Extended Abstarct Of
4th Iranian Conference on Applied Mathematics
10-12 March 2010
University Of Sistan & Balochestan

Variational iteration method for solving Seventh order integro-differential equations

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Abstract

In this paper, the variational iteration method is applied to solve boundary value problems for Seventh order integro-differential equations. Only one iteration is needed, and the obtained solutions are of remarkable accuracy.By giving two examples and comparing with the exact solution, the efficiency of the method will be shown .

Keyword:variational iteration; boundary-value problems; integro-differential equation

1. Introduction

In the recent years many different method were proposed to solve boundary value problems(BVPs), such as homotopy perturbation method (HPM)[1,2], variational iteration method (VIM)[3,4] and Modified Decomposition method (MDM)[5]. Recently Sweilam [6] implemented the VIM to solve fourth order integro-differential equations. In this paper, we apply the variational iteration method proposed by Ji-Huan He [7-10] to find approximate solutions for boundary value problems of seventh order integro-differential equations .

To illustrate the basic idea of VIM, we consider following general nonlinear system:

$$Lu + Nu = g(x), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and g(x)is a inhomogeneous forcing term. According to the variational iteration method [7-10], we can construct a correction functional for the system , as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\{Lu_n(s) + N\tilde{u}_n(s) - g(s)\}ds, \tag{2}$$

where λ is a Lagrange multiplier, which can be identified optimally via the variational theory [11], the subscripts n denotes the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n$=0.

We consider the general boundary value problem of the following type, to solve by using VIM:

$$y^{(vii)}(x) = g(x) + \int_0^x f(t, y(t), y'(t), y''(t), \ldots, y^{(vii)}(t))dt, \tag{3}$$

with suitable boundary conditions .

2. Applications

According to VIM, the correction functional for (3) can be constructed as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\{y_n^{(vii)}(s) - g(s) - \int_0^s \tilde{f}(t, y_n(t), y_n'(t), \ldots, y_n^{(vii)}(t))dt\}ds, \tag{4}$$

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where λ is general Lagrange multiplier, $\tilde{y}_n$ denote restricted variation i.e. $\delta\tilde{y}_n$ =0. Making the above correction functional stationary, we obtain the following stationary conditions:

$$1+\lambda^{(vi)}(x)=0, \qquad \lambda^{(v)}(x)=0, \qquad \lambda^{(iv)}(x)=0, \qquad \lambda'''(x)=0,$$
$$\lambda''(x)=0, \qquad \lambda'(x)=0, \qquad \lambda(x)=0, \qquad \lambda^{(vii)}(s)=0.$$

The Lagrange multiplier, therefore, can be obtain in the following form:

$$\lambda(s)=-\frac{(s-x)^6}{6!}. \tag{5}$$

Therefore equation (4) can be rewritten as:

$$y_{n+1}(x)=y_n(x)-\int_0^x \frac{(s-x)^6}{6!}\{y_n^{(vii)}(s)-g(s)-\int_0^s f(t,y_n(t),y_n'(t),\ldots,y_n^{(vii)}(t))dt\}ds. \tag{6}$$

Now, to demonstrate the accuracy of the variational iteration method we consider two following examples with known exact solutions.

2.1. Linear integro-differential equation

First,we consider the following integro-differential equation:

$$y^{(vii)}(x)=2-8e^x+\int_0^x y(t)dt, \qquad 0\le x\le 1 \tag{7}$$

subject to the boundary conditions:

$$y(0)=1, \quad y'(0)=0, \quad y''(0)=-1, \quad y'''(0)=-2,$$
$$y(1)=0, \quad y'(1)=-e, \quad y''(1)=-2e. \tag{8}$$

The exact solution of (7) is $y(x)=(1-x)e^x$. According to (6), we have the following iteration formulation:

$$y_{n+1}(x)=y_n(x)-\int_0^x \frac{(s-x)^6}{6!}\{y_n^{(vii)}(s)-2+8e^s-\int_0^s y_n(t)dt\}ds. \tag{9}$$

Now, starting with the initial solution

$$y_0(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6, \tag{10}$$

where a_0,a_1,a_2,a_3x,a_4,a_5 and a_6 which are unknown constants to be further determinined.
By the iteration formula (9), we obtian the following first-order approximation:

$$\begin{aligned} y_1(x)&=y_0(x)-\int_0^x \frac{(s-x)^6}{6!}\{y_0^{(vii)}(s)-2+8e^s-\int_0^s y_0(t)dt\}ds \\ &=(8+a_0)+(8+a_1)x+(a_2+4)x^2+(\frac{4}{3}+a_3)x^3+(\frac{1}{3}+a_4)x^4+(\frac{1}{15}+a5)x^5+(\frac{1}{90}+a_6)x^6 \\ &\quad+\frac{1}{2520}x^7+\frac{a_0}{40320}x^8+\frac{a_1}{362880}x^9+\frac{a_2}{1814400}x^{10}+\frac{a_3}{6652800}x^{11}+\frac{a_4}{19958400}x^{12}+\frac{a_5}{51891840}x^{13} \\ &\quad+\frac{a_6}{121080960}x^{14}-8e^x. \end{aligned} \tag{11}$$

Incorporating the boundary conditions, Eq.(8), into $y_1(x)$, we get

$$y_1(0) = a_0 = 1, \qquad y_1'(0) = a_1 = 0, \qquad y_1''(0) = 2a_2 = -1, \qquad y_1'''(0) = 6a_3 = -2,$$

$$y(1) = \frac{874638257}{39916800} + \frac{19958401}{19958400}a_4 + \frac{51891841}{51891840}a_5 + \frac{121080961}{121080960}a_6 - 8e = 0,$$

$$y'(1) = \frac{5968259}{3024400} + \frac{6652801}{1663200}a_4 + \frac{19958401}{3991680}a_5 + \frac{51891841}{8648640}a_6 - 7e = 0,$$

$$y''(1) = \frac{6780301}{362880} + \frac{1814401}{151200}a_4 + \frac{6652801}{332640}a_5 + \frac{19958401}{665280}a_6 - 6e = 0,$$

Solving the above system simultaneously, we obtian the unknowns as follows:

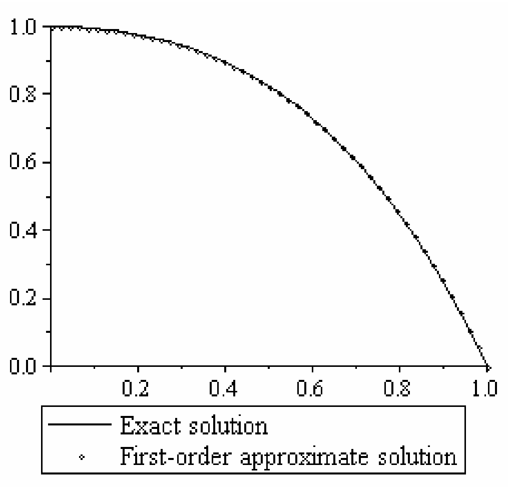
$$\begin{aligned}
&a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{-1}{2}, \quad a_3 = \frac{-1}{3},\\
&a_4 = -\frac{3001266767266598497485402 9}{12540092483933185558464 1} + \frac{1103527685870509261824000 0}{12540092483933185558464 1}e,\\
&a_5 = \frac{621809780656953475456257 7}{16946070924234034538465} - \frac{457543629986091323875200}{33892141848468069076 93}e,\\
&a_6 = -\frac{112493890519394664098718263}{75240555490359911335078 46} - \frac{68970445658572485464236 80}{125400924839331855584641}e.
\end{aligned} \qquad (12)$$

This gives us the approximate solution for (7) and (8) .
The numerical results of this application compared with the exact solution are presented in Table 1 and Fig 1. We observe the higher accuracy is obtianed without any difficulty.

Table 1 . Comparison of the first- order approximate solution with the exact solution.

x	y_E	y_1	Absolute error
0	1	1	0.00e-00
0.1	0.9946538262	0.9946538275	1.26e-09
0.2	0.9771222064	0.9771222018	4.60e-09
0.3	0.9449011656	0.944901164	2.00e-09
0.4	0.8950948188	0.895094819	1.00e-09
0.5	0.8243606355	0.824360631	4.00e-09
0.6	0.7288475200	0.728847543	2.30e-08
0.7	0.6041258121	0.60412578	3.00e-08
0.8	0.4451081856	0.44510815	4.00e-08
0.9	0.2459603111	0.24596027	4.00e-08
1	0	0	0.00e-00

Fig 1 . comparison of the approximate solution with the exact one.



2.1. Nonlinear integro-differential equation

Now we consider the following nonlinear integro-differential equation:

$$y^{(vii)}(x) = 1 + \int_0^x e^{-x} y^2(t)dt, \qquad 0 \le x \le 1 \tag{13}$$

with boundary conditions

$$\begin{aligned} &y(0) = y'(0) = y''(0) = y'''(0) = 1, \\ &y(1) = y'(1) = y''(1) = e. \end{aligned} \tag{14}$$

The exact solution of (13) is $y(x) = e^x$. According to (6), we have the following iteration formulation:

$$y_{n+1}(x) = y_n(x) - \int_0^x \frac{(s-x)^6}{6!} \{ y_n^{(vii)}(s) - 1 - \int_0^s e^{-t} y_n{}^2(t)dt \} ds. \tag{15}$$

We start with the following initial approximate

$$y_0(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6, \tag{16}$$

where $a_0, a_1, a_2, a_3 x, a_4, a_5$ and a_6 are unknown constants to be further determinined.
After we apply the iteration formula (15) to the BVP, we get the following first-order approximation

$$y_1(x) = y_0(x) - \int_0^x \frac{(s-x)^6}{6!} \{ -1 - \int_0^s e^{-t} y_0{}^2(t)dt \} ds \tag{17}$$

Incorporating the boundary conditions, Eq. (14), into $y_1(x)$, we have

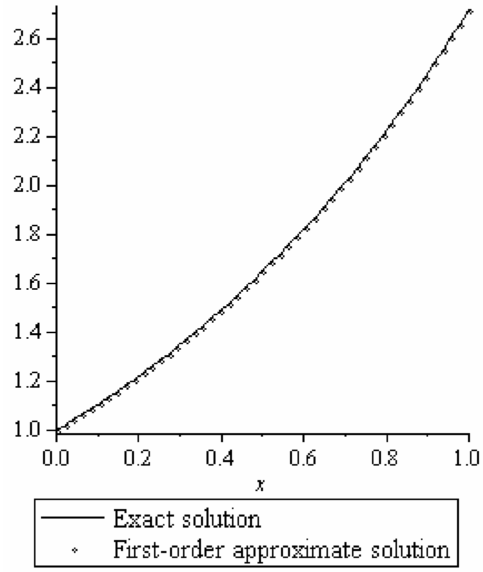
$$\begin{aligned} &a_0 = 1, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{6}, \\ &a_4 = \frac{4816098}{115585067}, \quad a_5 = \frac{578307}{69403294}, \quad a_6 = \frac{218465}{157257279}. \end{aligned} \tag{18}$$

Putting the values of a_i , i=0,1,…,6 one can get the approximate solution.
Comparision of the first-order approximate solution whit the exact one is shown in Table 2 and Fig 2.

Table 2 . Comparison of the first- order approximate solution with the exact solution.

x	y_E	y_1	Absolute error
0	1	1	0.00e-00
0.1	1.1051709181	1.1051710761	1.57e-07
0.2	1.2214027582	1.2214028349	7.67e-08
0.3	1.3498588076	1.3498589739	1.66e-07
0.4	1.4918246976	1.491824729	3.13e-08
0.5	1.6487212707	1.648721216	5.47e-08
0.6	1.8221188004	1.822118855	5.46e-08
0.7	2.0137527075	2.013752699	8.47e-09
0.8	2.2255409285	2.225540953	2.45e-08
0.9	2.4596031112	2.459603164	5.28e-08
1	2.7182818284	2.718281828	1.96e-08

Fig 2 . comparison of the approximate solution with the exact one.



Remark. The VIM algorithm is coded in the computer package Maple11. The Maple environment variable Digits contorolling the number of significant digits is set to 15 in calculations done in non-linear example 2.

3. Concolusion

In this paper, variational iteration method is employed to solve the linear and nonlinear BVPs for seventh order integro-differential equations**.** The method is applied in a direct way without using linearization, transformation, discretization .The numerical results in the Tables 1-2, show that the present method provides highly accurate numerical solutions for solving this type of the BVPs.

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