Free Vibration of Thin-Walled Cylindrical Shells

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Abstract
A method is presented for calculating the free vibrations of an isentropic thin-walled circular cylindrical shell with different boundary conditions. The method is based on Love's theory of shells. An experimental study was conducted on a circular cylindrical shell with simply supported boundary conditions. The objective was to evaluate the validity of the theory. To investigate the effects of a shell's dimension and boundary conditions on its natural frequencies, various models were studied. Shells with different lengths and radius/thickness ratios were compared. Moreover, shells with simply supported and clamped boundary conditions were modeled and analyzed. By comparing such varying terms interesting results were obtained, especially from designing point of view.

Keywords: circular cylindrical shell, natural frequency, mode shape, Love theory.

Introduction
Vibration analysis of circular cylindrical shells is of considerable importance as they are extensively used in many fields, such as; aerospace, mechanical, civil and marine engineering structures. A thin-walled shell is a three dimensional body bounded by two closely spaced curved surfaces. Compared with other dimensions, the distance of these two surfaces is small. Many regard shells as generalizations of flat plates; however, a flat plate is a special case of a shell with zero curvature. Furthermore, because of the inability to separate the bending terms of a shell from its stretching terms, static and dynamic analyses of shells has been of great complexity.

The literature concerning the vibration of shells is extremely extensive and readers can refer to Leissa [1] or more recently Amabili and Paidoussis [2] for comprehensive reviews of models and results presented in the literature.

Love [3] modified the Kirchhoff hypothesis for plates and established the assumptions used in the so-called classic theory of thin shells. These assumptions, which have now become the foundations of nearly all shell theories, are commonly known as Love's approximation of the first kind. Soedel [4] introduced a set of three closed form solutions for the natural frequencies of cylindrical shells and also obtained mode shape coefficients of a simply supported cylindrical shell by applying normal solutions to the Love theory [3].

Therefore, apart from the theoretical aspects of shell vibrations, there has also been many experimental studies made, see, e.g., References [5, 6, and 7], for comparison with theoretical predictions.

The main aim of the present paper is to investigate the limits of validation of the Love theory. These limits include the type of boundary conditions employed, the type of vibration applied to the shell and the dimensions of the shell. A general approach was made towards the Love theory, by applying it to; simply supported-simply supported and clamped-clamped thin-walled cylindrical shells with varying dimensions. Hence, a long cylindrical shell was first chosen for experimental and analytical studies. Analytical and experimental results were compared, in order to verify the validation of the theory and the experimental setup. Analytical and experimental results showed great adjacency, especially at certain frequencies and modes. Finally, several shell models were investigated to understand a shell's vibration behavior towards dimensional and structural variations. By applying such variations good agreements and great conclusions were yield.

Formulation of Problem
Consider a shell of constant thickness h, mean radius R, axial length L, with a Poisson's ratio of υ, density of ρ and Young's modulus of elasticity E. Shell coordinates in longitudinal and circumferential directions are x and θ, respectively, as shown in Figure 1.

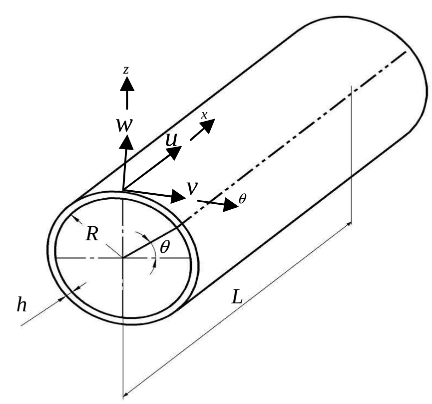

Figure1 : Circular cylindrical shell: coordinate system and dimensions.

The equations of motion for a cylindrical shell according to the Love theory can be written in matrix form as followed:

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = 0 \qquad (1)$$

where *u*, *v* and *w* are orthogonal directions respectively in the axial, circumferential and radial directions, shown in Figure 1, and $L_{ij}(i, j=1,2,3)$ are differential operators with respect to x and θ. If the shell has a uniform wall thickness, the allowable spatial form of distortion of a cross section must be periodic in the length of the circumference. The axial, tangential and radial displacements of the wall must vary with the axial position x and angle θ as:

$$\begin{cases} u = A\cos(k_m x)\cos(n\theta)\cos(\omega t) \\ v = B\sin(k_m x)\sin(n\theta)\cos(\omega t) \\ w = C\sin(k_m x)\cos(n\theta)\cos(\omega t) \end{cases} \tag{2}$$

in which k_m and n are the axial wavenumber and the circumferential wave parameter respectively, A, B and C are respectively, the modal (wave) amplitudes in the x, θ and z directions, and ω is the circular driving frequency.

Direct Solution of the Love Theory

Substituting equation (2) into the Love theory discussed earlier, a set of homogenous equations having the following matrix form is yield:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3}$$

in which $\lfloor C_{ij} \rfloor_F (i, j=1,2,3)$ are functions of n, k_m and ω. For the nontrivial solution the determinant of the coefficient matrix in equation (3) must be zero:

$$\det\left(\lfloor C_{ij} \rfloor_F\right) = 0\,,\ i, j = 1, 2, 3\,. \tag{4}$$

A characteristic equation is obtained from the expansion of equation (4) in which three unknowns are present; n, k_m and ω. In order to solve the characteristic equation according to the exact solution, one has to undergo several complicated operations. Therefore, according to the direct solution analysis, the axial wave number k_m and the circumferential wave parameter n of the corresponding natural frequency are assumed to be known. The circumferential wave parameter n, should always be represented by an integer number, independent from the type of boundary conditions opposed. However, k_m strongly depends upon the boundary conditions employed. For simplicity the flexural mode shapes of cylindrical shells in the axial direction are assumed to be of the same form as a transversely vibrating beam, with the same boundary conditions. Therefore according to beam functions for a shell with simply supported boundary conditions one has:

$$k_m = \frac{m\pi}{l}\,, \tag{5}$$

and similarly for a clamped shell one has:

$$k_m = \frac{(2m+1)\pi}{2l}\,. \tag{6}$$

In the above equations *m* is the axial wave parameter. By applying the direct solution method a new characteristic equation is yield for the Love theory, having the following form [1]:

$$\Omega^6 + a_4\Omega^4 + a_2\Omega^2 + a_0 = 0 \tag{7}$$

in which

$$\Omega^2 = \frac{\rho(1-\upsilon^2)R^2\omega^2}{E}\,. \tag{8}$$

Ω is called the non-dimensional frequency parameter. The solutions of equation (7) are [5]:

$$\Omega_{1mn}^2 = -\frac{2}{3}\sqrt{a_4^2 - 3a_2}\,\cos\frac{\alpha}{3} - \frac{a_4}{3} \tag{9}$$

$$\Omega_{2mn}^2 = -\frac{2}{3}\sqrt{a_4^2 - 3a_2}\,\cos\frac{\alpha + 2\pi}{3} - \frac{a_4}{3} \tag{10}$$

$$\Omega_{3mn}^2 = -\frac{2}{3}\sqrt{a_4^2 - 3a_2}\,\cos\frac{\alpha + 4\pi}{3} - \frac{a_4}{3} \tag{11}$$

in which

$$\alpha = \cos^{-1}\frac{27a_0 + 2a_4^3 - 9a_4a_2}{2\sqrt{\left(a_4^2 - 3a_2\right)^3}}\,. \tag{12}$$

For any combination of k_m and n, bi-cubic equation (7) would have three positive roots. A shell of a given length may vibrate in any of these three frequencies with all of them having the same longitudinal and circumferential wave numbers. Therefore in a cylindrical shell every mode shape has three distinct natural frequencies, however, the modes associated with each of these frequencies can be classified as primarily radial (or flexural), longitudinal (or axial) or circumferential (or torsional). In the above equations Ω_{1mn} is the lowest, and Ω_{3mn} is the highest natural frequency of a shell. Usually the lowest frequency is associated with a motion that is primarily radial. Nevertheless, depending upon shell dimensions, some low frequency modes are recognized as primarily longitudinal or circumferential rather than radial. As an example, for long shells which behave more like a beam rather than a ring, some low frequency modes are associated with axial dominant motions. Such modes produce bending motions in a long shell.

Experimental Setup

Tests were conducted on a circular cylindrical shell made of aluminum with a total length of 1830mm and radii of 76.2mm. Material properties of the shell were; E=68.2GPa (young's modulus), $\rho = 2700\,\mathrm{Kg/m^3}$ (density) and $\upsilon = 0.33$ (Poisson's ratio). Two steel disks with external radii of 76.2mm were inserted into each end of the shell. The disks were fastened tightly by two fastening chains to the aluminum shell, in order to produce simply supported boundary conditions on the two ends.

The cylinder was then subjected to a non-contact acoustical excitation, by means of two external loudspeakers. The experimental setup is shown in Figure 2. The whole shell setup was then put into a reverberant room in order to make the sound pressure equal at all surface points, inside and outside the shell, so that the acoustical excitation could easily excite axisymmetric (*n*=0) modes of the shell. 60 measurement points were uniformly distributed over the shell's length and circumference in order to identify various mode shapes. The shell response was then measured using two accelerometers.

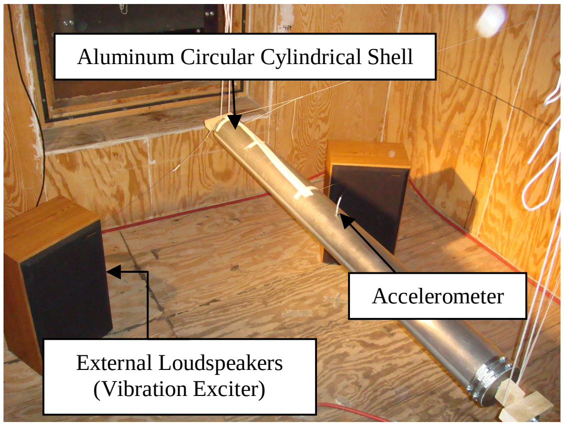

Figure 2: Experimental setup in a reverberant room.

It is interesting here to point out that, many experiments have been carried out regarding flow induced vibration of circular cylindrical shells. In the majority of these experiments the shell is either filled or surrounded by water from inside or outside. In these experiments the shaker or the hammer should be kept dry. This is usually very time consuming, or even at some circumstances an unaccomplished task to do. Whereas by using an acoustical excitation this problem is completely solved and the measurements of empty and fluid-filled shells could be done simultaneously without any change in the experimental setup.

Results and Discussions

A shell with such dimensions; $\frac{L}{R} = 22.67, \frac{R}{h} = 51.72$, is more complex than other cases, because this system exhibits; (1) axisymmetric (n=0), (2) beam-like (n=1) and (3) asymmetric (n>1) modes, all in low ranges of frequency (0 Hz-1000 Hz). Therefore the type of excitation employed and its capability in exciting all such modes is of great importance. In order to demonstrate the validity of the experiment and the type of excitation employed, an analytical approximate method was selected for comparison which was the direct solution of the Love theory.

In Table 1, the experimental and approximate analytical results, along with their relative errors are presented.

The mode shape identification is of crucial importance. Indeed, the high modal density in the case of shells makes it difficult to compare experimental and theoretical modes using natural frequencies only, therefore, the visualization is mandatory.

Comparisons reported in Table 1 show that the Love theory produces great results, especially in the case of the axisymmetric (*n*=0) and beam-like (*n*=1) modes, which are of great importance acoustically, and even from vibration point of view. Furthermore, it is remarkable how the theory predicts the fundamental frequency so accurately, with an error of less than 6%. Generally, the experimental and analytical results converge better at higher frequencies. Therefore it can be concluded that the Love theory is more accurate, for higher numbers of axial (*m*) and circumferential (*n*) waveparameter combinations. However, the theory endures its highest errors at *n*=2 and *n*=3 modes.

Table 1: Experimental results with their relative errors.

Longitudinal Wave Parameter	Circumferential Wave Parameter	Frequency (Hz)		
m	n	Experiment	Love	Error
1	1	138.40	138.93	5.9
1	2	190.30	172.94	0.4
2	2	310.50	244.78	11.5
3	2	496.60	424.15	9.1
1	3	502.20	471.46	21.2
2	3	477.00	481.44	14.6
3	3	558.90	514.47	1.2
2	1	464.70	518.02	6.1
4	3	638.30	586.64	0.9
4	2	679.80	688.23	8.0
5	3	782.00	706.05	8.1
6	3	833.80	870.62	9.7
1	0	842.50	892.08	4.4
1	4	884.40	902.29	2.0
2	4	887.00	906.75	2.2
3	4	981.60	917.34	6.6
4	4	945.80	938.28	0.8
5	4	964.70	974.47	1.0

In Figure 3 the lowest natural frequency of a shell f_{1mn}, is plotted as a function of the axial wave parameter/length ratio $\frac{m}{L}$. Two sets of data have been plotted in this figure: (1) the experimented shell with a thickness parameter (radius/thickness) of $\frac{R}{h} = 51.72$ (the solid line), and (2) a shell with a thickness parameter of $\frac{R}{h} = 500$ (the dashed line).

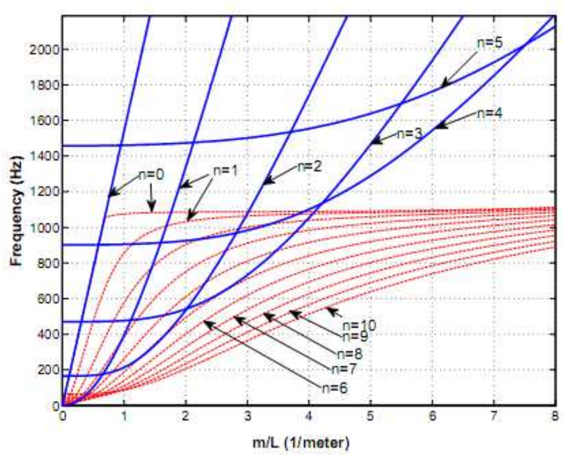

Figure 3: variation of the lowest natural frequency f_{1mn} with $\frac{m}{L}$: (1) $\frac{R}{h} = 51.72$ (____), (2) $\frac{R}{h} = 500$ (-----).

It is obvious from Figure 3 that, for two shells of the same length the one with a higher thickness parameter $\frac{R}{h}$ (larger radius or less thickness), yields lower natural frequencies. Moreover, as the length increases the magnitudes of the natural frequencies decrease likewise.

Meaning that, shorter shells have lower natural frequencies. It is also interesting to note that, for shells with larger thickness ratios frequencies are more closely gapped and the frequency lines rapidly converge to a certain frequency. Consequently in Figure 3 it can be observed for the thinner shell ($\frac{R}{h} = 500$), that after a certain frequency, namely 1200 Hz or less, there are no natural frequencies available. However on the other hand, shells with smaller thickness ratios, similar to that of the experiment, yield a wider variety of natural frequencies which are further gapped. Furthermore for shells of smaller thickness ratios, natural frequencies converge much slower. As a result, for such shells, there are numerous natural frequencies present at high frequency ranges. Therefore, shells with larger thickness ratios have higher modal densities compared to shells with smaller thickness ratios.

According to Figure 3, it is also interesting point out that the fundamental (lowest) frequency does not necessarily occur at the lowest or a certain *n*, but, the circumferential wave parameter *n*, of the fundamental mode, varies depending upon the length or radius of the shell.

In Figure 4 the lowest natural frequency of a shell f_{1mn}, is plotted versus circumferential wave parameter *n,* for shells with dimensions similar to that of the experiment, though, with two different boundary conditions: (1) simply supported and (2) clamped. Experimental results are also shown in Figure 4.

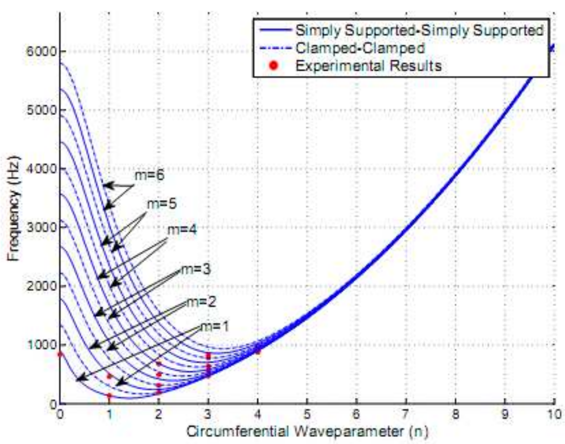

Figure 4: variation of the lowest natural frequency f_{1mn} with *n*.

As it can be seen in this figure, simply supported and clamped shells possess the same behavior, though with different frequencies. According to Figure 4, a shell with simply supported boundary conditions (experimented) yields lower natural frequencies compared to a shell with clamped boundary conditions. Moreover, the fundamental frequency always occurs at *m*=1, unlike the *n* component of the fundamental mode which is a variable, dependent upon the length/radius ratio. Furthermore as the frequency increases the frequency lines tend to converge and get closer. Hence, the modal density of a shell structure increases as the frequency increases. Therefore the mode identification process is much harder and complex at higher frequency ranges, where the modal density is higher.

Conclusion

The basic behavior of circular cylindrical shells has been examined using Love's equations. First, an aluminum cylindrical shell was chosen for experimental and analytical analysis. A shell with a long length was intentionally chosen, so that all three mode groups; (1) axisymmetric, (2) beam-like and (3) asymmetric were present at low frequency ranges. Therefore the validity of the Love theory and the experiment were confirmed for all these mode groups. It was also discovered that as *m* and *n* increase the accuracy of the theory increases likewise, resulting in errors less than 1% for high frequencies. Thus the theory could be used as an exact solution at high frequencies.

Moreover the effects of length and radius variations of a shell, on its natural frequencies, were studied in Figure 3. It was observed that, as the length or the radius/thickness ratio increase, the lowest natural frequencies of a shell decrease conversely. Moreover, shells with smaller radius/length ratios were found to have higher modal densities, especially at lower frequencies.

Finally the effects of boundary conditions were studied on a cylindrical shell. It was found out that clamped shells have higher natural frequencies compared to simply supported ones.

References

[1] W. Leissa, Vibration of Shells (NASA SP-288), US Government Printing Office, Washington DC, 1973.

[2] M. Amabili, M. P. Paidoussis, Review of studies on geometrically nonlinear vibrations and dynamics of circular cylindrical shells and panels, with and without fluid structure interaction, Applied Mechanics Reviews, vol. 56, 2003, 349-381.

[3] A. E. H. Love, On the small free vibrations and deformations of thin shells, Philosophical Transactions of the Royal Society (London), vol. 179A, 1888.

[4] W. Soedel, Vibrations of Shells and Plates, 3rd ed., Marcel Dekker, Inc., 2004.

[5] R. N. Arnold, G. B. Warburton, The flexural vibrations of thin cylinders, Institution of Mechanical Engineers, Proceeding 167, 1953, 62-80.

[6] F. Pellicano, Vibrations of circular cylindrical shells: theory and experiments, Journal of Sound and Vibration, vol. 303, 2007, 154-170.

[7] M. Amabili, Nonlinear Vibrations and Stability of Shells and Plates, Cambridge University Press, 2008.