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Topological rings of bounded and compact group homomorphisms on a topological ring

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Abstract. Let X be a topological ring. In this paper, we consider the three classes of nr-bounded, br-bounded and cr-continuous group homomorphisms defined on X and denote these classes by $B_{nr}(X), B_{br}(X)$ and $B_{cr}(X)$, respectively. We show that the operations of addition and product in $B_{nr}(X), B_{br}(X)$ and $B_{cr}(X)$ are continuous with respect to the topology considered on them. Moreover, we introduce the classes of nr-compact and br-compact group homomorphisms, and we study the situations where a class of bounded group homomorphisms coincides with the corresponding class of compact group homomorphisms.

Keywords: Bounded group homomorphism; Continuous group homomorphism; Compact group homomorphism; Topological ring.

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1 Introduction

A *topological ring* is a ring X which is also a topological space such that both the addition and the multiplication are continuous as maps from $X \times X$, equipped with the product topology, into X. There are many examples of topological rings among which we are interested in ring of scalar valued, ring of bounded linear operators on a topological vector space and ring of continuous real-valued functions on some topological space (where the topology is given by pointwise convergence). A subset B of X is said to be *bounded* if for each zero neighborhood V, there exists a zero neighborhood U such

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that $BU \subseteq V$ and $UB \subseteq V$. Also, $D \subseteq X$ is said to be *relatively compact* if its closure is compact.

In this paper, taking idea from [5], we define some different notions of boundedness for group homomorphisms between topological rings. We also consider some types of compact group homomorphisms between topological rings. Each class of bounded group homomorphisms enjoys a natural topology that makes it to a topological ring. We obtain some similar results for continuous group homomorphisms. Finally, we investigate some relations between bounded group homomorphisms and compact ones. For ample information about topological rings and the related notions, the reader is referred to [1, 2, 4] and [6].

Throughout the paper X and Y will denote topological rings. Every ring is assumed to be associative.

2 bounded group homomorphisms

Definition 2.1. Let X and Y be two topological rings. A group homomorphism $T : X \to Y$ is said to be

i. nr-bounded if there exists a zero neighborhood $U \subseteq X$ such that $T(U)$ is bounded in Y.

ii. br-bounded if for every bounded subset $B \subseteq X$, $T(B)$ is bounded in Y.

The class of all nr-bounded group homomorphisms from X into Y is denoted by $B_{nr}(X, Y)$ and is equipped with the topology of uniform convergence on some zero neighborhood, namely, a net (S_α) of nr-bounded group homomorphisms uniformly converges to zero on some zero neighborhood $U \subseteq X$ if for each zero neighborhood $V \subseteq Y$ there is an α_0 such that $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$. The class of all br-bounded group homomorphisms from X into Y is denoted by $B_{br}(X, Y)$ and is equipped with the topology of uniform convergence on bounded sets. Recall that a net (S_α) of br-bounded group homomorphisms uniformly converges to zero on a bounded subset $B \subseteq X$ if for each zero neighborhood $V \subseteq Y$ there is an α_0 such that $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$.

We use the notations $B_{nr}(X)$ and $B_{br}(X)$ for $B_{nr}(X, X)$ and $B_{br}(X, X)$, respectively. Since the sum of two bounded subsets in a topological ring is bounded (see [4]), it is not difficult to see that $B_{nr}(X)$ and $B_{br}(X)$ are subrings of the ring of all group homomorphisms on X. Notice that the above statements are not equivalent in general. To see this, consider the following examples.

Example 2.1. Let $X = \mathbb{R}^{\mathbb{N}}$, the space of all real sequences, with the coordinate-wise topology. Since the rectangular zero neighborhoods

$$\aleph = \{(-\varepsilon_1, \varepsilon_1) \times \ldots \times (-\varepsilon_k, \varepsilon_k) \times \mathbb{R} \times \mathbb{R} \times \ldots\}_{\varepsilon_i > 0, k \in \mathbb{N}},$$

forms a zero local basis for X, we only need to consider those zero neighborhoods in X that are of this form. It is easy to see that with the pointwise product, X is a topological ring. Consider the identity group homomorphism I on X. Indeed, I is br-bounded. But it is not nr-bounded. Suppose on a contrary, there is a zero neighborhood U which is bounded. Suppose that U is of the form

$$U = (-\gamma_1, \gamma_1) \times \ldots \times (-\gamma_r, \gamma_r) \times \mathbb{R} \times \mathbb{R} \times \ldots,$$

in which $\gamma_i > 0$. Consider the zero neighborhood

$$V = (-1, 1) \times \ldots (-1, 1) \times \mathbb{R} \times \mathbb{R} \times \ldots,$$

for which the number of intervals of the form $(-1, 1)$ is $r + 1$ times. Now, suppose that W is an arbitrary zero neighborhood in X. We can assume that W is of the form

$$W = (-\delta_1, \delta_1) \times \ldots \times (-\delta_m, \delta_m) \times \mathbb{R} \times \mathbb{R} \times \ldots,$$

that $\delta_i > 0$. So, it is not difficult to verify that $UW \nsubseteq V$ and $WU \nsubseteq V$. This shows that I is not an nr-bounded group homomorphism.

Example 2.2. Consider l^∞, the space of all bounded real sequences. Let X be l^∞ with the zero multiplication and topology induced by the uniform norm and Y be l^∞ with the pointwise product and the uniform norm topology. It is easy to see that X and Y are topological rings. Consider the identity group homomorphism $I : X \to Y$. I is nr-bounded. For, if $N_1^{(0)}$ is the unit ball with center zero, then it is a zero neighborhood in X which is bounded in Y. But we show that I is not a br-bounded group homomorphism. Suppose that $(a_n) \subseteq l^\infty$ is the sequence defined by $a_n = (1, 2, \ldots, n, 0, 0, \ldots)$. Indeed, (a_n) is bounded in X but it is not bounded in Y. For, if $\varepsilon > 0$ is arbitrary, then $(a_n)N_\varepsilon^{(0)} \nsubseteq N_1^{(0)}$.

Note that when X is a topological algebra with unity, we have two notions for boundedness since X is a topological vector space and a topological ring, too. In the following proposition, we show that these concepts are in fact the same. Recall that if X is a topological vector space, $B \subseteq X$ is said to be bounded if for each zero neighborhood V there is an $\alpha > 0$ such that $B \subseteq \alpha V$. For a study about topological vector spaces the reader is referred to [3].

Proposition 2.1. Let X be a topological algebra with unity. Then the two concepts of boundedness will be equivalent.

Proof. Consider X as a topological vector space and suppose that $B \subseteq X$ is bounded. Let W be an arbitrary zero neighborhood. So there is a zero neighborhood V such that $VV \subseteq W$. There is an $\alpha > 0$ such that $B \subseteq \alpha V$. Therefore, $BV \subseteq (\alpha V)V \subseteq \alpha W$, so that $B(\frac{V}{\alpha}) \subseteq W$. For the converse, consider X as a topological ring and let $B \subseteq X$ be bounded. Suppose that W is an arbitrary zero neighborhood. So, there is a zero neighborhood V such that $BV \subseteq W$. We claim that there is an $\alpha > 0$ such that $B \subseteq \alpha W$. Suppose on a contrary, for each $n \in \mathbb{N}$, $B \nsubseteq nW$. Thus, there is a sequence $(x_n) \subseteq B$ such that $\frac{x_n}{n} \notin W$. Since $X = \cup_{n=1}^{\infty} nV$, there is an $m \in \mathbb{N}$ such that $1 \in mV$ (1 is the unit of X). So, $\frac{1}{m} \in V$. This means $\frac{x_m}{m} \in BV \subseteq W$, so that $\frac{x_m}{m} \in W$ which is a contradiction. □

Remark 2.1. If X is a topological algebra which has no unity, then the result of Proposition 2.1 is not necessarily true. Put $X = \mathbb{R}^{\mathbb{N}}$ as a topological algebra with the coordinate-wise topology and the zero multiplication. Then every subset of X is bounded if X is considered as a topological ring but indeed X is not a bounded subset if we consider X as a topological vector space. Also, if A is normed algebra with unity and $B(A)$ is the set of all bounded group homomorphisms on A, we can equip $B(A)$ with the topology induced by the uniform norm. It this case, it is easy to see that the spaces $B(A)$, $B_{nr}(A)$ and $B_{br}(A)$ and their corresponding topologies will be equivalent.

In the following, Some parts of the proofs are similar to the proofs of Theorems 2.1 and 2.4 in [7] for $B_n(X)$ and $B_b(X)$, respectively.

Theorem 2.1. *The operations of addition and product in $B_{nr}(X)$ are continuous with respect to the topology of uniform convergence on some zero neighborhood.*

Proof. Suppose that two nets (T_α) and (S_α) of nr-bounded group homomorphisms converge to zero uniformly on some zero neighborhoods $U_1, U_2 \subseteq X$, respectively. Find a zero neighborhood U such that $U \subseteq U_1 \cap U_2$. Let W be an arbitrary zero neighborhood in X. So, there is a zero neighborhood V such that $V + V \subseteq W$. There are some α_0 and α_1 such that $T_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(U) \subseteq V$ for each $\alpha \geq \alpha_1$. Choose an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(T_\alpha + S_\alpha)(U) \subseteq T_\alpha(U) + S_\alpha(U) \subseteq V + V \subseteq W$. Thus, the addition is continuous. Now, we show the continuity of the product. Find an α_3 such that $S_\alpha(U) \subseteq U$ for each $\alpha \geq \alpha_3$. Take an α_4 such that $T_\alpha(U) \subseteq W$ for each $\alpha \geq \alpha_4$. Choose an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. If $\alpha \geq \alpha_5$ then $T_\alpha S_\alpha(U) \subseteq T_\alpha(U) \subseteq W$, as asserted. □

Remark 2.2. The class $B_{nr}(X)$ can contain a Cauchy sequence whose limit is not an nr-bounded group homomorphism. To see this, let $X = \mathbb{R}^{\mathbb{N}}$ with the topology of coordinate-wise convergence. Let P_n be the projection on the first n components. Each P_n maps the zero neighborhood

$$U_n = \{(x_i)_{i=1}^{\infty}, |x_i| < 1 \text{ for } i = 1, 2, \ldots, n\},$$

into a bounded set. For, if W is an arbitrary zero neighborhood, we can assume that W is of the form

$$W = (-\varepsilon_1, \varepsilon_1) \times \ldots \times (-\varepsilon_r, \varepsilon_r) \times \mathbb{R} \times \mathbb{R} \times \ldots,$$

in which $\varepsilon_i > 0$. Put $V = W$, then $P_n(U_n)V \subseteq W$ and $V P_n(U_n) \subseteq W$. On the other hand, $(P_n - P_m)(X) \subseteq W$ for each $m, n > r$. This shows that (P_n) is a Cauchy sequence in $B_{nr}(X)$. Also, (P_n) converges uniformly to the identity group homomorphism I on X. But we have seen in Example 2.1 that I is not nr-bounded.

Theorem 2.2. *The operations of addition and product in $B_{br}(X)$ are continuous with respect to the topology of uniform convergence on bounded sets.*

Proof. Suppose that two nets (T_α) and (S_α) of *br*-bounded group homomorphisms converge uniformly to zero on bounded sets. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. Find a zero neighborhood V such that $V + V \subseteq W$. There are some α_0 and α_1 such that $T_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(B) \subseteq V$ for each $\alpha \geq \alpha_1$. Take an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$ then $(T_\alpha + S_\alpha)(B) \subseteq T_\alpha(B) + S_\alpha(B) \subseteq V + V \subseteq W$. So, the addition is continuous. Now, we show the continuity of the product. Since the net (S_α) converges uniformly on B, it is eventually bounded (see [4]). So, there is an α_3 such that $E = \cup_{\alpha \geq \alpha_3} S_\alpha(B)$ is bounded. find an α_4 such that $T_\alpha(E) \subseteq W$ for each $\alpha \geq \alpha_4$. Choose an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. Thus, for all $\alpha \geq \alpha_5$, $T_\alpha S_\alpha(B) \subseteq T_\alpha(E) \subseteq W$, as we wanted. □

Proposition 2.2. Suppose that a net (h_α) of *br*-bounded group homomorphisms converges to a group homomorphism h uniformly on bounded sets. Then h is also *br*-bounded.

Proof. Fix a bounded set $B \subseteq X$. Let W be an arbitrary zero neighborhood in X. There is a zero neighborhood V such that $V + V \subseteq W$. Choose a zero neighborhood $V_1 \subseteq V$ such that $V_1 V_1 \subseteq V$. There is an α_0 such that $(h_\alpha - h)(B) \subseteq V_1$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. So, there is a zero neighborhood $V_2 \subseteq V_1$ such that $h_\alpha(B)V_2 \subseteq V_1$ and $V_2 h_\alpha(B) \subseteq V_1$. Therefore,

$$h(B)V_2 \subseteq h_\alpha(B)V_2 + V_1 V_2 \subseteq V_1 + V_1 V_1 \subseteq V + V \subseteq W.$$

Similarly, $V_2 h(B) \subseteq W$, as desired. □

3 Continuous group homomorphisms

The class of all continuous group homomorphisms from X into Y is denoted by $B_{cr}(X, Y)$ and is equipped with the topology of *cr*-convergence that means a net (S_α) of continuous group homomorphisms *cr*-converges to zero if for each zero neighborhood $W \subseteq Y$ there is a zero neighborhood $U \subseteq X$ such that for each zero neighborhood $V \subseteq Y$ there is an α_0 such that $S_\alpha(U) \subseteq VW$ for each $\alpha \geq \alpha_0$. We use the symbol $B_{cr}(X)$ for $B_{cr}(X, X)$. It is obvious that $B_{cr}(X)$ is a subring of the ring of all group homomorphisms on X. First note that there is no relation between continuous group homomorphisms and bounded ones. Consider the following example.

Example 3.1. Consider c_o, the space of all real vanishing sequences. Suppose that X is c_o as a topological ring with the pointwise multiplication and the coordinate-wise topology. Let Y be c_o as a topological ring with the zero multiplication and the uniform norm topology. Consider the identity group homomorphism $I : X \to Y$. Indeed, I is *br*-bounded and *nr*-bounded but it is not continuous.

Theorem 3.1. *The operations of addition and product in $B_{cr}(X)$ are continuous with respect to the topology of cr-convergence.*

Proof. Suppose that two nets (T_α) and (S_α) of continuous group homomorphisms *cr*-converge to zero. Let W be an arbitrary zero neighborhood in X. Choose a zero neighborhood V such that $V + V \subseteq W$. There is a zero neighborhood U such that for any zero neighborhood V_0 there are some α_0 and α_1 such that $T_\alpha(U) \subseteq V_0V$ for each $\alpha \geq \alpha_0$ and $S_\alpha(U) \subseteq V_0V$ for each $\alpha \geq \alpha_1$. Find an α_2 such that $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. If $\alpha \geq \alpha_2$, then $(T_\alpha + S_\alpha)(U) \subseteq T_\alpha(U) + S_\alpha(U) \subseteq V_0V + V_0V = V_0(V + V) \subseteq V_0W$. So, the addition is continuous. Now, we show that the product is continuous. Choose a zero neighborhood V_1 such that $V_1V_1 \subseteq U$. There are a zero neighborhood V_2 and an α_3 such that $S_\alpha(V_2) \subseteq V_1V_1$ for each $\alpha \geq \alpha_3$. Also, there is an α_4 such that $T_\alpha(U) \subseteq V_0W$ for all $\alpha \geq \alpha_4$. Find an α_5 such that $\alpha_5 \geq \alpha_3$ and $\alpha_5 \geq \alpha_4$. If $\alpha \geq \alpha_5$, then $T_\alpha S_\alpha(V_2) \subseteq T_\alpha(V_1V_1) \subseteq T_\alpha(U) \subseteq V_0W$, as desired. □

Proposition 3.1. Suppose that a net (h_α) of continuous group homomorphisms *cr*-converges to a group homomorphism h. Then h is also continuous.

Proof. Let $W \subseteq X$ be an arbitrary zero neighborhood. Choose a zero neighborhood V such that $V + VV \subseteq W$. There are a zero neighborhood U_0 and an α_0 such that $(h_\alpha - h)(U_0) \subseteq VV$ for each $\alpha \geq \alpha_0$. Fix an $\alpha \geq \alpha_0$. So, there is a zero neighborhood $U_1 \subseteq U_0$ such that $h_\alpha(U_1) \subseteq V$ and therefore $h(U_1) \subseteq h_\alpha(U_1) + VV \subseteq V + VV \subseteq W$, as we wanted. □

4 Compact group homomorphisms

Taking idea for compact operators on topological vector spaces in [5], a group homomorphism $T : X \to Y$ is said to be *nr*-compact if there is a zero neighborhood $U \subseteq X$ such that $T(U)$ is relatively compact in Y. Also, T is *br*-compact if T maps bounded subsets into relatively compact sets. Since every compact subset is bounded (see [4]), indeed, every *nr*-compact group homomorphism is *nr*-bounded and every *br*-compact group homomorphism is *br*-bounded. We use the notations $K_{nr}(X)$ and $K_{br}(X)$ to show the set of all *nr*-compact group homomorphisms, the set of all *br*-compact group homomorphisms on a topological ring X.

Proposition 4.1. $K_{br}(X) = B_{br}(X)$ if and only if X has the Heine-Borel property.

Proof. $K_{br}(X) = B_{br}(X) \Leftrightarrow I \in K_{br}(X) \Leftrightarrow X$ has the Heine-Borel property,
in which I means the identity group homomorphism on X. □

Remark 4.1. Note that when X has the Heine-Borel property, then $K_{nr}(X) = B_{nr}(X)$. Let $X = \mathbb{R}^{\mathbb{N}}$ with the coordinate-wise topology and the pointwise product. If $B \subseteq X$ is bounded, it is the product of bounded subsets of $\mathbb{R}^p$ that $p \in \mathbb{N}$. Using the Tychonoff theorem to show that the closure of any bounded set is compact. This shows that $K_{br}(X) = B_{br}(X)$. So, X has the Heine-Borel property and therefore $K_{nr}(X) = B_{nr}(X)$.

Compact group homomorphisms are not closed in the topologies induced by the corresponding bounded group homomorphisms. To see these interesting results, consider the following examples.

Example 4.1. $K_{nr}(X)$ is not a closed subring of $B_{nr}(X)$, in general. Let X be $\mathbb{R}^{\mathbb{N}}$ with the coordinate-wise topology and the zero multiplication. Suppose that P_n is the projection on the first n components. Each P_n is nr-compact because it maps the zero neighborhood

$$U_n = \{(x_i)_{i=1}^{\infty}, |x_i| < 1 \text{ for } i = 1, 2, \ldots, n\},$$

into a relatively compact set. Also, (P_n) converges uniformly to the identity group homomorphism I on X. But it is not difficult to see that I is nr-bounded but it is not nr-compact.

Example 4.2. $K_{br}(X)$ fails to be closed in $B_{br}(X)$, in general. Let X be c_0, the space of real vanishing sequences, with the coordinate-wise topology and the pointwise product. It is easy to see that X is a topological ring. Let P_n be the projection on the first n components. By using the Tychonoff theorem, we can conclude that for each $n \in \mathbb{N}$, P_n is br-compact. Also, (P_n) converges uniformly on bounded sets to the identity group homomorphism I. We show that I is not br-compact. Suppose that B is the sequence (a_n) defined by $a_n = (1, 1, \ldots, 1, 0, 0, \ldots)$ in which 1 is appeared n times. B is Cauchy in c_0, so that it is bounded. Also note that $\overline{B} = B$. Now, if $I \in K_{br}(c_0)$, then B should be compact. Since B is not complete, this is impossible.

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