A stopping rule for an iterative algorithm in systems of linear integral equations

Mortaza Gachpazan, Omid Baghani

Department of Applied Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad, Iran. *gachpazan@math.um.ac.ir*

General area of research: Analysis and Numerical Analysis

Abstract: Consider the second kind Volterra integral equations system of the form:

$$\mathbf{U}(x) = \mathbf{F}(x) + \int_a^x \mathbf{K}(x,t)\mathbf{U}(t)dt \equiv \mathcal{K}\mathbf{U}, \qquad (a \leq x \leq b,\ \mathbf{U} \in X), \tag{1}$$

where

$$\begin{aligned}
\mathbf{U}(x) =&[u_1(x), u_2(x), \ldots, u_m(x)]^T, \\
\mathbf{F}(x) =&[f_1(x), f_2(x), \ldots, f_m(x)]^T, \\
\mathbf{K}(x,t) =& [k_{ij}(x,t)], \quad i,j = 1,2,\ldots,m.
\end{aligned}$$

which $\mathcal{K}$ be integral operator on complete metric space $(X, d := \|.\|_\infty)$, $X := C([a,b], \mathbb{R}^m)$, $m \geq 1$. The vector function $\mathbf{F}$ and the matrix function $\mathbf{K}$ are given, and $\mathbf{U}$ is the vector function of the solution of system (1) that will be determined. We assume that $\mathbf{F}$ and $\mathbf{K}$ are continuous on the interval $[a,b]$ and triangular $D := \{(x,t) :\ x \in [a,b]\ ,\ t \in [a,x]\}$ respectively.

First, we show that there exist a positive constant such that M which the following inequality is satisfied for every $U, V \in X$:

$$d(\mathcal{K}^n\mathbf{U}, \mathcal{K}^n\mathbf{V}) \leq \frac{M^n(b-a)^n}{n!} d(\mathbf{U}, \mathbf{V}),$$

So, the mapping $\mathcal{K}^n$ is contractive when n is sufficiently large.

Second, by fixed point theorem we show that the contraction mapping $T : X \to X$ has a unique fixed point $\mathbf{U}^*$ and $\{T^n(\mathbf{U})\}_1^\infty$ converges to $\mathbf{U}^*$ for each $\mathbf{U} \in X$.

And finally, we determine number of iteration for obtaining the desired approximation and give some numerical examples.

Keywords: Integral operator, Successive approximation method, Fixed point theorem.