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Legendre Method for a Class of Nonlinear Optimal Control Problems

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Abstract

This paper introduces a numerical technique for solving a class of optimal control problems containing nonlinear dynamical system and functional of state variables. This numerical method consists of two major parts. In the first part, using linear combination property of intervals, we convert the nonlinear dynamical system into an equivalent linear system. And in the second part, which we are dealing with a linear dynamical system, using Legendre expansions for approximating both the state and associated control together with discretizing the constraints over the Chebyshev-Gauss-Lobatto points, the optimal control problem is transformed into a corresponding NLP problem which is diretly solved. The proposed idea is illustrated by several numerical examples.

Keywords: Optimal Control, Legendre Polynomials, Linear Combination Property of Intervals, Chebyshev-Gauss-Lobatto Points, Nonlinear Programming

1. Introduction

Optimal control theory is widely applied in aerospace, engineering, economics and other areas of science and has received considerable attention of researchers. During the past two decades, enormous effort has been spent on the development of computational methods for generating solutions of optimal control problems [1-8]. Although many computational methods have been developed and proposed, modification of the existing methods and development of new methods should yet be explored to obtain accurate solutions successfully.

The approaches to numerical solutions of optimal control problems may be divided into two major classes: the indirect methods and the direct methods. The indirect methods are based on the pontryagin maximum principle and require the numerical solution of boundary value problems that result from the necessary conditions of optimal control [9]. For many practical optimization problems, these boundary value problems are quite difficult to solve. In fact, the manner in which Pontryagin maximum principle is used differs so significantly from one type of problem to another that no standard solution procedure can be devised. Therefore, one has to devise direct computational algorithms to solve optimal control problems.

Direct optimization methods transcribe the (infinite-dimensional) continuous problem to a finite-dimensional nonlinear programming problem (NLP) through some parametrization of the state and/or control variables. In the direct methods, initial guesses have to be provided only for physically intuitive quantities such as the states and possibly controls. However, continuous advances in NLP algorithms and software have made these the methods of choice in many applications [10].

In this paper, we present a direct approach that based upon linear combination property of intervals and Legendre polynomials approximations together with the Chebyshev-Gauss-Lobatto (CGL) points as the collocation nodes to determine the optimal trajectories of high-order nonlinear (possibly discontinuous) dynamic systems. The most important reason of CGL points consideration instead of Legendre-Gauss-Lobatto (LGL) points is that CGL's have a closed-form formula, but, LGL's have no analytic forms. Our method consists of two major parts. In the first part, using linear combination property of intervals, we transform nonlinear dynamical system into a corresponding linear system. And in the second part, using general ideas of Razzaghi [11] (*i.e.*, Legendre method for linear systems), and applying the CGL points as collocation nodes for discretizing the latter linear dynamical system and inequality state constraints, the optimal control problem is converted into an NLP problem, which its parameters are the unknown Legendre coefficients. We also apply high-order Gauss--lobatto quadrature rules [6] for approximating the integral involved in the performance index in the discretization procedure. The advantages of recasting the optimal control problem as an NLP are:

1) the proposed method eliminates the requirement of solving a (2PBVP);

2) state and control inequality are easier to handle.

Numerical examples are given to demonstrate the applicability of the proposed technique. Moreover, a comparison is made with optimal solutions obtained by the presented approach and a collocation method [12].

2. Preliminaries

2.1. Properties of the Shifted Legendre Polynomials

The Legendre polynomials which are orthogonal in the interval [−1,1] are

$$P_{i+1}(x)=\frac{2i+1}{i+1}xP_i(x)-\frac{i}{i+1}P_{i-1}(x),\ i\geq 1 \tag{2.1.1}$$

with $P_0(x)=0$ and $P_1(x)=x$.

In order to use these polynomials on the interval $[0,h]$, one can apply the change of variables $x=\frac{2t}{h}-1$ in (2.1.1). Therefore, the shifted Legendre polynomials are constructed as follows

$$\hat{P}_i(t)=P_i\left(\frac{2t}{h}-1\right)\quad t\in[o,h]. \tag{2.1.2}$$

The orthogonal property of shifted Legendre polynomials is given by

$$\int_0^h \hat{P}_i(t)\hat{P}_j(t)\mathrm{d}t=\begin{cases}0 & i\neq j\\ \dfrac{h}{2i+1} & i=j\end{cases} \tag{2.1.3}$$

A function, $f(t)$, which is absolutely integrable within $0\leq t\leq h$ may be expressed in terms of a shifted Legendre series as

$$f(t)=\sum_{i=0}^{\infty}f_i\hat{P}_i(t) \tag{2.1.4}$$

where

$$f_i=\frac{2i+1}{h}\int_0^h f(t)\hat{P}_i(t)\mathrm{d}t. \tag{2.1.5}$$

If we assume that the derivative of $f(t)$ in Equation (2.1.4) is described by

$$\dot{f}(t)=\sum_{i=0}^{\infty}g_i\hat{P}_i(t) \tag{2.1.6}$$

the relationship between the coefficients f_i in (2.1.4) and g_i in (2.1.6) can be obtained as follows [11]

$$h\left[(2i+3)g_{i-1}-(2i-1)g_{i+1}\right]-2(2i-1)(2i+3)f_i=0 \\ i=1,2,\cdots \tag{2.1.7}$$

Further, the product of two shifted Legendre polynomials $\hat{P}_i(t)$ and $\hat{P}_j(t)$ can be approximated by

$$\hat{P}_i(t)\hat{P}_j(t)=\sum_{n=0}^{N}\gamma_{ijn}\hat{P}_n(t) \tag{2.1.8}$$

where

$$\gamma_{ijn}=\frac{2n+1}{h}\int_0^h\hat{P}_i(t)\hat{P}_j(t)\hat{P}_n(t)\mathrm{d}t,\ n=0,1,\cdots,N. \tag{2.1.9}$$

2.2. Linear Combination Property of Intervals

This property states that every uniform continuous function with a compact and connected domain can be written as a convex linear combination of its maximum and minimum. In other words, if and are the maximum and minimum of the uniform continuous function $H(x)$, one can write

$$H(x)=\lambda\alpha+(1-\lambda)\beta,\ 0\leq\lambda\leq 1. \tag{2.2.1}$$

3. Problem Statement

Consider the following nonlinear system

$$\dot{x}(t)=A(t)x(t)+H(t,u(t)) \tag{3.1}$$

with known initial and final conditions $x(0)=x_0$, $x(h)=x_h$, where $x(t)$ and $u(t)$ are $n\times 1$ and $q\times 1$ state and control vectors respectively, $A(t)\in R^{n\times n}$ and $u(t)\in U$ where U is a compact and connected subset of R^q. It is assumed that $n=q$ and $h(t,u(t))$ is a smooth or non-smooth continuous function over $[0,h]\times U$. Moreover, there exists a pair of state and control variables $(x(t),u(t))$ such that satisfies (4.1) and two point boundary conditions $x(0)=x_0$ and $x(h)=x_h$. The problem is to find the optimal control $u(t)$ and the corresponding state trajectory $x(t)$, $0\leq t\leq h$, satisfing Equation (3.1) while minimizing the cost functional

$$J=\int_0^h f(t,x(t))\mathrm{d}t. \tag{3.2}$$

Two special cases of $f(t,x(t))$ in (3.2) are $f(t,x(t))=c^T(t)x(t)$ and

$$f\left(t, x(t)\right) = \frac{1}{2} x^T(t) q(t) x(t).$$

Also, with the assumption of enough smoothness one can consider the following inequality state constraint

$$\Phi\left(t, x(t)\right) \le 0. \tag{3.3}$$

4. Linearization of the Dynamical System

Since $H:[0,h]\times U \to R^n$ is a continuous function and $[0,h]\times U$ is a compact and connected subset of R^{n+1}, then $\left\{H\left(t,u(t)\right): u \in U\right\}$ is a closed set in R^n clearly. Thus, $\left\{H_i\left(t,u(t)\right): u \in U\right\}$ for $i = 1,2,\cdots,n$ is closed in R. Now, suppose that the lower and upper bounds of the $\left\{H_i\left(t,u(t)\right): u \in U\right\}$ are $g_i(t)$ and $w_i(t)$ respectively. Therefore,

$$g_i(t) \le H_i\left(t,u(t)\right) \le w_i(t), \quad t \in [0,h]. \tag{4.1}$$

In other words

$$g_i(t) = Min_u \left\{H_i\left(t,u(t)\right): u \in U\right\}, \ t \in [o,h], \tag{4.2}$$

$$w_i(t) = Max_u \left\{H_i\left(t,u(t)\right): u \in U\right\}, \ t \in [o,h]. \tag{4.3}$$

Using linear combination property of intervals, that explained briefly in Section 2, $H_i\left(t,u(t)\right)$ can be expressed as a convex linear combination of its minimum $g_i(t)$ and maximum $w_i(t)$ as follows

$$\begin{aligned} H_i\left(t,u(t)\right) &= \lambda_i(t) w_i(t) + \left(1-\lambda_i(t)\right) g_i(t) \\ &= \lambda_i(t)\beta_i(t) + g_i(t), \end{aligned} \tag{4.4}$$

where $\beta_i(t) = w_i(t) - g_i(t)$ and $\lambda_i(t) \in [0,1]$.

Note that according to Equation (4.4), $\lambda_i(t)$ is the associated control variable.

Now, the main problem with the assumption of $f\left(t,x(t)\right) = c^T(t)x(t)$ is transformed into the following optimal control problem

$$\min \int_0^h c^T(t) x(t)\,\mathrm{d}t \tag{4.5}$$

Subject to

$$\dot{x}(t) = A(t)x(t) + \beta(t)\lambda(t) + g(t), \ \lambda(t) \in [0,1] \tag{4.6}$$

$$\Phi\left(t, x(t)\right) \le 0, \tag{4.7}$$

$$x(0) = x_0, \ x(h) = x_h. \tag{4.8}$$

For solving the above-mentioned problem one can apply the Legendre polynomials together with Chebyshev-Gauss-Lobatto (CGL) points (as collocation nodes). In the next section, the state variable $x(t)$ and associated control variable $\lambda(t)$ are expanded in terms of Legendre polynomials with unknown coefficients. Then, using CGL points as the collocation nodes the latter problem is converted to an NLP problem which its parameters are the unknown coefficients of the state and associated control.

5. Discretization

A discretization of the interval $0 = s_0 < s_1 < \cdots < s_N = h$ is chosen, where

$$s_i = \frac{h}{2} t_i + \frac{h}{2}, \ i = 0,1,\cdots,N \tag{5.1}$$

with $t_i = -\cos\left(\frac{i\pi}{N}\right)$. Trivially, s_i's are shifted CGL points in the interval $[0,h]$. We use the following expansions to approximate both $x(t)$ and associated control $\lambda(t)$

$$x(t) = x^N(t) = \sum_{i=0}^{N} a_i \hat{P}_i(t), \tag{5.2}$$

$$\lambda(t) = \lambda^N(t) = \sum_{i=0}^{N} b_i \hat{P}_i(t), \tag{5.3}$$

where $\hat{P}_i(t)$'s are the $i-th$ order shifted Legendre polynomials. To find the Legendre expansion coefficients c_i of the derivative $\dot{x}^N(t)$ such that

$$\dot{x}^N(t) = \sum_{i=0}^{N} a_i \dot{\hat{P}}_i(t) = \sum_{i=0}^{N} c_i \hat{P}_i(t) \tag{5.4}$$

we use the recurrence relation (2.1.7).

Using CGL points for discretizing dynamical system (4.6) together with the inequality state constraints (4.7) and boundedness of associated control $\lambda(t)$, the optimal control problem (4.5) - (4.8) is changed into the following NLP problem

$$\min L\left(a_0, a_1, \cdots, a_N\right) \tag{5.5}$$

Subject to

$$\dot{x}^N(s_i) = A(s_i)x^N(s_i) + \beta(s_i)\lambda(s_i) + g(s_i), \quad i = 0,\cdots,N \tag{5.6}$$

$$\Phi\left(s_i, x^N(s_i)\right) \le 0, \ 0 \le \lambda^N(s_i) \le 1, \ i = 0,\cdots,N \tag{5.7}$$

$$x^N(s_0) = \sum_{i=0}^{N} (-1)^i a_i = x_0, \ x^N(h) = \sum_{i=0}^{N} a_i = x_h. \tag{5.8}$$

$$\left[a_1, a_2, \cdots, a_N\right]^{\mathrm{T}} = M\left[c_0, c_1, \cdots, c_N\right]^{\mathrm{T}}, \tag{5.9}$$

where $L\left(a_0, a_1, \cdots, a_N\right)$ is a linear objective function, $\hat{P}_i(0) = \hat{P}_i(s_0) = (-1)^i$ and $\hat{P}_i(h) = \hat{P}_i(s_N) = 1$.

Note that the constraints of (5.9) arise from the following relations

$$a_i = \frac{h}{2}\left[\frac{c_{i-1}}{2i-1} - \frac{c_{i+1}}{2i+3}\right], \ i = 1,2,\cdots,N. \tag{5.10}$$

where $c_{N-1} = c_N = 0$.

Hint. After obtaining optimal state $x^*(t)$ and associated control $\lambda^*(t)$, for evaluating optimal control $u^*(t)$ we use the following equation

$$H\left(t, u^*(t)\right) = \beta(t)\lambda^*(t) + g(t). \tag{5.11}$$

6. Illustrative Examples

In this section, we conduct two numerical examples to illustrate the effectiveness of the proposed method. We use the method that stated in Sections 5 and 6 to transform the main problems into the equivalent NLP problems, and comparisons of our solutions with a collocation method solutions [12] are presented. All the problems are programmed in MAPLE 12 and run on a PC with 1.8 GHz and 1 GB RAM.

Example 6.1 We first consider a problem containing non-smooth function $H(t,u(t))$ which indirect approaches (base upon Pontryagin maximum principle) can not dealing with this case in a proper way. The problem is to find the control $u(t)$ and the state $x(t)$ which minimize

$$J = \int_0^1 \left(\left|\sin(2\pi t)\right| - \mathrm{e}^{-t}\right) x(t)\,\mathrm{d}t \tag{6.1}$$

subject to

$$\dot{x}(t) = \left(t^5 - t^2 + t\right)x(t) - \left|u(t)\right|^3 \mathrm{e}^{\sin(2\pi t)} \tag{6.2}$$

with $u(t) \in [-1,1]$, $x(0) = 0.9$ and $x(1) = 0.4$. Here $H(t,u(t)) = -\left|u(t)\right|^3 \mathrm{e}^{\sin(2\pi t)}$, and according to the (4.2) and (4.3) we have
$g(t) = \min_u \left\{H_i(t,u(t)) : u \in [-1,1]\right\} = -\mathrm{e}^{\sin(2\pi t)}$ and
$w(t) = \max_u \left\{H_i(t,u(t)) : u \in [-1,1]\right\} = 0$, thus
$\beta(t) = w(t) - g(t) = \mathrm{e}^{\sin(2\pi t)}$. In **Figures 1** and **2** we plot the optimized state $x(t)$ and control $u(t)$ for $N = 11$. Also, the numerical results compared with the collocation method [12] are listed in **Table 1**. From **Table 1**. one can see that our method achieves good result with a relatively smaller of nodes than [12].

Example 6.2 Find the control $u(t)$ and the state $x(t)$, which minimize

$$J = \frac{1}{2}\int_0^1 \left(\mathrm{e}^{-t} - 2t\right) x(t)\,\mathrm{d}t \tag{6.3}$$

subject to

$$\dot{x}(t) = -tx(t) + \ln\left(u(t) + t + 3\right) \tag{6.4}$$

with $u(t) \in [-1,1]$, $x(0) = 0$ and $x(1) = 0.8$. Here $H(t,u(t)) = \ln\left(u(t) + t + 3\right)$, and according to the (4.2) and (4.3) we have

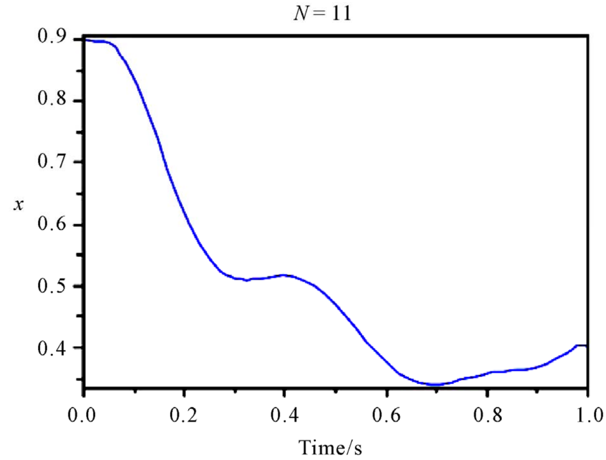

Figure 1. Optimal state $x^*(.)$ of example 6.1.

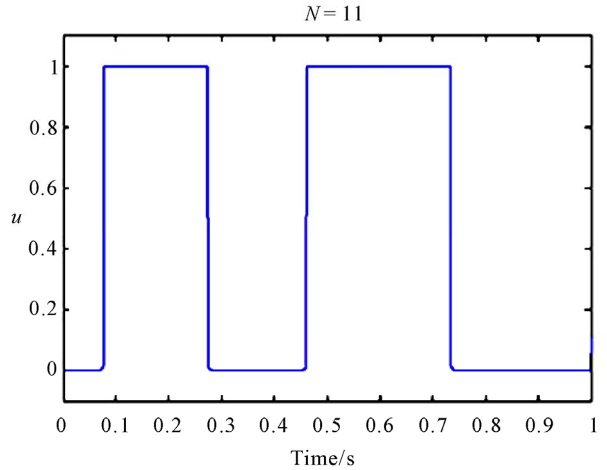

Figure 2. Optimal control $u^*(.)$ of example 6.1.

Table 1. Comparison of J^* between methods.

N	Legendre Method	Collocation Method [12]
6	−0.0331943700	−0.0354240398
8	−0.0346216260	−0.0356414811
10	−0.0370224908	−0.0368552070
11	−0.0387730644	−0.0377178920

$g(t) = \min_u \left\{H_i(t,u(t)) : u \in [-1,1]\right\} = \ln(t+2)$ and
$w(t) = \max_u \left\{H_i(t,u(t)) : u \in [-1,1]\right\} = \ln(t+4)$, thus

$$\beta(t) = w(t) - g(t) = \ln(t+4) - \ln(t+2) = \ln\frac{t+4}{t+2}$$

In **Figures 3** and **4** we plot the optimized state $x(t)$ and control $u(t)$ for $N = 12$. Also, the numerical results compared with the collocation method [12] are listed in **Table 2**. From **Table 2**. we see that the performance

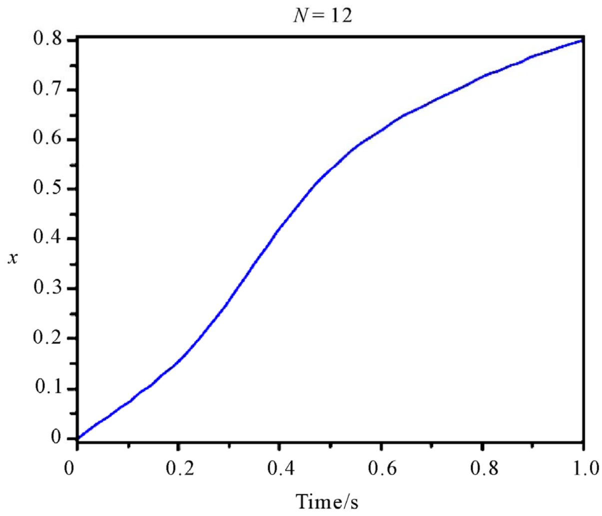

Figure 3. Optimal state $x^*(.)$ of example 6.2.

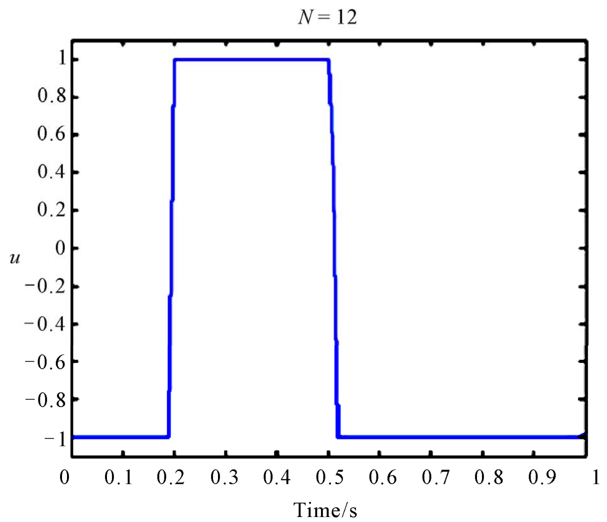

Figure 4. Optimal control $u^*(.)$ of example 6.2.

Table 2. Comparison of J^* between methods.

N	Legendre Method	Collocation Method [12]
7	−0.1820295698	−0.1815318828
9	−0.1825806940	−0.1821287778
11	−0.1826754624	−0.1823052407
12	−0.1828354838	−0.1826714837

index got by our approach are better than those obtained by the method in [12].

7. Conclusions

The aim of the present work is the determination of the optimal control and state vectors by a direct method of solution based upon linear combination property of intervals and shifted Legendre series expansions together with the CGL points as collocation nodes respectively. The method is based upon reducing a nonlinear optimal control problem to an NLP. The unity of the weight function of orthogonality for shifted Legendre series and the simplicity of the discretization are merits that make the approach very attractive. Moreover, only a small number of shifted Legendre series is needed to obtain a very satisfactory solution. The given numerical examples supports this claim.

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