draper and Guttman (1987), Fairbanks et al. (1982), Gupta and Kundu (2006) and Jeong et al. (1996).

One of the drawbacks of the traditional Type-I, Type-II or hybrid censoring schemes is that they do not have the flexibility of allowing removal of units at points other than the terminal points of the experiment. One censoring scheme known as Type-II progressive censoring scheme, which has this advantage can be described as follows: consider n units in a study and suppose $m<n$ is fixed before the experiment. At the time of the first failure, R_1 units are randomly removed. Similarly, at the time of the second failure, R_2 units from the remaining $n-2-R_1$, units are randomly removed. The test

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continues until the m-th failure at which time all the remaining $R_m = n-m-R_1-R_2-\cdots-R_{m-1}$ units are removed. The R_i's are fixed prior to the study. For more details the readers may refer to the Balakrishnan and Aggarwala (2000) and Balakrishnan (2007).

The drawback of the Type-II progressive censoring, similar to the conventional Type-II censoring (time censoring), is that it can take a lot of time to get to the m-th failure time. Recently, Kundu and Joarder (2006) proposed a censoring scheme called Type-II progressively hybrid censoring scheme which is a mixture of Type-II progressive and hybrid censoring schemes. It can be described as follows. Consider n identical items which put on a test. Each unit in a randomly selected sample is subjected under identical environmental conditions. The lifetimes of the sample units are independent and identically distributed (i.i.d) random variables. The integer $m<n$ and $R_1,\ldots,R_m$ satisfying $R_1+\cdots+R_m+m=n$ are fixed at the beginning of the experiment. The time point T is also fixed before hand. At the time of first failure, $Y_{1:m:n}$, R_1 of the remaining units are randomly removed. Similarly at the time of the second failure, $Y_{2:m:n}$, R_2 of the remaining units are removed and so on. If the m-th failure, $Y_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $Y_{m:m:n}$. On the other hand if the m-th failure does not occur before time point T and only J failure occur before the time point T, where $(0 \le J < m)$, then at the time point T all the remaining R_J^* units are removed and the experiment terminates at the time point T. Note that $R_J^* = n-(R_1-\cdots-R_J)-J$.

We denote these two cases as Case I and Case II, respectively, and call these censoring scheme as the progressively Type-II hybrid censoring schemes. Therefore, in the presence of progressively Type-II hybrid censoring schemes, we have one of the following types of observations;

$$\text{Case I}: \{X_{1:m:n},\ldots,X_{m:m:n}\} \quad \text{if } X_{m:m:n} < T,$$

$$\text{Case II}: \{X_{1:m:n},\ldots,X_{J:m:n}\} \quad \text{if } X_{J:m:n} < T < X_{J+1:m:n}.$$

In spite of the applicability of the hybrid censoring scheme, it is somewhat surprising to observe that limited attention has been paid in analyzing hybrid censored lifetime data when the lifetimes are not exponential. The main concern is the analysis becomes too difficult and may not be tractable. Weibull distribution is one of the most common distributions which is used to analyze several lifetime data. The density function of the Weibull distribution can take different shapes and also its hazard function can be increasing, decreasing and constant depending on the shape parameter. Because of the shape parameter, it has lots of flexibility compared to exponential distribution. Kundu (2007) presented the statistical inference on Weibull parameters when the data are hybrid censored, while Balakrishnan and Kateri (2008) discussed the existence and uniqueness of the maximum likelihood estimators of the Weibull parameters based on different forms of censored data.

In this article we consider the Type-II progressively hybrid censored lifetime data, when the lifetime follows two parameter Weibull distribution. First we provide the maximum likelihood estimators of the unknown parameters. It is observed that the maximum likelihood estimators do not have explicit forms. They can be obtained by solving a non-linear equation. We also suggest approximate maximum likelihood estimators, which have explicit expressions. Because the exact distributions of the MLE are not easy derived, we propose to use the asymptotic distributions of the MLE to construct the approximate confidence intervals.

We also consider the Bayes estimates under the assumptions of independent gamma priors on the scale and shape parameters. Based on these priors, the Bayes estimates cannot be obtained explicitly, so we use the Gibbs sampling procedure to compute the Bayes estimates and also to compute the highest posterior density (HPD) credible intervals. We compare the performances of the different methods by Monte Carlo simulations. One data set is analyzed for illustrative purposes.

The rest of the paper as follows. In Section 2, we describe the model and the available data. The maximum likelihood estimator (MLE) and approximate maximum likelihood estimator (AMLE) are presented in Sections 3 and 4, respectively. Bayesian inferences are provided in Section 5. Simulation results and data analysis are provided in Section 6. One data set is analyzed and the results are presented in Section 7. Finally we conclude the paper on Section 8.

2. Model description

Suppose the lifetime random variable Y has a Weibull distribution with shape and scale parameters as α and λ, respectively, and with probability density function (pdf) as

$$f_Y(y;\alpha,\lambda) = \alpha\lambda y^{\alpha-1} e^{-\lambda y^{\alpha}}, \quad y>0,\ \alpha>0,\ \lambda>0. \tag{1}$$

If the random variable Y has its pdf as in (1), then $X=\ln Y$ has the extreme value distribution with pdf

$$f_X(x;\mu,\sigma) = \frac{1}{\sigma} e^{(x-\mu)/\sigma - e^{(x-\mu)/\sigma}}, \quad -\infty < x < \infty, \tag{2}$$

where $\mu = -\ln\lambda/\sigma$ and $\sigma = 1/\alpha$ are the location and shape parameters, respectively.

The densities in (1) and (2) are equivalent models in the sense that procedures developed under one model can be easily used for the other model. Although, they are equivalent models, sometimes it is easier to work with the extreme value model in (2) than the Weibull model in (1), since the two parameters appear as location and scale parameters.

In fact, it is due to this the approximate maximum likelihood estimators can be obtained quite easily for model (2) than for model (1). When $\mu=0$ and $\sigma=1$, the model in (2) becomes the standard extreme value distribution with pdf:

$$f_Z(z,0,1)=e^{z-e^z},\quad -\infty<z<\infty. \tag{3}$$

3. Maximum likelihood estimators

In this section, we provide the MLEs of the unknown parameters α and λ in (1). Based on the observed data, the likelihood function for Case I is given by

$$L(\alpha,\lambda)=C(\alpha\lambda)^m\prod_{i=1}^{m}(y_{i:m:n})^{\alpha-1}e^{-\lambda\sum_{i=1}^{m}y_{i:m:n}^{\alpha}(1+R_i)}, \tag{4}$$

while for Case II, it is

$$L(\alpha,\lambda)=C(\alpha\lambda)^J\prod_{i=1}^{J}(y_{i:m:n})^{\alpha-1}e^{-\lambda\sum_{i=1}^{J}y_{i:m:n}^{\alpha}(1+R_i)-\lambda T^{\alpha}R_J^*}\quad \text{if } J>0,$$

$$L(\alpha,\lambda)=Ce^{-\lambda T^{\alpha}n}\quad \text{if } J=0. \tag{5}$$

The correspond log-likelihood functions are obtained from (4) and (5) as

$$l(\alpha,\lambda)=\log L(\alpha,\lambda)=const.+m\ln\alpha+m\ln\lambda+(\alpha-1)\sum_{i=1}^{m}\ln y_{i:m:n}-\lambda\sum_{i=1}^{m}y_{i:m:n}^{\alpha}(1+R_i) \tag{6}$$

and

$$l(\alpha,\lambda)=\log L(\alpha,\lambda)=const.+J\ln\alpha+J\ln\lambda+(\alpha-1)\sum_{i=1}^{J}\ln y_{i:m:n}-\lambda\sum_{i=1}^{J}y_{i:m:n}^{\alpha}(1+R_i)-\lambda T^{\alpha}R_J^*, \tag{7}$$

respectively. Taking derivatives with respect to α and λ in (6) and equating them to zero, we obtain the likelihood equations as

$$\frac{\partial l}{\partial\lambda}=\frac{m}{\lambda}-\sum_{i=1}^{m}y_{i:m:n}^{\alpha}(1+R_i)=0, \tag{8}$$

$$\frac{\partial l}{\partial\alpha}=\frac{m}{\alpha}+\sum_{i=1}^{m}\ln y_{i:m:n}-\lambda\sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}=0. \tag{9}$$

From (8), we obtain

$$\hat{\lambda}(\alpha)=\frac{m}{\sum_{i=1}^{m}y_{i:m:n}^{\alpha}(1+R_i)}. \tag{10}$$

Upon using (10) in (9), it becomes

$$\frac{m}{\alpha}+\sum_{i=1}^{m}\ln y_{i:m:n}-\hat{\lambda}(\alpha)\sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}=0. \tag{11}$$

Note that (11) can be written in the form

$$\alpha=h(\alpha), \tag{12}$$

where

$$h(\alpha)=\frac{m}{\hat{\lambda}(\alpha)\sum_{i=1}^{m}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n}-\sum_{i=1}^{m}\ln y_{i:m:n}}. \tag{13}$$

From (12), we propose a simple iterative scheme to solve for α. It has been proposed in the literature by Kundu (2007), Banerjee and Kundu (2008) and Pareek et al. (2009). Start with an initial guess of α, say $\alpha^{(0)}$, then obtain $\alpha^{(1)}=h(\alpha^{(0)})$ and proceed in this way iteratively to obtain $\alpha^{(n+1)}=h(\alpha^{(n)})$. Stop the iterative procedure, when $|\alpha^{(n+1)}-\alpha^{(n)}|<\epsilon$, some pre-assigned tolerance limit.

In case of (7), the likelihood equation can be expressed as

$$\hat{\lambda}(\alpha)=\frac{J}{\sum_{i=1}^{J}y_{i:m:n}^{\alpha}(1+R_i)+T^{\alpha}R_J^*}$$

and $\alpha = h(\alpha)$, where

$$h(\alpha) = \frac{J}{\hat{\lambda}(\alpha)\sum_{i=1}^{J}(1+R_i)y_{i:m:n}^{\alpha}\ln y_{i:m:n} + \hat{\lambda}(\alpha)R_J^* T^{\alpha}\ln T - \sum_{i=1}^{J}\ln y_{i:m:n}}.$$

A simple iterative procedure to the one described above can be used to solve for α in this case as well. Note that when $J=0$, the maximum likelihood estimator for λ is zero and for α do not exist. Since the MLEs do not exist in explicit forms, we suggest the following approximate MLEs which have explicit expressions. This method proposed by Balakrishnan and Varadan (1991) (see also, Balakrishnan et al., 2003, 2004; Banerjee and Kundu, 2008) to develop AMLEs.

4. Approximate maximum likelihood estimators

We use the following notation; $x_{i:m:n} = \ln y_{i:m:n}$ and $S = \ln T$. Then, the likelihood function of the observed data $x_{i:m:n}$ for Case I is given by

$$L(\mu,\sigma) = Const.\left(\frac{1}{\sigma}\right)^m \prod_{i=1}^{m} g(z_{i:m:n})(\overline{G}(z_{i:m:n}))^{R_i}, \tag{14}$$

while for Case II, it is given by

$$L(\mu,\sigma) = Const.\left(\frac{1}{\sigma}\right)^J \prod_{i=1}^{J} g(z_{i:m:n})(1-G(z_{i:m:n}))^{R_i}(1-G(V))R_J^*, \tag{15}$$

where $g(x) = e^{x-e^x}$, $\overline{G}(x) = e^{-e^x}$, $z_{i:m:n} = (x_{i:m:n}-\mu)/\sigma$, $V = (S-\mu)/\sigma$, $\mu = -(1/\alpha)\ln\lambda$, $\sigma = 1/\alpha$. In the case of (14), ignoring the constant, we have the log-likelihood as

$$l(\mu,\sigma) = \log L(\mu,\sigma) = -m\ln\sigma + \sum_{i=1}^{m}\ln g(z_{i:m:n}) + \sum_{i=1}^{m} R_i \ln(1-G(z_{i:m:n})). \tag{16}$$

Taking now derivatives with respect to μ and σ, we get the likelihood equations as

$$\frac{\partial l}{\partial\mu} = -\frac{1}{\sigma}\sum_{i=1}^{m}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + \frac{1}{\sigma}\sum_{i=1}^{m} R_i \frac{g(z_{i:m:n})}{1-G(z_{i:m:n})} = 0, \tag{17}$$

$$\frac{\partial l}{\partial\sigma} = -\frac{m}{\sigma} - \frac{1}{\sigma}\sum_{i=1}^{m} z_{i:m:n}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + \sum_{i=1}^{m} R_i \frac{g(z_{i:m:n})}{1-G(z_{i:m:n})}\frac{z_{i:m:n}}{\sigma} = 0. \tag{18}$$

Eqs. (17) and (18) can be written equivalently as

$$-\sum_{i=1}^{m}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + \sum_{i=1}^{m} R_i \frac{g(z_{i:m:n})}{1-G(z_{i:m:n})} = 0, \tag{19}$$

$$-m - \sum_{i=1}^{m} z_{i:m:n}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})} + \sum_{i=1}^{m} R_i z_{i:m:n}\frac{g(z_{i:m:n})}{i-G(z_{i:m:n})} = 0. \tag{20}$$

Clearly, (19) and (20) do not have explicit solutions. So, we expand the functions $g'(z_{i:m:n})/g(z_{i:m:n})$ and $g(z_{i:m:n})/\overline{G}(z_{i:m:n})$ in Taylor series around the point μ_i, where

$$\mu_i = G^{-1}(p_i) = \ln(-\ln(1-q_i)), \tag{21}$$

and $p_i = 1-q_i = 1-\prod_{j=m-i+1}^{m}\alpha_j$, also $\alpha_j = (j+\sum_{i=m-j+1}^{m} R_i)/(1+j+\sum_{i=m-j+1}^{m} R_i)$ for $j=1,\ldots m$, see Balakrishnan and Aggarwala (2000).

We may then consider the following approximations:

$$\frac{g'(z_{i:m:n})}{g(z_{i:m:n})} \approx \alpha_i - \beta_i z_{i:m:n}, \tag{22}$$

$$\frac{g(z_{i:m:n})}{\overline{G}(z_{i:m:n})} \approx 1-\alpha_i + \beta_i z_{i:m:n}, \tag{23}$$

where, for $i=1,\ldots,m$,

$$\alpha_i = 1 + \ln q_i[1-\ln(-\ln q_i)], \tag{24}$$

$$\beta_i = \ln q_i. \tag{25}$$

Upon using the approximations in (25) and (26) into (19) and (20), we obtain the approximate likelihood equations as

$$-\sum_{i=1}^{m}(\alpha_i-\beta_i z_{i:m:n})+\sum_{i=1}^{m}R_i(1-\alpha_i+\beta_i z_{i:m:n})=0, \tag{26}$$

$$-m-\sum_{i=1}^{m}(\alpha_i-\beta_i z_{i:m:n})z_{i:m:n}+\sum_{i=1}^{m}z_{i:m:n}R_i(1-\alpha_i+\beta_i z_{i:m:n})=0. \tag{27}$$

From (29), we obtain the AMLE of μ as

$$\hat{\mu}_I=A_I-B_I\hat{\sigma}_I, \tag{28}$$

where

$$A_I=\frac{\sum_{i=1}^{m}x_{i:m:n}\beta_i(1+R_i)}{\sum_{i=1}^{m}\beta_i(1+R_i)}, \tag{29}$$

$$B_I=\frac{\sum_{i=1}^{m}\alpha_i-\sum_{i=1}^{m}R_i(1-\alpha_i)}{\sum_{i=1}^{m}\beta_i(1+R_i)}. \tag{30}$$

From (30), we obtain the approximate likelihood equation for σ as

$$G_I\sigma^2+D_I\sigma-E_I=0, \tag{31}$$

where

$$C_I=m+B_I^2\left[\sum_{i=1}^{m}\beta_i(1+R_i)\right]-B_I^2\left(\sum_{i=1}^{m}\beta_i(1+R_i)\right)=m,$$

$$D_I=\sum_{i=1}^{m}\alpha_i(x_{i:m:n}-A_I)-\sum_{i=1}^{m}R_i(x_{i:m:n}-A_I)-\sum_{i=1}^{m}R_i\alpha_i(A_I-x_{i:m:n}),$$

$$E_I=\sum_{i=1}^{m}\beta_i R_i(x_{i:m:n}-A_I)^2+\sum_{i=1}^{m}\beta_i(x_{i:m:n}-A_I)^2>0.$$

Therefore, we obtain the AMLE of σ from (34) as

$$\hat{\sigma}_I=\frac{-D_I+\sqrt{D_I^2+4mE_I}}{2m},$$

which is the only positive root.

Similarly, for Case II, expanding $g(V)/(1-G(V))$ around the point μ_{d^*}, we obtain

$$\frac{g(V)}{1-G(V)}=1-\alpha_{d^*}+\beta_{d^*}V.$$

Here, $\alpha_{d^*}=1+\ln q_{d^*}(1-\ln(-\ln q_{d^*}))$, $\beta_{d^*}=-\ln q_{d^*}$ and $\mu_{d^*}=\ln(-\ln q_{d^*})$. Following the same procedure as above, we obtain the AMLE of μ as

$$\hat{\mu}_{II}=A_{II}-B_{II}\hat{\sigma}_{II},$$

where

$$A_{II}=\frac{\sum_{i=1}^{J}x_{i:m:n}\beta_i(1+R_i)+SR_J^*\beta_{d^*}}{R_J^*\beta_{d^*}+\sum_{i=1}^{J}\beta_i(1+R_i)},$$

$$B_{II}=\frac{\sum_{i=1}^{J}\alpha_i-\sum_{i=1}^{J}R_i(1-\alpha_i)-R_J^*(1-\alpha_i)}{R_J^*\beta_{d^*}+\sum_{i=1}^{J}\beta_i(1+R_i)}.$$

Moreover, $\hat{\sigma}_{II}$ can be obtained as

$$\hat{\sigma}_{II}=\frac{-D_{II}+\sqrt{D_{II}^2+4JE_{II}}}{2J},$$

where

$$E_{II}=-\sum_{i=1}^{J}\beta_i(x_{i:m:n}-A_{II})^2+\sum_{i=1}^{J}\beta_i R_i(x_{i:m:n}-A_{II})^2+R_J^*\beta_{d^*}(S-A_{II})^2>0,$$

$$D_{II} = \sum_{i=1}^{J} \alpha_i x_{i:m:n} - A_{II} b_{II} \left(R_J^* \beta_{d^*} + \sum_{i=1}^{J} \beta_i (1+R_i) \right) - \sum_{i=1}^{J} x_{i:m:n} R_i (1-\alpha_i) - S R_J^* (1-\alpha_{d^*}).$$

5. Bayes estimates and credible intervals

In this section, we describe the Bayes estimates of the unknown parameters as well as the corresponding highest posterior density (*HPD*) credible intervals when the shape parameter is unknown. For computing the Bayes estimates, we have assumed squared error loss functions, but any other loss function (SEL) can be easily incorporated. This method has also been used by Kundu (2007, 2008) and Banerjee and Kundu (2008).

5.1. Prior and posterior distributions

In this subsection, we need to assume some prior distributions for the unknown parameters. Following the approach of Berger and Sun (1993), we assume that λ has a gamma prior, Gamma(a,b), with $a>0$, and $b>0$, while we do not assume any specific form for the prior of α, say $\pi_2(\alpha)$. We only assume that $\pi_2(\alpha)$ has the support on $(0,+\infty)$, and is independent of the prior of λ. Using then the joint prior distribution of α and λ, we obtain the joint density of the data, α and λ for the two cases as follows.

Case I:

$$L(\alpha,\lambda,data) \propto \alpha^m \lambda^{m+a-1} \pi_2(\alpha) \prod_{i=1}^{m} y_{i:m:n}^{\alpha-1} e^{-\lambda[\sum_{i=1}^{m} y_{i:m:n}^{\alpha}(1+R_i)+b]}. \tag{32}$$

Case II:

$$L(\alpha,\lambda,data) \propto \alpha^J \lambda^{J+a-1} \pi_2(\alpha) \prod_{i=1}^{J} y_{i:m:n}^{\alpha-1} e^{-\lambda[\sum_{i=1}^{J} y_{i:m:n}^{\alpha}(1+R_i)+b+T^{\alpha}R_J^*]}. \tag{33}$$

Based on $L(\alpha,\lambda,data)$, the joint posterior density function of α and λ, given the data, is given by

$$L(\alpha,\lambda|data) = \frac{L(data|\alpha,\lambda) \times \pi_1(\lambda|a,b) \times \pi_2(\alpha)}{\int_0^\infty \int_0^\infty L(data|\alpha,\lambda) \times \pi_1(\lambda|a,b) \times \pi_2(\alpha)\, d\alpha\, d\lambda}. \tag{34}$$

Therefore, if $g(\alpha,\lambda)$ is any function of α and λ, its Bayes estimate under the squared error loss function is given by

$$\hat{g}(\alpha,\lambda) = E_{\alpha,\lambda|data}(g(\alpha,\lambda)) = \frac{0^\infty \int_0^\infty g(\alpha,\lambda) L(data,\alpha,\lambda)\, d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty L(data,\alpha,\lambda)\, d\alpha\, d\lambda}. \tag{35}$$

Since, it is not possible to compute (34) and therefore (35) analytically even when $\pi_2(\alpha)$ is known explicitly, we adopt two different procedures to approximate (35): (a) Lindley's approximation and (b) Gibbs sampling procedure.

Although we can compute the approximate Bayes estimates of α and λ using Lindley's approximation, it is not possible to compute the credible interval from here. Therefore, we propose the following Markov Chain Monte Carlo (MCMC) method to draw samples from the posterior density function and then to compute the Bayes estimates and HPD credible intervals.

5.2. Gibbs sampling

We use the Gibbs sampling procedure to generate a sample from the posterior density function $L(\alpha,\lambda|data)$ and then to compute the Bayes estimates and HPD credible intervals.

We assume that the prior $\pi_1(\lambda|a,b)$ is Gamma(a,b) and the prior $\pi_2(\alpha)$ is log-concave, and that they are independent. We then have the following results.

Theorem 1. *The conditional density of λ, given α and the data, is Gamma $(a+m, \sum_{i=1}^{m} y_{i:m:n}^{\alpha}(1+R_i)+b)$ for Case I, and is Gamma $(a+J, \sum_{i=1}^{J} y_{i:m:n}^{\alpha}(1+R_i)+T^{\alpha}R_J^*+b)$ for Case II.*

Proof. It follows readily from the joint density function. □

Theorem 2. *The conditional density of α, given data, is log-concave.*

Proof. The proof is presented in Appendix. □

Using the idea of Geman and Geman (1984) and Theorems 1 and 2, it is possible to generate (α,λ) from the posterior density function and in turn obtain the Bayes estimates and credible intervals. We use the following algorithm for this purpose.

Algorithm.

Step 1: Generate α_1 from the log-concave density $L(\cdot|data)$, as given in (36) and (37) depending on the case, by using the method proposed by Devroye (1984).

Step 2: Generate λ_1 from $\pi_1(\cdot|\alpha_1,data)$ as given in Theorem 1.

Step 3: Repeat Steps 1 and 2, M times, and obtain α_i and λ_i for $i=1,\ldots,M$.

Step 4: The Bayes estimates of α and λ with respect to the squared error loss function are then

$$\hat{\alpha}_{Bayes} = \hat{E}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}\alpha_k$$

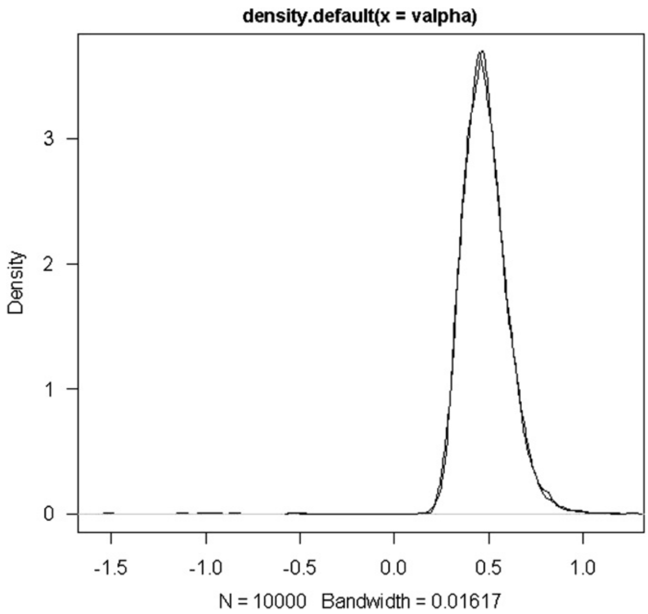

Fig. 1. Comparison of the distribution of $\hat{\alpha}/\alpha$ when $\alpha=1$ and 2 for $T=1$ in MLE.

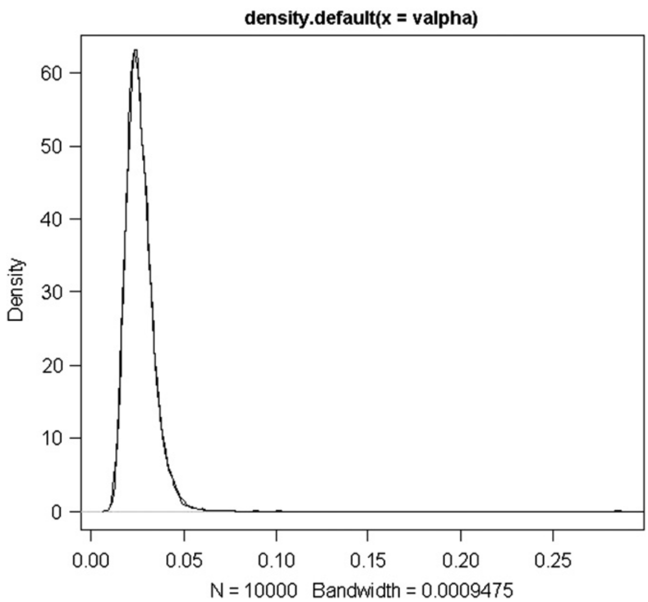

Fig. 2. Comparison of the distribution of $\hat{\alpha}/\alpha$ when $\alpha=1$ and 2 for $T=1$ in AMLE.

and

$$\hat{\lambda}_{Bayes} = \hat{E}(\lambda|data) = \frac{1}{M}\sum_{k=1}^{M}\lambda_k.$$

Step 5: The posterior variances of λ and α are

$$\hat{V}(\alpha|data) = \frac{1}{M}\sum_{k=1}^{M}\{\alpha_k - \hat{E}(\alpha|data)\}^2$$

and

$$\hat{V}(\lambda|data) = \frac{1}{M}\sum_{k=1}^{M}\{\lambda_k - \hat{E}(\lambda|data)\}^2.$$

Table 1
Bias of the AMLEs for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	Schemes	$T=0.5$		$T=1$		$T=2$	
		$(R_1,\ldots,R_m)$	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.62	0.25	0.6	0.26	0.64	0.25
	5	25,0,0,0,0	−0.03	−0.21	−0.31	−0.29	−0.11	−0.34
	5	0,25,0,0,0	−0.04	−0.23	−0.28	−0.61	−0.32	−0.54
50	10	0,0,0, …,0,40	0.23	0.26	0.22	0.27	0.23	0.27
	10	40,0,0,0, …,0	−0.07	−0.23	−0.11	−0.73	−0.31	−0.35
	10	0,40,0, … 0,0,0	−0.1	−0.25	−0.27	−0.61	−0.39	−0.54
100	10	0,0,0, …,0,90	0.23	0.25	0.23	0.23	0.23	0.25
	10	90,0,0, …,0,0	−0.07	−0.24	−0.21	−0.6	−0.49	−0.23
	10	0,90,0,0, …,0,0	−0.07	−0.25	−0.23	−0.60	−0.35	−0.74
	20	0,0,0, …,0,0,0,80	0.1	0.22	0.1	0.23	0.1	0.23
	20	80,0,0,0, …,0,0,0	−0.1	−0.25	−0.23	−0.61	−0.002	−0.16
	20	0,80,0,0, …,0,0	−0.1	−0.26	−0.25	−0.60	−0.21	−0.38
	50	0,0,0, …,0,0,0,50	0.21	0.55	0.036	0.05	0.036	0.045
	50	50,0,0,0, …,0,0,0	−0.2	−0.31	0.59	−0.78	0.72	−0.19
	50	0,50,0,0, …,0,0	−0.23	−0.75	0.47	−0.75	0.55	−0.14

Table 2
MSEs of the AMLEs for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	Schemes	$T=0.5$		$T=1$		$T=2$	
		$(R_1,\ldots,R_m)$	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	1.62	0.52	1.5	0.55	1.73	0.53
	5	25,0,0,0,0	0.17	0.1	0.15	0.53	0.11	0.31
	5	0,25,0,0,0	0.14	0.1	0.12	0.38	0.17	0.51
50	10	0,0,0, …,0,40	0.27	0.45	0.22	0.44	0.25	0.44
	10	40,0,0,0, …,0	0.1	0.09	0.06	0.57	0.31	0.32
	10	0,40,0, … 0,0,0	0.07	0.09	0.1	0.37	0.3	0.48
100	10	0,0,0, …,0,90	0.26	0.5	0.27	0.49	0.26	0.51
	10	90,0,0, …,0,0	0.07	0.08	0.07	0.36	0.38	0.84
	10	0,90,0,0, …,0,0	0.04	0.07	0.07	0.37	0.14	0.56
	20	0,0,0, …,0,0,0,80	0.07	0.31	0.08	0.32	0.08	0.32
	20	80,0,0,0, …,0,0,0	0.07	0.08	0.08	0.37	0.23	0.6
	20	0,80,0,0, …,0,0	0.05	0.08	0.08	0.36	0.30	0.58
	50	0,0,0, …,0,0,0,50	0.08	0.36	0.02	0.04	0.02	0.03
	50	50,0,0,0, …,0,0,0	0.09	0.1	0.48	0.71	0.61	0.06
	50	0,50,0,0, …,0,0	0.08	0.11	0.31	0.64	0.36	0.05

Step 6: Order $\alpha_1,\ldots,\alpha_M$ and $\lambda_1,\ldots,\lambda_M$ as $\alpha_{(1)}<\cdots<\alpha_{(M)}$ and $\lambda_{(1)},\ldots,\lambda_{(M)}$ to compute the credible intervals of α and λ. Then, the $100(1-2\beta)\%$ credible intervals for α and λ become

$$(\alpha_{(M\beta)},\alpha_{(M(1-\beta))})$$

and

$$(\lambda_{(M\beta)},\lambda_{(M(1-\beta))}).$$

Based on the generated M, α and λ values and using the method proposed by Chen and Shao (1999), the approximate HPD credible intervals of α and λ can then be constructed.

6. Simulations and data analysis

6.1. Simulations

In this section, we perform some simulations to evaluate the performance of all the methods we used earlier for different sample sizes, different sampling schemes, different parameter values, in terms of their bias and mean squared

Table 3
Bias of the MLEs for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		(R_1,…,R_m)	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.62	0.25	0.62	0.27	0.63	−0.26
	5	25,0,0,0,0	−0.31	−0.78	4.46	−0.52	3.42	−0.82
	5	0,25,0,0,0	−0.15	−0.48	−0.29	−0.7	−0.57	−0.82
50	10	0,0,0,…,0,40	0.24	0.28	0.23	0.27	0.23	0.28
	10	40,0,0,0,…,0	−0.48	−0.9	0.74	−0.73	0.22	−0.77
	10	0,40,0,…0,0,0	−0.4	−0.81	−0.51	−0.88	0.39	−0.85
100	10	0,0,0,…,0,90	0.24	0.26	0.23	0.26	0.24	0.25
	10	90,0,0,…,0,0	0.001	−0.8	−0.14	−0.9	−0.97	−0.94
	10	0,90,0,0,…,0,0	−0.21	−0.73	−0.3	−0.85	−0.41	−0.91
	20	0,0,0,…,0,0,0,80	0.1	0.23	0.1	0.22	0.1	0.23
	20	80,0,0,0,…,0,0,0	−0.58	−0.95	−0.64	−0.96	0.24	−0.84
	20	0,80,0,0,…,0,0	−0.53	−0.92	−0.60	−0.94	0.14	−0.83
	50	0,0,0,…,0,0,0,50	0.02	0.03	0.037	0.04	0.036	0.05
	50	50,0,0,0,…,0,0,0	−0.68	−0.86	0.88	−0.69	1.05	−0.67
	50	0,50,0,0,…,0,0	−0.65	−0.96	0.26	−0.7	0.74	−0.61

Table 4
MSEs of the MLEs for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		(R_1,…,R_m)	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	1.61	0.53	1.66	0.55	1.63	0.55
	5	25,0,0,0,0	0.19	0.62	0.26	0.29	1.28	0.72
	5	0,25,0,0,0	0.12	0.3	0.14	0.49	1.04	0.67
50	10	0,0,0,…,0,40	0.28	0.45	0.27	0.45	0.26	0.44
	10	40,0,0,0,…,0	0.26	0.82	0.72	0.54	0.20	0.61
	10	0,40,0,…0,0,0	0.19	0.67	0.28	0.77	1.18	0.74
100	10	0,0,0,…,0,90	0.27	0.51	0.27	0.52	0.27	0.50
	10	90,0,0,…,0,0	0.07	0.64	0.06	0.81	1.5	0.88
	10	0,90,0,0,…,0,0	0.16	0.57	0.17	0.88	0.30	0.70
	20	0,0,0,…,0,0,0,80	0.08	0.31	0.08	0.31	0.08	0.32
	20	80,0,0,0,…,0,0,0	0.35	0.92	0.42	0.94	0.2	0.70
	20	0,80,0,0,…,0,0	0.29	0.84	0.37	0.88	0.22	0.70
	50	0,0,0,…,0,0,0,50	0.02	0.05	0.02	0.03	0.02	0.04
	50	50,0,0,0,…,0,0,0	0.48	0.76	1.06	0.50	1.17	0.45
	50	0,50,0,0,…,0,0	0.43	0.92	0.21	0.51	0.61	0.38

error (MSE). We also compare the average width of the asymptotic confidence intervals and credible intervals and their coverage percentages.

All the programs are written in *R*. Since λ is a scale parameter, we have taken in all cases $\lambda=1$ without loss of generality. For simulation purposes, we present the results when T is of the form $T^{1/\alpha}$. The reason to choose T in that form is the following; if $\hat{\alpha}$ represents the MLE or AMLE of α, then by simulation we show that the distribution of $\hat{\alpha}/\alpha$ becomes independent of α in that case for $\lambda=1$ (Figs. 1 and 2). For that purpose we report the result only for $\alpha=1$ without loss of generality. But these results can be used for any other α also.

We have used three different sampling schemes, as follows:

Scheme 1: $R_1=\cdots=R_{m-1}=0$ and $R_m=n\text{-}m$;
Scheme 2: $R_1=n-m$ and $R_2=\cdots=R_m=0$;
Scheme 3: $R_1=0$, $R_2=n-m$ and $R_3=\cdots=R_m=0$.

We have also used different n, m and T. In each case, we compute the MLEs, AMLEs and the Bayes estimates of the unknown parameters. For computing the Bayes estimators, we assumed that α and λ have Gamma (a,b) and Gamma (γ,δ) priors, respectively. Moreover, we used the non-informative priors for both α and λ, by considering $a=b=\gamma=\delta=0$. For comparison

Table 5
Bias of the Bayes estimators for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		(R_1,…,R_m)	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.48	1.83	0.49	1.81	0.49	1.60
	5	25,0,0,0,0	0.15	0.93	−0.04	−0.03	−0.05	−0.54
	5	0,25,0,0,0	0.13	−0.42	−0.59	−0.68	−0.16	−0.76
50	10	0,0,0,…,0,40	0.36	1.01	0.32	0.80	0.32	0.73
	10	40,0,0,0,…,0	−0.03	−0.14	−0.13	−0.53	0.13	−0.78
	10	0,40,0,…0,0,0	−0.10	−0.75	−0.21	−0.85	−0.13	−0.81
100	10	0,0,0,…,0,90	0.32	1.30	0.36	1.41	0.37	1.50
	10	90,0,0,…,0,0	−0.11	−0.25	−0.18	−0.56	−0.24	−0.73
	10	0,90,0,0,…,0,0	−0.16	−0.80	−0.26	−0.87	−0.29	−0.91
	20	0,0,0,…,0,0,0,80	0.27	0.56	0.23	0.51	0.24	0.49
	20	80,0,0,0,…,0,0,0	−0.14	−0.60	−0.19	−0.77	0.34	−0.84
	20	0,80,0,0,…,0,0	−0.23	−0.89	−0.31	−0.99	−0.31	−0.92
	50	0,0,0,…,0,0,0,50	0.15	0.12	0.14	0.11	0.17	0.12
	50	50,0,0,0,…,0,0,0	−0.16	−0.83	0.66	−0.71	0.68	−0.57
	50	0,50,0,0,…,0,0	−0.28	−0.95	0.40	−0.72	0.63	−0.56

Table 6
MSEs of the Bayes estimators for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		(R_1,…,R_m)	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.34	0.79	0.34	0.59	0.33	0.36
	5	25,0,0,0,0	0.11	0.93	0.03	0.007	0.05	0.33
	5	0,25,0,0,0	0.12	0.19	0.04	0.47	0.05	0.59
50	10	0,0,0,…,0,40	0.20	1.97	0.18	1.81	0.19	1.33
	10	40,0,0,0,…,0	0.04	0.03	0.03	0.03	0.11	0.61
	10	0,40,0,…0,0,0	0.05	0.57	0.05	0.73	0.06	0.66
100	10	0,0,0,…,0,90	0.21	4.03	0.21	4.56	0.21	6.40
	10	90,0,0,…,0,0	0.03	0.06	0.04	0.32	0.06	0.54
	10	0,90,0,0,…,0,0	0.04	0.64	0.07	0.76	0.09	0.83
	20	0,0,0,…,0,0,0,80	0.13	0.72	0.10	0.65	0.11	0.7
	20	80,0,0,0,…,0,0,0	0.04	0.36	0.04	0.59	0.24	0.72
	20	0,80,0,0,…,0,0	0.06	0.79	0.10	0.86	0.10	0.86
	50	0,0,0,…,0,0,0,50	0.04	0.06	0.04	0.06	0.55	0.04
	50	50,0,0,0,…,0,0,0	0.04	0.69	0.47	0.51	0.47	0.33
	50	0,50,0,0,…,0,0	0.08	0.90	0.23	0.53	0.40	0.32

purposes, we have considered informative priors also. For example we have tried: $a=b=\gamma=\delta=0.0001$. The Bayes estimators were computed under the squared error loss function. It is observed that if we have proper priors information then the Bayesian inferences are not significantly different than corresponding inferences obtained using non-proper priors.

For comparison purpose, we replicated the process 10,000 times and report the Bias, MSEs, the average confidence/credible width and coverage percentage for $(D>1)$.

The average bias of the MLEs, AMLEs and Bayes estimators and the corresponding MSEs are reported in Tables 1–8. The average confidence/credible width and the corresponding coverage percentages are reported in Tables 9–12.

From Tables 1–8, we observed that the $(\hat{\alpha},\hat{\lambda})$ based on the AMLE give smaller biases and MSEs compare to those based on the MLE and Bayesian methods.

In studying the effect of different censoring schemes, we observed that the bias in Scheme 3 is smaller than other two schemes and it is larger in Scheme 1 than Schemes 2 and 3.

When comparing in terms of interval estimation, Bayesian credible interval provides a good balance between the coverage probabilities as well as average credible widths. Therefore, in general we would recommend to use the Bayesian credible interval with non-informative prior if no prior information about the parameters is available. It is observed that if we have proper prior information then the Bayesian inference has a clear advantage over the classical inference in case of interval estimation.

If one wants to guarantee the coverage probability is above the nominal level and the width of the credible interval is not the major concern, then AMLEs confidence interval is proposed in most cases.

Table 7
Bias of the Bayes estimators under proper priors for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		$(R_1,\ldots,R_m)$	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.052	2.23	0.44	1.33	0.42	1.41
	5	25,0,0,0,0	0.08	0.84	−0.05	−0.02	−0.01	−0.57
	5	0,25,0,0,0	0.12	−0.04	−0.59	−0.68	−0.16	−0.78
50	10	0,0,0, …,0,40	0.38	1.14	0.30	0.86	0.31	0.50
	10	40,0,0,0, …,0	0.00002	−0.12	−0.13	−0.53	0.144	−0.80
	10	0,40,0, … 0,0,0	−0.30	−0.76	−0.24	−0.85	−0.10	−0.80
100	10	0,0,0, …,0,90	0.31	1.16	0.37	1.35	0.36	1.36
	10	90,0,0, …,0,0	−0.11	−0.25	−0.19	−0.56	−0.24	−0.73
	10	0,90,0,0, …,0,0	−0.17	−0.80	−0.26	−0.87	−0.29	−0.91
	20	0,0,0, …,0,0,0,80	0.28	0.61	0.28	0.66	0.21	0.39
	20	80,0,0,0, …,0,0,0	−0.13	−0.60	−0.19	−0.76	0.32	−0.86
	20	0,80,0,0, …,0,0	−0.22	−0.89	−0.32	−0.99	−0.30	−0.94
	50	0,0,0, …,0,0,0,50	0.14	0.15	0.13	0.04	0.18	0.11
	50	50,0,0,0, …,0,0,0	−0.15	−0.83	0.69	−0.69	0.70	−0.58
	50	0,50,0,0, …,0,0	−0.28	−0.94	0.46	−0.71	0.64	−0.57

Table 8
MSEs of the Bayes estimators under proper priors for progressively Type-II hybrid censored samples when the sample sizes are n=30, 50, 100.

n	m	Schemes	T=0.5		T=1		T=2	
		$(R_1,\ldots,R_m)$	α	λ	α	λ	α	λ
30	5	0,0,0,0,25	0.35	1.34	0.30	0.19	0.28	0.45
	5	25,0,0,0,0	0.07	0.74	0.03	0.005	0.04	0.35
	5	0,25,0,0,0	0.09	0.21	0.04	0.47	0.05	0.61
50	10	0,0,0, …,0,40	0.18	2.60	0.17	1.10	0.18	0.05
	10	40,0,0,0, …,0	0.04	0.02	0.03	0.28	0.12	0.64
	10	0,40,0, … 0,0,0	0.05	0.58	0.06	0.73	0.07	0.63
100	10	0,0,0, …,0,90	0.18	2.41	0.22	3.33	0.21	4.30
	10	90,0,0, …,0,0	0.04	0.06	0.05	0.31	0.06	0.54
	10	0,90,0,0, …,0,0	0.04	0.64	0.06	0.76	0.09	0.83
	20	0,0,0, …,0,0,0,80	0.12	0.80	0.12	0.99	0.09	0.51
	20	80,0,0,0, …,0,0,0	0.04	0.36	0.04	0.59	0.22	0.74
	20	0,80,0,0, …,0,0	0.05	0.79	0.10	0.86	0.01	0.86
	50	0,0,0, …,0,0,0,50	0.04	0.07	0.04	0.03	0.55	0.04
	50	50,0,0,0, …,0,0,0	0.04	0.69	0.49	0.49	0.47	0.35
	50	0,50,0,0, …,0,0	0.08	0.90	0.27	0.51	0.40	0.33

Table 9
Average confidence width (ACW) and the (coverage percentage (CP)) of AMLEs estimators for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	Schemes ($R_1,\ldots,R_m$)	$T=0.5$ ACWα, ACWλ(CPα,CPλ)	$T=1$ ACWα, ACWλ(CPα,CPλ)	$T=2$ ACWα, ACWλ(CPα,CPλ)
30	5	0,0,0,0,25	3.19, 2.49(0.85,0.87)	3.18, 2.51(0.85,0.86)	3.26, 2.48(0.84,0.86)
	5	25,0,0,0,0	6.72, 5.45(1.00,1.00)	2.5, 2.23(1.00,1.00)	2.67, 1.82(0.99,0.63)
	5	0,25,0,0,0	3.57, 2.86(0.99,1.00)	2.67, 1.43(0.99,1.00)	2.27, 1.41(0.99,0.69)
50	10	0,0,0, . . . ,0,0,40	1.63, 1.69(0.90,0.77)	1.61, 1.68(0.91,0.77)	1.62, 1.68(0.91,0.77)
	10	40,0,0,0, . . . ,0,0	6.41, 5.29(1.00,1.00)	2.89, 0.97(1.00,0.66)	1.53, 1.35(0.75,0.67)
	10	0,40,0, . . . 0,0,0	3.40, 2.79(1.00,1.00)	2.71, 1.46(1.00,0.99)	1.55, 1.03(0.68,0.54)
100	10	0,0,0,0, . . . ,0,90	1.63, 1.65(.90,0.69)	1.63, 1.63(0.90,0.69)	1.63, 1.65(0.90,0.68)
	10	90,0,0, . . . ,0,0,0	6.45, 5.27(1.00,1.00)	2.46, 2.72(1.0,1.00)	1.64, 1.93(0.61,0.40)
	10	0,90,0,0, . . . ,0,0	3.46, 2.8(1.00,1.00)	2.81, 1.46(1.00,1.00)	2.39, 0.94(0.97,0.45)
	20	0,0,0, . . . ,0,0,0,80	0.99, 1.10(0.93,0.72)	1.00, 1.11(0.93,0.71)	0.99, 1.11(0.93,0.71)
	20	80,0,0,0, . . . ,0,0,0	6.28, 5.19(1.0,1.00)	1.32, 1.45(1.0,1.00)	1.83, 1.25(0.85,0.54)
	20	0,80,0,0, . . . ,0,0,0	3.35, 2.76(1.00,1.00)	2.78, 1.46(1.00,0.99)	1.57, 0.95(0.67,0.48)
	50	0,0,0, . . . ,0,0,0,50	0.77, 0.87(0.80,0.10)	0.58, 0.59(0.95,0.90)	0.58, 0.58(0.95,0.90)
	50	50,0,0,0, . . . ,0,0,0	5.49, 4.78(0.99,1.00)	1.89, 0.49(0.64,0.32)	1.14, 0.53(0.14,0.68)
	50	0,50,0,0, . . . ,0,0,0	2.84, 2.51(1.00,1.00)	1.77, 0.747(0.76,0.50)	1.03, 0.56(0.23,0.82)

Table 10
Average confidence width (ACW) and the (coverage percentage (CP)) of MLEs estimators for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	schemes ($R_1,\ldots,R_m$)	$T=0.5$ ACWα, ACWλ(CPα,CPλ)	$T=1$ ACWα, ACWλ(CPα,CPλ)	$T=2$ ACWα, ACWλ(CPα,CPλ)
30	5	0,0,0,0,25	3.23, 2.5(0.85,0.87)	3.23, 2.53(0.85,0.87)	3.23, 2.51(0.85,0.86)
	5	25,0,0,0,0	4.75, 1.47(0.99,1.00)	3.74, 1.76(0.99,1.00)	2.35, 0.43(0.59,0.23)
	5	0,25,0,0,0	3.16, 1.91(0.99,0.92)	2.63, 1.12(0.99,0.70)	2.56, 0.64(0.81,0.13)
50	10	0,0,0, . . . ,0,40	1.64, 1.69(0.90,0.77)	1.63, 1.68(0.90,0.77)	1.63, 1.69(0.90,0.77)
	10	40,0,0,0, . . . ,0,0	3.61, 0.63(0.99,0.004)	6.04, 0.87(1.00,0.27)	3.11, 0.47(0.99,0.011)
	10	0,40,0, . . . 0,0,0	2.22, 0.68(0.99,0.03)	1.80, 0.45(0.99,0.001)	3.23, 0.34(0.79,0.02)
100	10	0,0,0,0, . . . ,0,90	1.64, 1.66(0.90,0.68)	1.63, 1.66(0.90,0.68)	1.63, 1.65(0.90,0.69)
	10	90,0,0, . . . ,0,0,0	6.93, 1.37(1.00,0.97)	5.94, 0.67(1.00,0.00)	1.55, 0.38(0.22,0.00)
	10	0,90,0,0, . . . ,0,0	2.94, 0.98(0.99,0.40)	2.61, 0.53(0.99,0.00)	2.31, 0.338(0.96,0.00)
	20	0,0,0, . . . ,0,0,0,80	1.004, 1.11(0.93,0.71)	0.99, 1.11(0.93,0.71)	1.003, 1.11(0.92,0.71)
	20	80,0,0,0, . . . ,0,0,0	2.89, 0.28(0.99,0.00)	2.47, 0.20(0.99,0.00)	2.16, 0.26(0.92,0.00)
	20	0,80,0,0, . . . ,0,0,0	1.74, 0.29(0.99,0.00)	1.46, 0.21(0.97,0.00)	2.07, 0.26(0.99,0.00)
	50	0,0,0, . . . ,0,0,0,50	0.65, 0.65(0.96,0.87)	0.58, 0.58(0.95,0.90)	0.58, 0.58(0.95,0.90)
	50	50,0,0,0, . . . ,0,0,0	2.14, 0.31(0.98,0.01)	2.17, 0.33(0.38,0.05)	1.38, 0.21(0.002,0.002)
	50	0,50,0,0, . . . ,0,0,0	1.28, 0.16(0.88,0.00)	1.45, 0.32(0.82,0.002)	1.15, 0.25(0.04,0.00)

Based on our final results, totally, all three methods are almost at the same level. In some cases AMLE is proposed as in some other cases MLE or Bayesian methods are proposed.

Now we explain how we can use the results for any other α values also. For example when $\alpha=2$, then for $n=30$, $m=5$, $T=0.5$ and the Scheme is R_1 (Tables 3 and 4), the bias of the MLEs for α will be 2×0.62 and the MSEs will be 4×1.61, the average confidence width will be 2×3.23 and the coverage percentage will be 0.85.

6.2. Data analysis

For illustrative purposes, we present here a data analysis using the proposed methods. The following data set (Linhart and Zucchini, 1986, p. 69) are failure times of the air conditioning system of an airplane. This data set was analyzed by Gupta and Kundu (2001). One question arises whether the data fit Weibull distribution or not. To check for goodness of fit we provide PP plot in Fig. 3. Also we compute the Anderson–Darling statistic, it is 0.552 and the associated p value is 0.159. Since the p value is quite high, we cannot reject the null hypothesis that the data are coming from the Weibull distribution.

We created an artificial data by progressive Type-II hybrid censoring.

The ordered data are as follows: 1, 3, 5, 7, 11, 11, 11, 12, 14, 14, 14, 16, 16, 20, 21, 23, 42, 47, 52, 62, 71, 71, 87, 95, 90, 120, 120, 225, 246, 261.

Table 11
Average confidence width (ACW) and the (coverage percentage (CP)) of the Bayes estimators for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	Schemes (R_1,…,R_m)	$T=0.5$ ACWα, ACWλ(CPα,CPλ)	$T=1$ ACW(α), ACWλ(CPα,CPλ)	$T=2$ ACWα, ACWλ(CPα,CPλ)
30	5	0,0,0,0,25	2.89, 1.70(1.00,0.99)	2.86, 1.52(1.00,.97)	0.49, 1.60(0.33,0.90)
	5	25,0,0,0,0	1.93, 8.38(1.00,0.99)	1.39, 1.22(1.00,1.00)	1.28, 0.43(1.00,0.21)
	5	0,25,0,0,0	1.68, 2.65(0.99,0.96)	1.08, 0.34(1.00,0.00)	0.88, 0.11(0.91,0.00)
50	10	0,0,0, . . . ,0,40	2.18, 1.77(0.99,0.98)	2.04, 3.41(1.00,0.98)	2.08, 2.04(1.00,0.98)
	10	40,0,0,0, . . . ,0	1.42, 2.21(0.99,1.00)	1.16, 0.52(1.00,0.00)	1.67, 0.20(0.99,0.00)
	10	0,40,0, . . . 0,0,0	1.09, 0.52(0.97,0.07)	0.83, 0.10(0.92,0.00)	0.90, 0.08(0.94,0.00)
100	10	0,0,0, . . . ,0,90	2.12, 1.57(0.99,0.95)	2.19, 2.63(0.98,0.97)	2.22, 1.90(1.00,0.98)
	10	90,0,0, . . . ,0,0	1.17, 1.78(0.97,1.00)	0.99, 0.53(0.97,0.00)	0.83, 0.26(0.96,0.00)
	10	0,90,0,0, . . . ,0,0	0.91, 0.43(0.95,0.01)	0.68, 0.11(0.82,0.00)	0.60, 0.04(0.78,0.00)
	20	0,0,0, . . . ,0,0,0,80	1.76, 1.42(0.99,1.00)	0.23, 0.51(0.10,0.65)	1.68, 2.27(0.99,0.98)
	20	80,0,0,0, . . . ,0,0,0	1.14, 0.78(1.00,0.37)	0.99, 0.24(0.97,0.00)	2.26, 0.17(0.99,0.00)
	20	0,80,0,0, . . . ,0,0	0.79, 0.15(0.98,0.00)	0.57, 0.04(0.67,0.32)	0.57, 0.04(0.67,0.00)
	50	0,0,0, . . . ,0,0,0,50	1.32, 0.37(1.00,0.99)	1.27, 0.88(1.00,0.98)	1.37, 0.95(1.00,0.95)
	50	50,0,0,0, . . . ,0,0,0	1.11, 0.28(0.98,0.01)	3.47, 0.07(1.00,0.00)	3.48, 0.47(1.00,0.00)
	50	0,50,0,0, . . . ,0,0	0.71, 0.04(0.9,0.00)	2.29, 0,06(0.99,0.81)	3.14, 0.44(1.00,0.00)

Table 12
Average confidence width (ACW) and the (coverage percentage (CP)) of the Bayes estimators under proper priors for progressively Type-II hybrid censored samples when the sample sizes are $n=30$, 50, 100.

n	m	Schemes (R_1,…,R_m)	$T=0.5$ ACWα, ACWλ(CPα,CPλ)	$T=1$ ACW(α), ACWλ(CPα,CPλ)	$T=2$ ACWα, ACWλ(CPα,CPλ)
30	5	0,0,0,0,25	2.90, 2.06(1.00,0.99)	2.61, 0.35(1.00,0.00)	0.38, 1.08(0.33,0.83)
	5	25,0,0,0,0	1.76, 7.67(1.00,0.96)	1.26, 1.21(0.99,1.00)	1.32, 0.40(1.00,0.21)
	5	0,25,0,0,0	1.65, 2.38(1.00,0.94)	1.07, 0.333(0.98,0.00)	0.88, 0.11(0.95,0.00)
50	10	0,0,0, . . . ,0,40	2.19, 1.53(0.99,0.99)	1.94, 3.97(0.99,0.96)	2.03, 1.48(1.00,0.98)
	10	40,0,0,0, . . . ,0	1.55, 2.53(1.00,1.00)	1.17, 0.52(1.00,0.00)	1.73, 0.20(1.00,0.00)
	10	0,40,0, . . . 0,0,0	1.05, 0.47(0.93,0.07)	0.75, 0.10(0.91,0.00)	0.90, 0.08(0.89,0.00)
100	10	0,0,0, . . . ,0,90	2.05, 1.38(1.00,0.95)	2.19, 2.23(0.98,0.97)	2.17, 1.32(0.98,0.97)
	10	90,0,0, . . . ,0,0	1.18, 1.83(0.99,1.00)	0.96, 0.53(0.97,0.00)	0.84, 0.26(0.97,0.00)
	10	0,90,0,0, . . . ,0,0	0.87, 0.41(0.94,0.03)	0.71, 0.12(0.87,0.00)	0.58, 0.04(0.82,0.00)
	20	0,0,0, . . . ,0,0,0,80	1.77, 1.75(1.00,0.99)	1.67, 1.94(0.99,0.99)	1.60, 1.81(1.00,0.98)
	20	80,0,0,0, . . . ,0,0,0	1.18, 0.79(1.00,0.35)	0.99, 0.24(0.97,0.00)	2.25, 0.15(0.98,0.00)
	20	0,80,0,0, . . . ,0,0	0.80, 0.15(0.97,0.00)	0.56, 0.04(0.60,0.00)	1.20, 0.10(0.90,0.00)
	50	0,0,0, . . . ,0,0,0,50	1.29, 1.37(1.00,0.99)	1.25, 0.86(1.00,0.94)	1.38, 0.95(1.00,0.94)
	50	50,0,0,0, . . . ,0,0,0	1.12, 0.28(1.00,0.00)	3.51, 0.07(1.00,0.00)	3.41, 0.46(1.00,0.00)
	50	0,50,0,0, . . . ,0,0	0.71, 0.04(0.82,0.00)	2.42, 0.06(0.99,0.00)	3.14, 0.44(1.00,0.00)

Example 1. In this case we have $n = 30$ and we took $m = 10$, $T = 80$, $R_1 = R_2 = \cdots = R_9 = 2, R_{10} = 8$. Thus, the progressive Type-II hybrid censored sample is 1, 3, 5, 7, 11, 12, 16, 20, 23, 71. From the above sample, corresponding to Case I, we obtain the MLEs, AMLEs and Bayes estimates of α and λ as (0.711,0.029), (0.712,0.031), (0.829,0.039), respectively. The average confidence widths of α and λ based on the MLEs are (0.938,0.489). Similarly, using the AMLEs, we have (0.940,0.490). Also we compute the average credible length of α and λ and they are (0.720,0.113).

Example 2. Now consider $m = 10$ and $T = 19$ and R_i's to be the same as before. In this case, the progressively Type-II hybrid censored sample obtained 1, 3, 5, 7, 11, 12, 16. It is observed that the data correspond to Case II and $D=J=7$. Based on the sample, we obtain the MLEs, AMLEs and Bayes estimates of α and λ as (0.968,0.017), (1.112,0.027) and (0.63,0.048), respectively. The average confidence lengths of α and λ based on the MLEs are (1.569,0.798). Similarly, based on AMLEs they are (1.80,0.044) and the average credible lengths of α and λ are (1.490,0.189).

7. Conclusions

In this paper, we have discussed the classical and Bayesian inferential procedures for the progressively Type-II hybrid censored data from the Weibull distribution. It is shown that the maximum likelihood estimator of the shape parameter can be obtained by using an iterative procedure. The proposed approximate maximum likelihood estimators of the shape

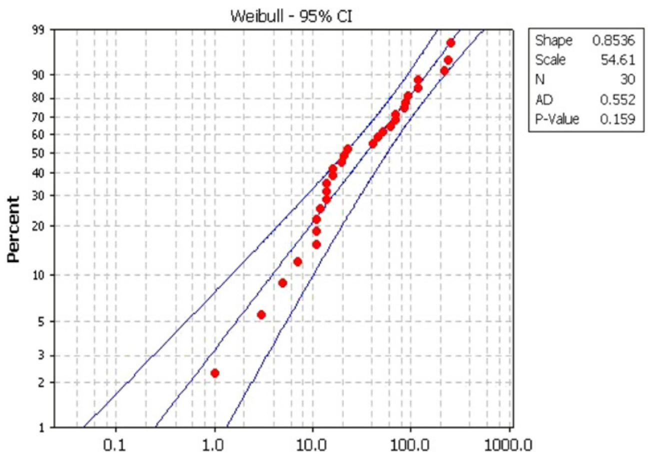

Fig. 3. PP plot for goodness of fit.

and scale parameters can be obtained in explicit forms. Bayes estimates of the unknown parameters can be obtained using Gibbs sampling methods. A comparison is made through MSEs, bias, average confidence/credible widths and the corresponding coverage percentages.

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Appendix

To prove Theorem 2, we need the following lemma.

Lemma 1. *For* $x_i \geq 0$ *and* $b \geq 0$, *define* $g(\alpha) = \sum_{i=1}^{n} x_i^{\alpha} + b$. *Then*, $(\partial^2/\partial\alpha^2)\ln g(\alpha) \geq 0$ (*see*, Kundu, 2007).

Proof of Theorem 2. We consider here only Case I, and Case II follows exactly in the same manner. The conditional density of α given the data is

$$L(\alpha|data) \propto \frac{\alpha^m \pi_2(\alpha) \prod_{i=1}^{m} y_{i:m:n}^{\alpha-1}}{(\sum_{i=1}^{m} y_{i:m:n}^{\alpha}(1+R_i)+b)^{a+m}}. \tag{36}$$

Therefore, ignoring the additive constant, the log-likelihood function of the posterior density function of α can be written as

$$\ln L(\alpha|data) = \ln\pi_2(\alpha) + m\ln\alpha + (\alpha-1)\sum_{i=1}^{m} \ln y_{i:m:n} - (a+m)\ln\left(\sum_{i=1}^{m} y_{i:m:n}^{\alpha}(1+R_i)+b\right).$$

Hence, by using Lemma 1 and the assumption on $\pi_2(\alpha)$, it easily follows that $L(\alpha|data)$ is log-concave.

We just provide the posterior density function of α for Case II. Note that, for $J>0$,

$$L(\alpha|data) \propto \frac{\alpha^J \pi_2(\alpha) \prod_{i=1}^{J} y_{i:m:n}^{\alpha-1}}{(\sum_{i=1}^{J} y_{i:m:n}^{\alpha}(1+R_i)+T^{\alpha}R_J^{*}+b)^{a+J}}. \tag{37}$$

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