THE NON-ABELIAN TENSOR SQUARE AND SCHUR MULTIPLIER OF GROUPS OF ORDERS p^2q AND p^2qr

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Abstract. The aim of this paper is to determine the non-abelian tensor square and Schur multiplier of groups of square free order and of groups of orders p^2q, pq^2 and p^2qr, where p, q and r are primes and $p < q < r$.

1. Introduction

The notion of non-abelian tensor product $G\otimes H$ of groups G and H, which was introduced by R. Brown and J.-L. Loday [2, 3], is a generalization of the usual tensor product of the abelian groups. Let there be actions

$$G \times H \longrightarrow G, (g, h) \longmapsto^{h} g \quad ; \quad H \times G \longrightarrow H, (h, g) \longmapsto^{g} h$$

in such a way that for all $g, g_1 \in G$ and $h, h_1 \in H$,

$$^{^{g_1}h}g = \ ^{g_1^{-1}}(^{h}(^{g_1}g)) \quad \text{and} \quad ^{^{h_1}g}h = \ ^{h_1^{-1}}(^{g}(^{h_1}h)), \qquad (\star)$$

where G acts on itself by conjugation $(g, g_1) \longmapsto^{g} g_1 = gg_1g^{-1}$, and H acts on itself similarly. Then the non-abelian tensor product $G\otimes H$ is defined to be the group generated by symbols $g \otimes h$, for all $g \in G, h \in H$, subject to the relations

$$g_1g \otimes h = (^{g_1}g \otimes \, ^{g_1}h)(g_1 \otimes h) \quad , \quad g \otimes h_1h = (g \otimes h_1) \ (^{h_1}g \otimes \, ^{h_1}h)$$

for all $g, g_1 \in G$, $h, h_1 \in H$. In the case $G = H$ and G acts on itself by conjugation, $G \otimes G$ is called the non-abelian tensor square of G.

Following the publications of Brown and Loday's work, a number of purely group theoretic papers have appeared on this topic. Some of them investigate structural properties of the

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tensor square, while the others are devoted to explicit descriptions for particular groups, for instance dihedral, quaternionic, symmetric and all groups of order at most 30 in [1].

Later, Hannebauer [7] determined the structure of tensor square of the linear groups $SL(2,q)$, $PSL(2,q)$, $GL(2,q)$ and $PGL(2,q)$ for all $q \geqslant 5$ and $q \neq 9$.

Ellis and Leonard [4] devised a computer algorithm for the computation of tensor square of finite groups which can handle much larger groups than those given in [1]. Using the CAYLEY-program, they compute the tensor square of $B(2,4)$, the 2-generator Burnside group of exponent 4, where $|B(2,4)| = 2^{12}$. Recently, Hannebauer's result is improved for the linear groups $SL(n,q), PSL(n,q), GL(n,q)$ and $PGL(n,q)$ for all $n, q \geq 2$ in [5]. These results extremely depend on knowing the order of $M(G)$, the Schur multiplier of a group G. Also, some computations of $G \otimes G$ for polycyclic groups have been done in [9].

In the present paper, we determine the non-abelian tensor square of the groups of square free order and groups of orders p^2q, pq^2 and p^2qr, where p, q and r are primes and $p < q < r$. Here we give some notations which will be used throughout the paper.

G^{ab}	Abelianisation of G,
$e(G)$	exponent of G,
Q_2	quaternion group of order 8,
A_4	alternating goup of order 12,
$(\mathbb{Z}_{p^t})^k$	direct product of k copies of the cyclic group of order p^t.

Theorem A. *Let G be a group of order n, where n is a square free number. Then*

$$G \otimes G \cong \mathbb{Z}_n.$$

Theorem B. *Let G be a group of order p^2q, where p and q are prime numbers and $p < q$. The structure of $G \otimes G$ is one of the following*

(i) If $G^{ab} = \mathbb{Z}_{p^2}$, then $G \otimes G \cong \mathbb{Z}_{p^2q}$.

(ii) If $G^{ab} = (\mathbb{Z}_p)^2$, then $G \otimes G \cong (\mathbb{Z}_p)^4 \times \mathbb{Z}_q$.

(iii) If $G^{ab} = \mathbb{Z}_3$, then $G \otimes G \cong \mathbb{Z}_3 \times Q_2$.

Theorem C. *Let G be a group of order pq^2, where p and q are prime numbers and $p < q$.*

The structure of $G \otimes G$ is one of the following

(i) If $G' = \mathbb{Z}_{q^2}$, then $G \otimes G \cong \mathbb{Z}_{pq^2}$.

(ii) If $G' = (\mathbb{Z}_q)^2$ and $M(G) = 0$ or if $G' = \mathbb{Z}_q$, then $G \otimes G \cong \mathbb{Z}_p \times (\mathbb{Z}_q)^2$.

(iii) If $G' = (\mathbb{Z}_q)^2$ and $M(G) = \mathbb{Z}_q$, then $G \otimes G \cong \mathbb{Z}_p \times H$, where H is an extra-special q-group of order q^3.

Theorem D. *Let G be a group of order p^2qr, where p, q and r are prime numbers $p < q < r$, $pq \neq 6$. The structure of $G \otimes G$ is one of the following*

(i) If $|G'| = q$, r or qr where $q \not\equiv 1 \pmod r$, then $G \otimes G \cong \mathbb{Z}_{p^2qr}$ when G^{ab} is cyclic, otherwise $G \otimes G \cong (\mathbb{Z}_p)^4 \times \mathbb{Z}_{qr}$.

(ii) If $|G'| = qr$ where $q \equiv 1 \pmod r$, then $G \otimes G \cong \mathbb{Z}_{p^2} \times G'$ when G^{ab} is cyclic, otherwise $G \otimes G \cong (\mathbb{Z}_p)^4 \times G'$.

2. Basic Results

In this section we recall some definitions and basic results on the tensor square which are necessary for our main theorems.

Let G be a group and $G \otimes G$ be the tensor square of G. The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1_{\otimes}$ $(g \in G)$ on $G \otimes G$. Moreover, we denote by $\nabla(G)$ the subgroup of $G \otimes G$ generated by all elements $g \otimes g$ for all $g \in G$. The commutator map induces homomorphisms $\kappa : G \otimes G \to G$ and $\kappa' : G \wedge G \to G$ sending $g \otimes h$ and $g \wedge h$ to $[g, h] = ghg^{-1}h^{-1}$. The kernel of κ is denoted by $J_2(G)$.

Results in [2, 3] give the following commutative diagram with exact rows and central extensions as columns

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\Gamma(G^{ab}) \arrow[r] \arrow[d, equal] & J_2(G) \arrow[r] \arrow[d] & M(G) \arrow[r] \arrow[d] & 0 \\
\Gamma(G^{ab}) \arrow[r] & G \otimes G \arrow[r] \arrow[d, "\kappa"'] & G \wedge G \arrow[r] \arrow[d, "\kappa'"'] & 1 \\
 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
 & 1 & 1 &
\end{tikzcd}
\tag{2.1}
$$

where Γ is the Whitehead's quadratic functor (see Whitehead [10]).

For the sake of convenience of the reader we state some known results which are used in the proof of the main theorems.

Theorem 2.1. [1, Proposition 8]. *If G is a group in which G' has a cyclic complement C, then $G \otimes G \cong (G \wedge G) \times G^{ab}$ and hence $|G \otimes G| = |G||M(G)|$.*

Theorem 2.2. [6, Theorem A]. *Let G be a group such that $G^{ab} = \prod_{i=1}^{n}\prod_{j=1}^{k_i} \mathbb{Z}_{p_i^{e_{ij}}}$ where $1 \le e_{i1} \le e_{i2} \le ... \le e_{ik_i}$ for all $1 \le i \le n$, $k_i \in \mathbb{N}$ and $p_i \neq 2$. Then*

$$|G \otimes G| = \prod_{i=1}^{n} p_i^{d_i}|G||M(G)|$$

in which $d_i = \sum_{j=1}^{k_i}(k_i - j)e_{ij}$.

3. Proof of Main Theorems

In this section we prove the main theorem as we mentioned earlier in section one.

Lemma 3.1. *Let G be a finite non-abelian group.*

(i) If G is a square free order group, then G' is cyclic.

(ii) If G is a group of order p^2q, then $G' = \mathbb{Z}_q$ or $G' = (\mathbb{Z}_2)^2$.

(iii) If G is a group of order pq^2, then $G' = \mathbb{Z}_q$, $G' = \mathbb{Z}_{q^2}$ or $G' = (\mathbb{Z}_q)^2$.

(iv) If G is a group of order p^2qr and $pq \neq 6$, then $|G'| = q, r$ or qr.

(v) If G is a group of order p^2qr and $pq = 6$, then $|G'| = 3, r, 3r, 4$ or $4r$.

Proof. One may use Sylow theorems for examples in case (ii) to show the number of Sylow p-subgroups of G is 1 or q and the number of Sylow q-subgroups of G is 1 or p^2. If G has one Sylow q-subgroup Q, then G/Q is abelian and so $G' = \mathbb{Z}_q$. Otherwise $p = 2$ and $q = 3$, hence $|G| = 12$ and we should have $G \cong A_4$.

The proof of other cases is similar and we omit it. □

Lemma 3.2. *Let G be a finite non-abelian group.*

(i) If G is a square free order group, then $M(G) = 0$.

(ii) If G is a group of order p^2q, then

$$M(G) = \begin{cases} 0 & if\ G' = \mathbb{Z}_q \text{ and } G^{ab} = \mathbb{Z}_{p^2} \\ \mathbb{Z}_p & if\ G' = \mathbb{Z}_q \text{ and } G^{ab} = (\mathbb{Z}_p)^2 \\ \mathbb{Z}_2 & if\ G' = (\mathbb{Z}_2)^2 \end{cases}$$

(iii) If G is a group of order pq^2, then

$$M(G) = \begin{cases} 0 & if\ G' = \mathbb{Z}_q \\ 0 & if\ G' = \mathbb{Z}_{q^2} \\ 0\ or\ \mathbb{Z}_q & if\ G' = (\mathbb{Z}_q)^2 \end{cases}$$

(iv) If G is a group of order p^2qr and $pq \neq 6$, then

$$M(G) = \begin{cases} 0 & if\ G^{ab}\ is\ cyclic \\ \mathbb{Z}_p & otherwise \end{cases}$$

(v) If G is a group of order p^2qr and $pq = 6$, then $M(G)$ is as the same as part (iv). Moreover if $|G'| = 4$ or $4r$, then $M(G) = \mathbb{Z}_2$.

Proof. (i) Since all Sylow subgroups of G are cyclic, so $M(G) = 0$. For (ii), if $G' = \mathbb{Z}_q$ *and* $G^{ab} = \mathbb{Z}_{p^2}$, then again all Sylow subgroups of G are cyclic and therefore $M(G) = 0$. In the case $G' = \mathbb{Z}_q$ *and* $G^{ab} = (\mathbb{Z}_p)^2$, we can see that $|G'|$ and $|G^{ab}|$ are coprime, so by Schur-Zassenhaus lemma, G' has a complement. The result follows from [8, Corollary 2.2.6]. The proof of the other parts is similar. □

Lemma 3.3. *Let G be a finite non-abelian group.*

(i) If G is a square free order group of order n, then $|G \otimes G| = n$.

(ii) If G is a group of order p^2q, then

$$|G \otimes G| = \begin{cases} p^2q & if\ G^{ab} = \mathbb{Z}_{p^2} \\ p^4q & if\ G^{ab} = (\mathbb{Z}_p)^2 \\ 24 & if\ G^{ab} = \mathbb{Z}_3 \end{cases}$$

(iii) If G is a group of order pq^2, then

$$|G \otimes G| = \begin{cases} pq^2 & if\ G' = \mathbb{Z}_{q^2} \\ pq^2 & if\ G' = \mathbb{Z}_q\ or\ G' = (\mathbb{Z}_q)^2\ and\ M(G) = 0 \\ pq^3 & if\ G' = (\mathbb{Z}_q)^2\ and\ M(G) = \mathbb{Z}_q \end{cases}$$

(iv) If G is a group of order p^2qr and $pq \neq 6$, then

$$|G \otimes G| = \begin{cases} p^2qr & if\ G^{ab}\ is\ cyclic \\ p^4qr & otherwise \end{cases}$$

(v) If G is a group of order p^2qr and $pq = 6$, then the order of $G \otimes G$ is similar to the part (iv). Moreover if $|G'| = 4$ or $4r$, then $|G \otimes G| = 24r$.

Proof. (i) Since $|G'|$ and $|G^{ab}|$ are coprime, G' has a cyclic complement and $|G \otimes G| = |G|$ by Schur-Zassenhaus lemma, Theorem 2.1 and Lemma 3.2.

(ii) Suppose that $G^{ab} = \mathbb{Z}_{p^2}$. In this case $|G'|$ and $|G^{ab}|$ are coprime and $|G \otimes G| = |G| = p^2q$ by Lemma 3.2.

Now assume that $G^{ab} = (\mathbb{Z}_p)^2$, if $p \neq 2$, then $|G \otimes G| = p^4q$ by Theorem 2.2 and Lemma 3.2. Otherwise $e(\nabla(G))$ divides $e(G) = 2q$ and $e(\Gamma(G^{ab})) = 4$. Hence $e(\nabla(G)) = 2$ and $\nabla(G) = (\mathbb{Z}_2)^3$ and it implies that $|G \otimes G| = 2^4q$.

We note that if $G^{ab} = \mathbb{Z}_3$, then $G = A_4$ and this case is computed in [1].

(iii) Suppose that G' is cyclic of order q^2, so we have $|G \otimes G| = pq^2$ by Theorem 2.1 and Lemma 3.2. The case that $G' = (\mathbb{Z}_q)^2$ and $M(G) = 0$ holds similarly.

Now if $|G'| = q$ and $p \neq 2$, then $|G \otimes G| = pq^2$ by Theorem 2.2 and Lemma 3.2. Also if $p = 2$, then it is easy to see that $|G \otimes G| = 2q^2$.

(iv) Suppose that G^{ab} is cyclic, so $|G \otimes G| = |G|$ by Theorems 2.1 and Lemma 3.2.

The other case is similar to the case (ii).

(v) It is straight forward. □

We are ready to prove main theorems. Notice that if G' is cyclic, then $G \otimes G$ is abelian.

Proof of Theorem A. Since G is a group of square free order, then $G \otimes G$ is abelian of order n by Lemmas 3.1 and 3.3.

Proof of Theorem B. (i) It is clear that $G \otimes G$ is an abelian group of order p^2q by Lemma 3.3. On the other hand, $e(G \otimes G)$ divides $|G \otimes G|$ and the epimorphism $\pi : G \otimes G \longrightarrow G^{ab} \otimes G^{ab}$ implies that $e(G \otimes G) = p^2q$. Hence $G \otimes G \cong \mathbb{Z}_{p^2q}$, as required.

(ii) It is as same as the case (i).

(iii) If $G' = (\mathbb{Z}_2)^2$, then $G = A_4$ and one may refer to the Table 1 given in [1].

Proof of Theorem C. (i) The exponent of $G \otimes G$ is equal to pq^2, so the proof follows by Lemma 3.3.

(ii) Since $G \otimes G/J_2(G) = G'$ is abelian, we have $(G \otimes G)' \subseteq J_2(G)$, where $|J_2(G)| = p$. In addition the epimorphism π implies that $(G \otimes G)'$ is in it's kernel which is of order q^2, so $(G \otimes G)' = 1$. The result holds by Lemma 3.3 and the fact that $e(G \otimes G) = pq$.

(iii) It is clear that $\nabla(G)$ and the Sylow q-subgroup of $G \otimes G$, say H, are normal. Thus by Lemma 3.3,

$$G \otimes G \cong \nabla(G) \times H,$$

in which $\nabla(G) = \mathbb{Z}_p$ and H is of order q^3.

Proof of Theorem D. (i) If G^{ab} is cyclic, then the epimorphism π implies that $e(G \otimes G) = p^2qr$, so the result follows by Lemma 3.3.

If G^{ab} is not cyclic, the proof is similar.

(ii) Assume that $G^{ab} = \mathbb{Z}_{p^2}$, then $|\nabla(G)| = |J_2(G)| = p^2$ and it can be easily seen that Ker π is isomorphic to G' and of order qr. Moreover since $e(G \otimes G) = p^2qr$, Lemma 3.3 implies that

$$G \otimes G \cong \nabla(G) \times \mathrm{Ker}\ \pi \cong \nabla(G) \times G' \cong \mathbb{Z}_{p^2} \times G'.$$

If $G^{ab} = (\mathbb{Z}_p)^2$, then $e(G \otimes G) = pqr$, $|J_2(G)| = p^4$ and Ker $\pi \cong G'$. Therefore it follows from Lemma 3.3 that

$$G \otimes G \cong J_2(G) \times \mathrm{Ker}\ \pi \cong (\mathbb{Z}_p)^4 \times G'.$$

Finally, we note that in Theorem D, if $pq = 6$, i.e. $|G| = 12r$, then $|G'| = 3, r, 3r, 4$ or $4r$. In the cases that $|G'| = 3, r$ or $3r$, the structure of $G \otimes G$ is similar to Theorem D. In other cases one may show that $G \otimes G \cong \mathbb{Z}_{3r} \times Q_2$.

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References

[1] R. Brown, D. L. Johnson, E. F. Robertson, Some Computations of Non-Abelian Tensor Products of Groups, *J. of Algebra.* **111** (1987) 177-202.

[2] R. Brown, J.-L. Loday, Excision homotopique en basse dimension. *C.R. Acad. Sci. Paris Sér. I Math* **298** (1984) 353-356.

[3] R. Brown, J.-L Loday, Van Kampen theorems for diagrams of spaces, With an appendix by M. Zisman, *Topology* **26** (1987) 311-335.

[4] G. J. Ellis, F. Leonard, Computing Schur multipliers and tensor products of finite groups, *Proc. Royal Irish Acad,* **95A** (1995) 137-147.

[5] A. Erfanian, R. Rezaei, S. H. Jafari, Computing the nonabelian tensor square of general linear groups, *Italian J. of Pure and Applied Math,* **24** (2008) 203-210.

[6] A. Erfanian, S. H. Jafari, R. Rezaei, On the order of nonabelian tensor square of finite groups, *Submitted.*

[7] T. Hannebauer, On non-abelian tensor squares of linear groups, *Arch. Math,* **55** (1990) 30-34.

[8] G. Karpilovsky, *The Schur multiplier,* London Math. Soc. Monographs, New Series no. 2, 1987.

[9] R. F. Mores, Advances in computing the non-abelian tensor square of polycyclic groups, *Irish Math. Soc. Bulletin,* **56** (2005) 115-123.

[10] J. H. C. Whitehead, A certain exact sequence, *Ann. of Math,* **52** (1950) 51-110.

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