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A piecewise-spectral parametric iteration method for solving the nonlinear chaotic Genesio system

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ABSTRACT

In this paper, an effective algorithm which is a combination of the spectral collocation method and the parametric iteration method (PIM) is proposed for simulating the solution of the chaotic Genesio system (CGS). Comparison with the fourth-order Runge–Kutta method (RK4) confirms the very high accuracy of the presented algorithm. The obtained results reveal that the proposed algorithm is a promising method for the solution of the CGS and more promising because it can further be readily employed to solve other chaotic systems.

1. Introduction

In many circumstances, a system of coupled, nonlinear ordinary differential equations will model the dynamical behavior of certain mechanical systems, whereby exact solutions or a closed form of analytical solutions are very difficult to obtain. We generally have to rely on numerical integration, some particular transformations, linearization or discretization in order to obtain their approximate solutions. Although numerical integration methods are flexible, they have their weaknesses. They react quite sensitively on the selection of time-step size to be dependable [1]. Also, they provide solutions in discretized form, only at two ends of the time interval.

Dynamical systems that demonstrate chaotic behavior are sensitive to initial conditions. Though these systems are deterministic through some description by mathematical rules, the behavior of chaotic systems appears to be random because of its sensitivity to initial conditions. Chaotic behavior can be observed in a variety of systems such as electrical circuits, lasers, fluid dynamics, mechanical devices, time evolution of the magnetic field of celestial bodies, population growth in ecology and the dynamics of molecular vibrations. The history of chaos theory has come a long way since Jacques Hadamard who in 1898 published a significant study of a free particle gliding frictionlessly on a surface of constant negative curvature which exhibits chaotic motion. We have learnt much of chaos today than before and certainly more will be revealed in the future. One of the many chaotic systems discovered in the past is the Genesio system. It was developed by Genesio and Tesi [2] where it observes unpredictable, chaotic behavior. The dynamical equations describe an uncomplicated square element and three straightforward ordinary differential equations that are dependent on three positive real parameters, such as

$$\begin{cases} x'(t) = y(t), \\ y'(t) = z(t), \\ z'(t) = -\gamma x(t) - \beta y(t) - \alpha z(t) + x^2(t), \end{cases} \tag{1}$$

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subjected to the initial conditions

$$x(0) = 0.2, \qquad y(0) = -0.3 \quad \text{and} \quad z(0) = 0.1, \tag{2}$$

where α, β and γ are positive constants, satisfying the inequality $\alpha\beta < \gamma$.

Recently, there has been much attention devoted to search for better and more efficient methods for determining solutions, approximate or exact, analytical or numerical, to these kinds of nonlinear models [3–9]. For example, Goh et al. in [9] applied the classical variational iteration method (VIM) and the multistage VIM (MVIM) to solve the nonlinear CGS of (1). However, the convergence region of the corresponding results is very small, as will shown later in this paper. Furthermore, the solution procedure is complicated. Besides, it is time-consuming, and tedious computational work is done by the MVIM to get an approximate solution.

We believe that an easy-to-use algorithm can be proposed in order to overcome these shortcomings. Therefore, the strategy that will be pursued in this work rests mainly on establishing an effective algorithm based on the spectral scheme [10,11] and the parametric iteration method (PIM) [12,13] for obtaining a highly accurate approximate solution of the chaotic system (1). The case analyzed in this paper shows that the newly developed algorithm is a very simple, effective and much more accurate to solve the CGS.

2. The basic idea of the PIM

In this section, we first describe the PIM for solving the CGS of (1). Then the local convergence is discussed.

2.1. Analysis of the PIM

The PIM provides the solution of (1) as a sequence of approximations. The method gives convergent successive approximations of the exact solution if such a solution exists; otherwise, approximations can be used for numerical purposes. The idea of the PIM is very simple and straightforward. Consider the following initial value problem of dimension n

$$\begin{aligned} &\mathbf{u}'(t) = \mathbf{f}(t, \mathbf{u}(t)), \quad \mathbf{u}(t_0) = \mathbf{u}^0, \\ &\mathbf{u} : \mathbb{R} \to \mathbb{R}^n, \qquad \mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad t \in [t_0, t_0 + T], \end{aligned} \tag{3}$$

with the usual assumption that $\mathbf{f}$ is continuous.

To explain the basic idea of the PIM, we first consider (3) as

$$L[\mathbf{u}(t)] + N[\mathbf{u}(t)] = g(t), \tag{4}$$

where L with the property $L[\mathbf{v}] \equiv 0$ when $\mathbf{v} \equiv 0$ denotes the so-called auxiliary linear operator with respect to $\mathbf{u}$, N is a nonlinear operator with respect to $\mathbf{u}$ and $g(t)$ is the source term. We then construct a family of iterative processes for Eq. (4) as follows, [12,13]:

$$L[\mathbf{u}^{k+1}(t) - \mathbf{u}^k(t)] = hH(t)A[\mathbf{u}^k(t)], \tag{5}$$

with the initial condition

$$\mathbf{u}^{k+1}(t_0) = \mathbf{u}^0, \quad \forall k \geq 0, \tag{6}$$

where

$$A[\mathbf{u}^k(t)] = L[\mathbf{u}^k(t)] + N[\mathbf{u}^k(t)] - g(t) \equiv \mathbf{u}'^k(t) - \mathbf{f}(t, \mathbf{u}^k(t)), \tag{7}$$

and the subscript k denotes the kth iteration, and $\mathbf{u}^0(t)$ is the initial guess, which can be freely chosen with possible unknown constants, or it can also be solved from its corresponding linear homogeneous equation or linear nonhomogeneous equation i.e.,

$$L[\mathbf{u}^0(t)] = 0 \quad \text{or} \quad L[\mathbf{u}^0(t)] = g(t). \tag{8}$$

The $h \neq 0$ and $H(t) \neq 0$ denote the so-called auxiliary parameter and auxiliary function, respectively, which can be identified easily and efficiently by the techniques proposed below. It should be emphasized that though we have the great freedom to choose the auxiliary linear operator L, the auxiliary parameter h, the auxiliary function $H(t)$ and the initial approximation $\mathbf{u}_0(t)$, which is fundamental to the validity and flexibility of the PIM, we can also assume that all of them are properly chosen so that solution of (3) exists, as will be shown in this paper later. Accordingly, the successive approximations $\mathbf{u}^k(t)$, $(k \geq 1)$ of the PIM iterative relation in the auxiliary parameter h will be readily obtained. Consequently, the exact solution may be obtained by using

$$\mathbf{u}(t) = \lim_{k \to \infty} \mathbf{u}^k(t). \tag{9}$$

Therefore, the $(k+1)$th-order PIM of Eq. (5) forms a set of linear ODEs and can be easily solved, especially by means of symbolic computation software such as Maple, Mathematica, Matlab and others.

2.2. Convergence theorem

The parametric iteration formula, (5), makes a recurrence sequence $\{\mathbf{u}^k(t)\}$. Obviously, the limit of the sequence will be the solution of (3) if the sequence is convergent. In the following, we give a new proof of convergence of the PIM, which details can be found in [13]. Here we suppose that for every k, $\mathbf{u}^k \in C^1[0, T]$ and $\{\mathbf{u}'^k\}$ is uniformly convergent.

Theorem 1. *If the sequence (9) converges, where $\mathbf{u}^k(t)$ is produced by the parametric iteration formulation of (5), then it must be the exact solution of (3).*

Proof. If the sequence $\{\mathbf{u}^k(t)\}$ converges, we can write

$$\mathbf{U}(t) = \lim_{k\to\infty} \mathbf{u}^k(t), \tag{10}$$

and it holds

$$\mathbf{U}(t) = \lim_{k\to\infty} \mathbf{u}^{k+1}(t). \tag{11}$$

Using (10) and (11) and the definition of L, we can easily gain

$$\lim_{k\to\infty} L[\mathbf{u}^{k+1}(t) - \mathbf{u}^k(t)] = L \lim_{k\to\infty} [\mathbf{u}^{k+1}(t) - \mathbf{u}^k(t)] = 0. \tag{12}$$

From (12) and according to (5), we obtain

$$hH(t) \lim_{k\to\infty} A[\mathbf{u}^k(t)] = L \lim_{k\to\infty} [\mathbf{u}^{k+1}(t) - \mathbf{u}^k(t)] = 0. \tag{13}$$

Since $h \neq 0$ and $H(t) \neq 0$ for all t, the relation (13) gives us

$$\lim_{k\to\infty} A[\mathbf{u}^k(t)] = 0. \tag{14}$$

From (14) and continuity property of the operator $\mathbf{f}$, it follows that

$$\begin{aligned} \lim_{k\to\infty} A[\mathbf{u}^k(t)] &= \lim_{k\to\infty} (\mathbf{u}'^k(t) - \mathbf{f}(t, \mathbf{u}^k(t))) \\ &= (\lim_{k\to\infty} \mathbf{u}^k(t))' - \mathbf{f}(t, \lim_{k\to\infty} \mathbf{u}^k(t)) \\ &= \mathbf{U}'(t) - \mathbf{f}(t, \mathbf{U}(t)). \end{aligned} \tag{15}$$

From Eqs. (14) and (15), we get

$$\mathbf{U}'(t) - \mathbf{f}(t, \mathbf{U}(t)) = 0, \quad a \leq t \leq b. \tag{16}$$

On the other hand, in view of (3), (6) and (11), it holds that

$$\mathbf{U}(t_0) = \lim_{k\to\infty} \mathbf{u}^{k+1}(t_0) = \mathbf{u}(t_0) = \mathbf{u}^0. \tag{17}$$

Hence, according to the two expressions, (16) and (17), $\mathbf{U}(t)$ must be the exact solution of the system (3) and this ends the proof. □

It is clear that the convergence of the sequence (9) depends upon the initial guess $\mathbf{u}^0(t)$, the auxiliary linear operator L, the auxiliary parameter h and the auxiliary function $H(t)$. Fortunately, the PIM provides us the great freedom of choosing those items. Thus, as long as $\mathbf{u}^0(t)$, L, h and $H(t)$ are so properly chosen that the sequence (9) converges in a region $a \leq t \leq b$, it must converge to the exact solution in this region. Therefore, the combination of the convergence theorem and the freedom of the choice of the above factors establishes the cornerstone of the validity and flexibility of the PIM.

3. Analysis of the new method

To solve nonlinear systems of ODEs like the CGS using the spectral collocation methods, because of the nonlinearity, it is no longer enough simply to invert the differentiation matrix. Instead, we can iteratively solve the system with a stopping criterion (see [11]). Note that this implementation of the method may lead to divergent results. On the other hand, by using the PIM algorithm (5) to solve nonlinear systems, we obtain a series solution in general, which in practice is a truncated series solution. This series solution gives a good approximation to the exact solution only in a small region of t. An easy and reliable way of ensuring the validity of approximations for large t is to determine the solution in a sequence of intervals of t, which are subject to continuity conditions at the end points of each interval. Therefore, in this section, the ideas of the spectral technique and the PIM are combined to establish a new piecewise-spectral PIM algorithm in solving nonlinear systems of ODEs. The proposed algorithm will provide us with a convenient way to modify and control the convergence

region and rate of the solution. In the next Section 3.1, a simple explicit Chebyshev spectral collocation method is presented for solving the nonlinear system of IVPs of (3). Afterwards, an explicit piecewise-spectral PIM is proposed to solve the system (3) in the Section 3.2.

3.1. A spectral collocation method

Assume that $u_i(t)$ and $u_i'(t) = f_i\big(t, u_1(t), \ldots, u_n(t)\big)$ $(i = 1, \ldots, n)$ are the ith component of the solution $\mathbf{u}$ and the ith equation of the system (3), respectively. Consider basis functions $\phi_{i,j}$ that are polynomials of degree $N-1$ satisfying $\phi_{i,j}(t_k) = \delta^i_{j,k}$ for the Chebyshev nodes (note that $t_1 = 1$ and $t_N = -1$)

$$t_k = \cos\left(\frac{(k-1)\pi}{N-1}\right), \quad k = 1, \ldots, N,\ i = 1, \ldots, n. \tag{18}$$

The polynomial (the unknown functions $u_i(t)$ $(i = 1, \ldots, n)$ is approximated as a truncated series of polynomials)

$$p_i(t) \simeq u_i(t) = \sum_{j=1}^{N} \phi_{i,j+1}(t) u_{i,j}, \tag{19}$$

interpolates the points $(t_j, u_{i,j})$, that is, $p_i(\mathbf{t}) = \mathbf{u}_i$. The values of the interpolating polynomial's rth derivative at the nodes are

$$p_i^{(r)}(\mathbf{t}) = D^{(r)}\mathbf{u}_i, \quad i = 1, \ldots, n, \tag{20}$$

where the k, jth element of the differentiation matrix $D^{(r)}$ is $\phi_{i,j}^{(r)}(t_k)$. Note that $D^{(r)} \neq (D^{(1)})^r$.

Generally, in order to solve the system (3) using a (pseudo)spectral collocation scheme, the interpolating polynomials $p_i(t)$ $(i = 1, \ldots, n)$ are required to satisfy the equations of the system at the interior nodes. The values of the interpolating polynomials at the interior nodes $(m = 1 : N-1)$ are $p_i(\mathbf{t}_m) = (\mathbf{u}_i)_m = I_{m,:}\,\mathbf{u}_i$ and the derivative values are $p_i^{(r)}(\mathbf{t}_m) = D^{(r)}_{m,:}\,\mathbf{u}_i$. The initial conditions that involve the values of the interpolating polynomials can be handled by using the formulas

$$p_i(\mathbf{t}_N) = (\mathbf{u}_i)_N = I_{N,:}\,\mathbf{u}_i. \tag{21}$$

The Differentiation Matrix Suite [14] provides two useful Matlab functions for spectral collocation, `chebdif` and `chebint`. The function call `[t, DM] = chebdif(N, M)` computes the differentiation matrices for $r = 1, 2, \ldots, M$, where $0 < M \leq N-1$ on the interval $[-1, 1]$, where the subarray `DM(:, :, r)` contains the $N \times N$ matrix $DM^{(r)}$. The column vector $\mathbf{t}$ contains the Chebyshev nodes with $\mathbf{t}_1 = 1$ and $\mathbf{t}_N = -1$. The function call `p = chebint(u, t)` evaluates the polynomial that interpolates the data vector `u` at the Chebyshev nodes. The polynomial is evaluated at the values given in vector `t` $\in [-1, 1]$. The codes are available at http://dip.sun.ac.za/~weideman/research/differ.html.

Fortunately, in order to avoid the change of variables and to simplify the procedure, the function call `[t, DM] = chebdif(N, M)` with the small changes as `t = 1/2 * ((b - a) * t + (b + a))` and `DM(:, :, ell) = (2/(b - a))ell * D` in the lines 38 and 57, respectively, can be converted to the function call `[t, DM] = chebdift(N, M, a, b)` to directly compute the transformed Chebyshev differentiation matrices and nodes on the arbitrary interval $[a, b]$. Moreover, the function call p=chebint(u,t) with the small change as `t = 1/2 * ((b - a) * t + (b + a))` in the line 27 can be converted to the function call `p = chebintt(u, t, a, b)` to directly evaluate the polynomial that interpolates the data vector `u` at the transformed Chebyshev nodes on the arbitrary interval $[a, b]$.

As a result, requiring no change of variables, we can straightforwardly solve the system (3). Here we present a simple explicit iterative procedure for solving the system (3).

For the interpolating polynomial to satisfy the ith differential equation of the system (3) at each interior node, the collocation equation

$$p_i'(\mathbf{t}_m) = f_i(\mathbf{t}_m, p_1(\mathbf{t}_m), \ldots, p_n(\mathbf{t}_m)), \quad i = 1, \ldots, n, \tag{22}$$

should be satisfied. Substituting the differentiation matrix relations, the collocation equation can be written as

$$D^{(1)}_{m,:}\,\mathbf{u}_i = I_{m,:} f_i(\mathbf{t}_m, \mathbf{u}_1, \ldots, \mathbf{u}_n), \quad i = 1, \ldots, n. \tag{23}$$

In view of (3), we have $\mathbf{u}(t_0) = \mathbf{u}_0$. Here we put $(\mathbf{u}_0)_i = \mathbf{u}_i^0$ $(i = 1, \ldots, n)$. Hence, the initial conditions $p_i(\mathbf{t}_N) = \mathbf{u}_i^0$ are satisfied when

$$I_{N,:}\,\mathbf{u}_i = \mathbf{u}_i^0, \quad i = 1, \ldots, n. \tag{24}$$

Consequently, the solution vector $\mathbf{u}$ can be explicitly found by solving the following equation:

$$\begin{bmatrix} D^{(1)}_{m,:} \\ I_{N,:} \end{bmatrix} \mathbf{u}_i^{k+1} = \begin{bmatrix} I_{m,:} f_i(\mathbf{t}_m, \mathbf{u}_1^k, \ldots, \mathbf{u}_n^k) \\ \mathbf{u}_i^0 \end{bmatrix}, \quad i = 1, \ldots, n, \tag{25}$$

with the fixed-point iterative method and the suitable initial iterate. Now, if we define $\mathbf{L}_i = [D^{(1)}_{m,:}; I_{N,:}]$ and $\mathbf{F}_i^k = [I_{m,:} f_i(\mathbf{t}_m, \mathbf{u}_1^k, \ldots, \mathbf{u}_n^k); \mathbf{u}_i^0]$, then we will have $\mathbf{u}_i^{k+1} = \mathbf{L}_i^{-1}\,\mathbf{F}_i^k$.

It should be noted that this implementation of the spectral collocation may lead to divergent results.

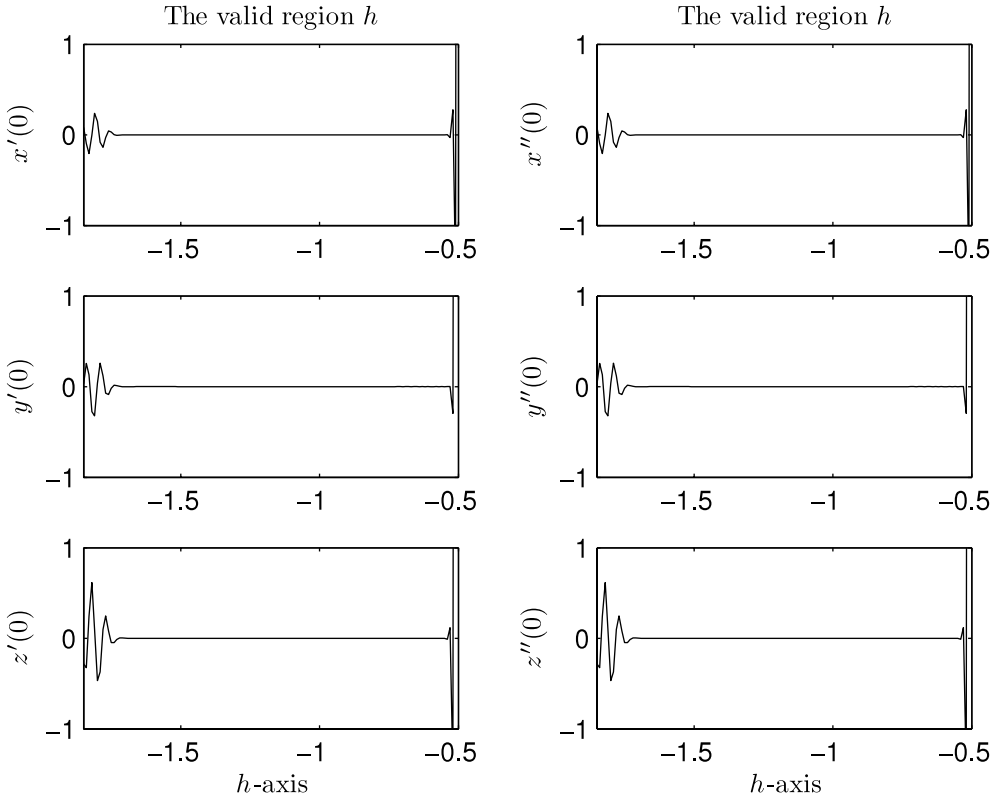

Fig. 1. The valid region of the auxiliary parameter h using the PSP algorithm for the system CGS 1.

3.2. A piecewise-spectral PIM

We note that, in general, the application of the PIM to solve the nonlinear systems of ODEs leads to the calculation of terms that are not needed and more time is consumed in repeated calculations for series solutions. The unneeded and repeated calculations may or may not lead to a faster convergence. Also, the successive iterates of the PIM may be very complex so that the resulting integrals in relation (5) may not be performed analytically. As mentioned above, a simple way of ensuring the validity of the approximations (5) for large t is to determine the solution in a sequence of equal intervals, which are subject to continuity conditions at the end points of each interval. Assume that the system (3) is to be solved in the interval $[0, T]$. Define the following set of disjoint intervals $I^s_{\Delta_s} = [t^s, t^{s+1})$ with $(\Delta t)_s \equiv \Delta_s = t^{s+1} - t^s$, $t^0 = 0$ and $t^M = T$ so that $\bigcup_{s=0}^{M-1} I^s_{\Delta_s} = [0, T]$, where M denotes the number of disjoint intervals of $[0, T]$. Assume that the points $t^s_1, \ldots, t^s_N$ are the transformed Chebyshev nodes on the interval $I^s_{\Delta_s}$ with $t^s_N = t^s$ and $t^s_1 = t^{s+1}$. Suppose also that the polynomial $p_{i,s+1}(t)$ interpolates the points $(t^s_j, u_{i,j})$ $(j = 1, \ldots, N)$, that is, $p_{i,s+1}(\mathbf{t}^s) = \mathbf{u}_{i,s+1}$.

Now in each $I^s_{\Delta_s}$, considering the continuity conditions at the end points of each interval, and by using Eqs. (18)–(21), the PIM equation (5) and the initial condition (6) are transformed into the following equations:

$$L_i[\, p^{k+1}_{i,s+1}(\mathbf{t}^s_m) - p^k_{i,s+1}(\mathbf{t}^s_m)] = hH_i(\mathbf{t}^s_m)A_i[p^k_{i,s+1}(\mathbf{t}^s_m)], \tag{26}$$

$$p^{k+1}_{i,s+1}(t^s_N) = p_{i,s+1}(t^s) = p_{i,s}(t^s), \quad s = 0, \ldots, M-1, \tag{27}$$

where $p_{i,0}(t^0) = \mathbf{u}^0_{i,0} = \mathbf{u}^0_i$.

Let K be the maximum number of iterations, i.e., $k = 0, 1, \ldots, K-1$. In view of (7) and the definitions of L and A, by substituting the differentiation matrix relations, the Eqs. (26) and (27) can be written as

$$\begin{bmatrix} L_i \\ I_{N,:} \end{bmatrix} \mathbf{u}^{k+1}_{i,s+1} = (I + hH_i(\mathbf{t}^s)) \begin{bmatrix} L_i \\ I_{N,:} \end{bmatrix} \mathbf{u}^k_{i,s+1} + hH_i(\mathbf{t}^s) \left(\begin{bmatrix} N_i[\mathbf{u}^k_{i,s+1}] \\ 0 \end{bmatrix} - \begin{bmatrix} g_i(\mathbf{t}^s_m) \\ (\mathbf{u}^K_{i,s})_1 \end{bmatrix} \right), \tag{28}$$

where I denotes the identity matrix of order N, $(\mathbf{u}^K_{i,s})_1$ the first component of $\mathbf{u}^K_{i,s}$, $H_i(\mathbf{t}^s) = \mathrm{diag}(H_i(t^s_1), \ldots, H_i(t^s_N))$ and $g_i(\mathbf{t}^s_m) = [g_i(t^s_1), \ldots, g_i(t^s_{N-1})]^T$.

Now, if we define $\mathbf{L}_i = [L_i, I_{N,:}]^T$, $\mathbf{N}_i = [N_i, 0]^T$, $H_i(\mathbf{t}^s) = \mathbf{H}_i$ and $\mathbf{g}_i = [g_i(\mathbf{t}^s_m), (\mathbf{u}^K_{i,s})_1]^T$, then we will have

$$\mathbf{L}_i\, \mathbf{u}^{k+1}_{i,s+1} = (I + h\mathbf{H}_i)\mathbf{L}_i\, \mathbf{u}^k_{i,s+1} + h\mathbf{H}_i(\mathbf{N}_i\, \mathbf{u}^k_{i,s+1} - \mathbf{g}_i), \quad i = 1, \ldots, n. \tag{29}$$

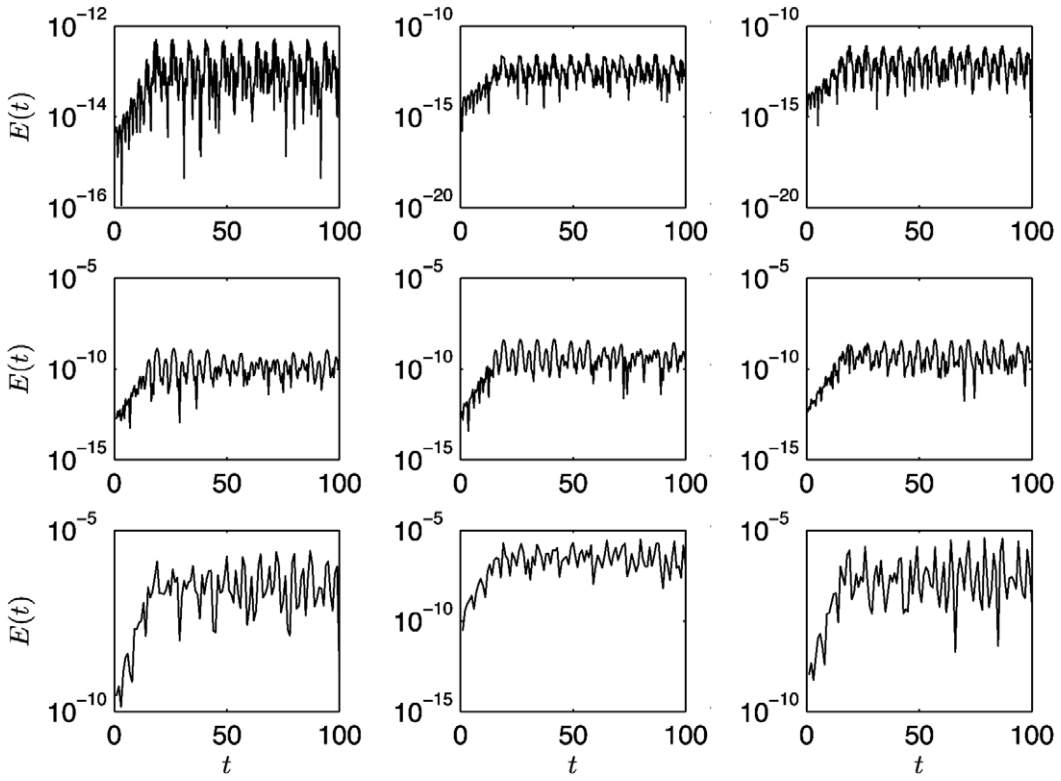

Fig. 2. The differences between the PSP solution when $N = 10$ and $h = -1$ for different values of Δt and the numerical RK4 ($\Delta t = 0.001$) solution (first column: $x(t)$, second column: $y(t)$ and third column: $z(t)$, and also first row: $\Delta t = 0.25$, second row: $\Delta t = 0.5$ and third row: $\Delta t = 1$) for the CGS of (1).

By taking $\mathbf{H}_i = I$ $(i = 1, \ldots, n)$ for simplicity, (29) is simplified to the following recurrence relation for finding $\mathbf{u}_{i,s+1}^{k+1}$, which is called the piecewise-spectral PIM (PSP):

$$\mathbf{u}_{i,s+1}^{k+1} = (1+h)\mathbf{u}_{i,s+1}^{k} + h\,\mathbf{L}_i^{-1}(\mathbf{N}_i\,\mathbf{u}_{i,s+1}^{k} - \mathbf{g}_i), \quad i = 1, \ldots, n. \tag{30}$$

Here the vector $\mathbf{u}_{i,s+1}^{k+1}$ is defined as

$$\mathbf{u}_{i,s+1}^{k+1} = \{u_{i,s+1}^{k+1}(t_1^s), u_{i,s+1}^{k+1}(t_2^s), \ldots, u_{i,s+1}^{k+1}(t_N^s)\}, \quad i = 1, \ldots, n. \tag{31}$$

In using the PSP algorithm above, we begin by choosing the best possible initial approximation that satisfies the initial condition (6). To this end, we may determine the initial approximation by solving the above linear system (8). Thus, starting from the initial approximation $\mathbf{u}^0(t)$, we can use the recurrence formula (30) to successively obtain directly $\mathbf{u}^{k+1}(t)$ for $k \geq 0$.

Now, by substituting $h = -1$ into (30), we will get the following piecewise-spectral collocation algorithm for solving the system of (3):

$$\mathbf{u}_{i,s+1}^{k+1} = \mathbf{L}_i^{-1}(\mathbf{g}_i - \mathbf{N}_i\,\mathbf{u}_{i,s+1}^{k}), \quad i = 1, \ldots, n, \tag{32}$$

As a result, the piecewise-spectral collocation algorithm proposed above is only a special case of the PSP algorithm given by (30) when $h = -1$.

4. Choosing the auxiliary parameter *h*

It is important to ensure that a solution series obtained using the PSP algorithm, which is always as a family of solution expressions in the auxiliary parameter h, is convergent in a large enough region whereby the convergence region and rate of solution series are dependent upon the auxiliary parameter h and thus can be enlarged by choosing a proper value for h. Most important, however, is how to choose the value of h to make sure that the solution series converges fast enough in a large enough region. Since we have a family of solution expressions in the auxiliary parameter h; hence, regarding h as an independent variable, a simple and practical way of selecting h is to plot the curve of solution's derivatives with respect to h in some points [13,15]. So, if the solution is unique, all of them converge to the same value and hence there exists a horizontal line segment in its figure that corresponds to a region of h called the valid region of h. Thus, if we set h any value in

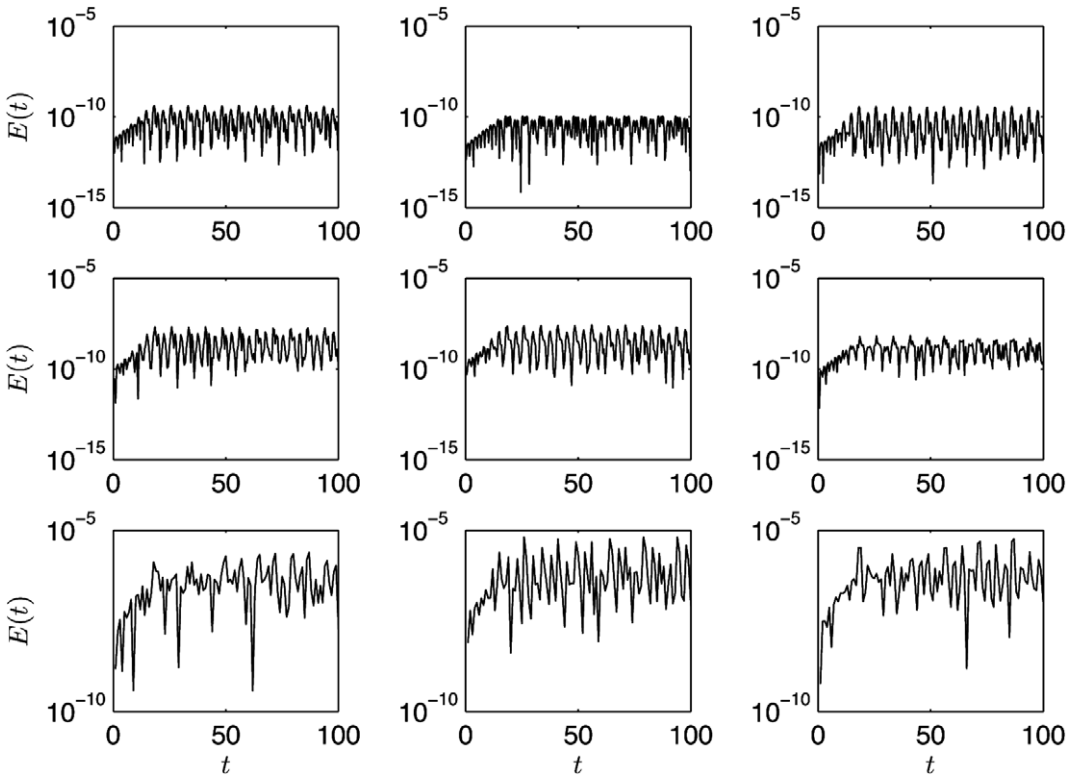

Fig. 3. The differences between the PSP solution when $N = 10$ and $h = -0.95$ for different values of Δt and the numerical RK4 ($\Delta t = 0.001$) solution (first column: $x(t)$, second column: $y(t)$ and third column: $z(t)$, and also first row: $\Delta t = 0.25$, second row: $\Delta t = 0.5$ and third row: $\Delta t = 1$) for the CGS of (1).

the so-called valid region of h, we are sure that the corresponding solution series converges. It is found that, for given initial approximation and the auxiliary function, the valid regions of h are often nearly the same for a given problem. In most cases, we can find a proper value of h to ensure that the solution series converges in the whole spatial/temporal regions. Therefore, the curves h provide us with an easy way to show the influence of h on the convergence region and rate of solutions.

Remark 1. It should be emphasized that implicitly we are dealing with a stability analysis of the PSP algorithm, which in our case the valid/stability function for given Δt and N depends on the auxiliary parameter h. Of interest is the set of all h such that (30) is numerically stable. Since the valid/stability region usually is difficult to characterize in general, here, we suggest the above geometric approach to discover the valid/stability region.

5. Numerical implementation

In this section, to give a clear overview of the content of this study, the CGS of (1) will be tested by the above-mentioned PSP algorithm, which will ultimately show that the approach is easy to implement and accurate when applied to the CGS. All the results are calculated by using the symbolic calculus software Matlab 7.

For the sake of comparison, we use the constants $\alpha = 1.2$, $\beta = 2.92$ and $\gamma = 6$ as in [9], where the system exhibits chaotic behavior, alongside with its initial conditions $x(0) = 0.2$, $y(0) = -0.3$ and $z(0) = 0.1$.

In order to solve the system (1) using the PSP equation (30), according to (4), we choose

$$\begin{aligned} &L_x = x', \quad N_x = -y, \quad g_x(t) = 0,\\ &L_y = y', \quad N_x = -z, \quad g_y(t) = 0,\\ &L_z = z' + \alpha z, \quad N_x = \gamma x + \beta y - x^2, \quad g_z(t) = 0. \end{aligned} \tag{33}$$

To investigate the valid region h of the solution obtained via the PSP algorithm, here we separately plot the curves of $x'(0), y'(0), z'(0), x''(0), y''(0)$ and $z''(0)$ with respect to h, as shown in Fig. 1. According to these curves, it is easy to discover the valid region of h. As mentioned before, the valid regions h are often nearly the same for a given problem, for instance, observe the valid regions h of $x'(0)$ and $x''(0)$. We point out that the valid region of h becomes more accurate as the number k increases. It is usually convenient to investigate the valid region h of the PSP algorithm by means of such kinds of the curves.

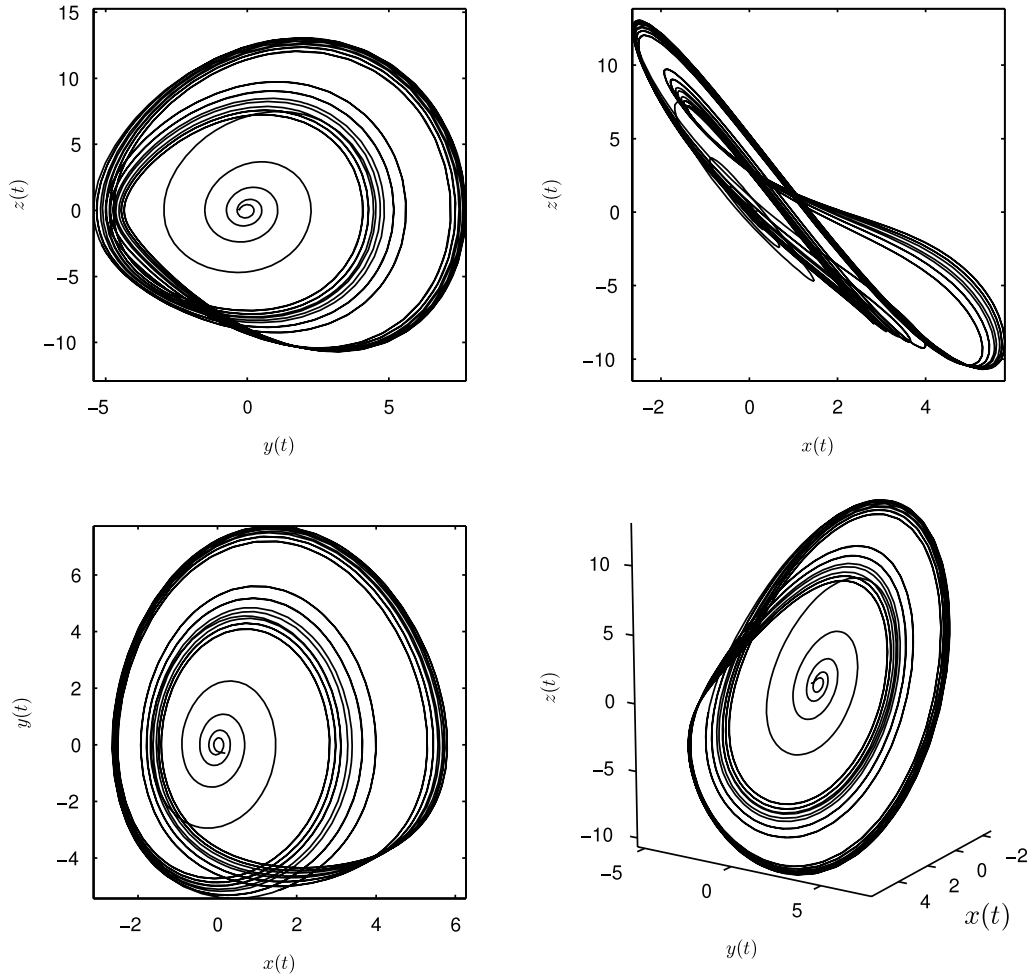

Fig. 4. Phase portraits using the PSP algorithm ($h = -1$) on $\Delta t = 0.5$ for $\alpha = 1.2$, $\beta = 2.92$ and $\gamma = 6$.

In order to show the accuracy and the efficacy of the PSP algorithm proposed, here, the differences ($E(t) = |S_{\mathrm{PSP}}(t) - S_{\mathrm{RK4}}(t)|$) between the solutions of the PSP algorithm when $h = -1$ and $h = -0.95$, and the numerical RK4 method with the time step 0.001 for different large values of the step size Δt have been given in Figs. 2 and 3. From the numerical results of this case analyzed, it is easy to conclude that the PSP algorithm is very effective to solve the CGS of (1).

Moreover, in Fig. 4, we produce an *xy*, *xz*, *yz* and *xyz* phase portrait of the CGS using the PSP solution on $\Delta t = 0.5$.

6. Numerical comparisons

Goh et al. in [9] applied the VIM and multistage VIM (MVIM) for solving the CGS (1). The above-mentioned methods are methods which have some limitations. Most important, however, it is the fact that the convergence region of the corresponding results is small, as shown below. In order to overcome the limitations of the above-named methods and also to enlarge the convergence region of the solution, in the present paper, we applied the PSP algorithm. Also, Goh et al. reported the obtained numerical results of the system (1) for the interval [0, 11.2], see Table 1. In the studies by Goh et al., much time was spent and boring operations were done by the VIM and MVIM to get an approximate solution. In our study, however, a rational approximate solution is computed easily using this technique. Generally speaking, the PSP algorithm is reliable and more efficient as compared to the VIM and MVIM. As a result, it is easy to conclude that the PSP algorithm is a powerful tool for solving the CGS.

7. Conclusions

In this paper, an effective piecewise-spectral parametric iteration method called the PSP algorithm has been proposed for solving the nonlinear systems of ODEs. A concrete modeling problem of chaotic systems has been tested by using the PSP algorithm proposed in the present work, and the obtained numerical results have shown noteworthy performance. For the

Table 1
Numerical comparison between VIM, MVIM and RK4 using the step size $\Delta t = 0.01$ in [9].

$t/$	$x(t)$		$y(t)$		$z(t)$	
	VIM	MVIM	VIM	MVIM	VIM	MVIM
1	8.6×10^{-5}	8.3×10^{-8}	1.0×10^{-4}	4.4×10^{-7}	2.5×10^{-4}	6.3×10^{-8}
5	4.0×10^{-1}	1.0×10^{-5}	3.1×10^{1}	3.3×10^{-5}	1.2×10^{1}	1.9×10^{-4}
10	2.0×10^{4}	2.0×10^{-3}	1.4×10^{5}	2.0×10^{-2}	1.2×10^{8}	7.1×10^{-2}
11	5.1×10^{4}	3.6×10^{-2}	4.6×10^{5}	2.8×10^{-1}	8.8×10^{8}	1.1×10^{0}

analyzed CGS, it has been shown that the presented procedure provides much more accurate results than those obtained using the classical and multistage VIM procedures. Generally speaking, the proposed approach can be further implemented to solve the other chaotic systems of physical interests with high accuracy.

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