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Some outer commutator multipliers and capability of nilpotent products of cyclic groups

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Abstract. In this paper, first we obtain an explicit formula for an outer commutator multiplier of nilpotent products of cyclic groups with respect to the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t})$ where $r_{i+1} \mid r_i \ \ (1 \leq i \leq t-1)$, $c_1 + c_2 + 1 \geq n$, $2c_2 - c_1 > 2n - 2$ and $(p, r_1) = 1$ for all prime less than or equal $c_1 + c_2 + n$, second we give a necessary condition for these groups to be $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable.

Keywords: Outer commutator; Nilpotent product; Baer invariant; Capability.

Mathematics Subject Classification 2010: 20E34, 20F12.

1 Introduction and preliminaries

Schur [27] in 1907 and Wiegold [30] in 1971 obtained a structure of the Schur multiplier of the direct product of two finite groups as follows:

$$M(A \times B) \cong M(A) \oplus M(B) \oplus \tfrac{[A,B]}{[A,B,A*B]}, \text{ where } \tfrac{[A,B]}{[A,B,A*B]} \cong A_{ab} \otimes B_{ab}.$$

In 1979, Moghaddam [22] and in 1998, Ellis [4], succeeded to extend the above result to obtain a structure of the c-nilpotent multiplier of the direct product of two groups, $\mathcal{N}_c M(A \times B)$. Also in 1997 M. R. R. Moghaddam and the second author in a joint paper [24] presented an explicit formula for the c-nilpotent multiplier of a finite abelian group.

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Tahara [29] and Haebich [6] concentrated on the Schur multiplier of semidirect products of groups. Also the second author worked on the Baer invariants of a semidirect product in [16, 19].

In 1972 Haebich [6] presented a formula for the Schur multiplier of a regular product of a family of groups. It is known that the regular product is a generalization of the nilpotent product and the last one is a generalization of the direct product, so Haebich's result is a vast generalization of the Schur's result. Also, Moghaddam [22], in 1979 gave a formula similar to Haebich's formula for the Schur multiplier of a nilpotent product. Moreover, in 1992, Gupta and Moghaddam [5] presented an explicit formula for the c-nilpotent multiplier of the nth nilpotent product $\mathbb{Z}_2 \overset{n}{*} \mathbb{Z}_2$.

In 2001, the second author [17] found a structure similar to Haebich's type for the c-nilpotent multiplier of a nilpotent product of a family of cyclic groups. The c-nilpotent multiplier of a free product of some cyclic groups was studied by the second author [18] in 2002.

Recently, the authors [21, 25] concentrated on the Baer invariant with respect to the variety of polynilpotent groups. We presented an explicit structure for some polynilpotent multipliers of the nth nilpotent product of some infinite cyclic groups [25] and also found explicit structures for all polynilpotent multipliers of finitely generated abelian groups [21].

In [26] the authors succeeded to determine an explicit structure of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z})$, where $c_1 + c_2 + 1 \geq n$ and $2c_2 - c_1 > 2n - 2$. Also the second author, Hokmabadi and Mohammadzadeh [20], succeeded to compute an explicit structure of $\mathfrak{N}_c M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t})$ and $\mathfrak{N}_{c_1,\ldots,c_t} M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t})$ under some conditions. Hokmabadi [10] presented the necessary and sufficient conditions for $\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ to be $\mathfrak{N}_c$-capable, where $r_{i+1} \mid r_i \quad (1 \leq i \leq t-1)$ and $(p, r_1) = 1$ for all prime less that or equal n. In this article we intend to extend the results of [26] and [20] to obtain an explicit structure of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t})$, where $r_{i+1} \mid r_i \quad (1 \leq i \leq t-1)$, $c_1 + c_2 + 1 \geq n$, $2c_2 - c_1 > 2n-2$ and $(p, r_1) = 1$ for all prime less that or equal $c_1 + c_2 + n$. Furthermore a necessary condition on $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability of $\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ is given.

Definition 1.1. Let G be any group with a free presentation $G \cong F/R$, then, after R. Baer [1], the *Baer invariant* of G with respect to a variety of groups $\mathfrak{V}$, denoted by $\mathfrak{V}M(G)$, is defined to be

$$\mathfrak{V}M(G) = \frac{R \cap V(F)}{[RV^*F]} ,$$

where V is the set of words of the variety $\mathfrak{V}$, $V(F)$ is the verbal subgroup of F with respect to $\mathfrak{V}$ and

$$[RV^*F] = < v(f_1, \ldots, f_{i-1}, f_i r, f_{i+1}, \ldots, f_n) v(f_1, \ldots, f_i, \ldots, f_n)^{-1} \mid$$
$$r \in R, 1 \leq i \leq n, v \in V, f_i \in F, n \in \mathbb{N} > .$$

In special case of the variety $\mathfrak{A}$ of abelian groups, the Baer invariant of G will be the well-known notion the *Schur multiplier*

$$\frac{R \cap F'}{[R, F]}.$$

If $\mathfrak{V}$ is the variety of nilpotent groups of class at most $c \geq 1$, $\mathfrak{N}_c$, then the Baer invariant of G with respect to $\mathfrak{N}_c$ which is called the *c-nilpotent multiplier* of G, will be

$$\mathfrak{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ {}_cF]},$$

where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F and $[R,\ {}_1F] = [R, F], [R,\ {}_cF] = [[R,\ {}_{c-1}F], F]$, inductively.

Lemma 1.2. ([11]). If u and w are any two words and $v = [u, w]$ and K is a normal subgroup of a group G, then

$$[Kv^*G] = [[Ku^*G], w(G)][u(G), [Kw^*G]].$$

Proof. See [11, Lemma 2.9]. □

Now, using the above lemma, let $\mathfrak{V}$ be the outer commutator variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, then the Baer invariant of a group G with respect to $\mathfrak{V}$, is as follows:

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{R \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]} \quad (\star).$$

Definition 1.3. *Basic commutators* are defined in the usual way. If X is a fully ordered independent subset of a free group, the basic commutators on X are defined inductively over their weight as follows:
(i) All the members of X are basic commutators of weight one on X;
(ii) assuming that $n > 1$ and that the basic commutators of weight less than n on X have been defined and ordered;
(iii) a commutator $[b, a]$ is a basic commutator of weight n on X if $wt(a) + wt(b) = n,\ a < b$, and if $b = [b_1, b_2]$, then $b_2 \leq a$. The ordering of basic commutators is then extended to include those of weight n in any way such that those of weight less than n precede those of weight n. The natural way to define the order on basic commutators of the same weight is lexicographically, $[b_1, a_1] < [b_2, a_2]$ if $b_1 < b_2$ or if $b_1 = b_2$ and $a_1 < a_2$.

The next two theorems are vital in our investigation.

Theorem 1.4. ([9]). Let $F =< x_1, x_2, \ldots, x_d >$ be a free group, then

$$\frac{\gamma_n(F)}{\gamma_{n+i}(F)} \quad , \qquad 1 \leq i \leq n$$

is the free abelian group freely generated by the basic commutators of weights $n, n+1, \ldots, n+i-1$ on the letters $\{x_1, \ldots, x_d\}$.

Theorem 1.5. ([9]). The number of basic commutators of weight n on d generators is given by the following formula:

$$\chi_n(d) = \frac{1}{n}\sum_{m|n}\mu(m)d^{n/m},$$

where $\mu(m)$ is the Möbius function.

Definition 1.6. Let $\mathfrak{V}$ be a variety of groups defined by a set of laws V. Then the *verbal product* of a family of groups $\{G_i\}_{i\in I}$ associated with the variety $\mathfrak{V}$ is defined to be

$$\mathfrak{V}\prod_{i\in I} G_i = \frac{\prod^* G_i}{V(G)\cap [G_i]^*},$$

where $G = \prod^*_{i\in I} G_i$ is the free product of the family $\{G_i\}_{i\in I}$ and $[G_i]^* = < [G_i, G_j]|i, j \in I, i \neq j >^G$ is the cartesian subgroup of the free product G.

The verbal product is also known as *varietal product* or simply $\mathfrak{V}$-*product*. If $\mathfrak{V}$ is the variety of all groups, then the corresponding verbal product is the free product; if $\mathfrak{V} = \mathfrak{A}$ is the variety of all abelian groups, then the verbal product is the direct product. Also, if $\mathfrak{V} = \mathfrak{N}_c$ is the variety of nilpotent groups of class at most $c \geq 1$, then the verbal product is called the *cth nilpotent product* of the G_i's.

2 Main results

Let $G_i = \langle x_i | x_i^{k_i}\rangle \cong \mathbb{Z}_{k_i}$ be the cyclic group of order k_i if $k_i \geq 1$, $(m+1 \leq i \leq m+t)$ and the infinite cyclic group if $k_i = 0$, $(1 \leq i \leq m)$. Let

$$1 \to R_i = \langle x_i^{k_i}\rangle \to F_i = \langle x_i\rangle \to G_i \to 1$$

be a free presentation for G_i, so the nth nilpotent product of the family $\{G_i\}_{i\in I}$ is defined as follows:

$$\overset{n}{\underset{i\in I}{\prod^*}} G_i = \frac{\prod^*_{i\in I} G_i}{\gamma_{n+1}(\prod^*_{i\in I} G_i)\cap [G_i]^*_{i\in I}}.$$

It is easy to show that the cartesian subgroup is the kernel of the natural homomorphism from $\prod^*_{i\in I} G_i$ to the direct product $\prod^\times_{i\in I} G_i$. Since G_is are cyclic, it is easy to see that $\gamma_{n+1}(\prod^*_{i\in I} G_i) \subseteq [G_i]^*$ and hence $\prod^n_{i\in I} G_i = \prod^*_{i\in I} G_i/\gamma_{n+1}(\prod^*_{i\in I} G_i)$. Also, it is routine to check that a free presentation for the nth nilpotent product of $\prod^*_{i\in I} G_i$ is as follows:

$$1 \to R = S\gamma_{n+1}(F) \to F = \overset{*}{\underset{i\in I}{\prod}} F_i \to \overset{n}{\underset{i\in I}{\prod^*}} G_i \to 1,$$

where $S = \langle x_i^{k_i} | i \in I \rangle^F$ and F is the free group on the set

$$X = \{x_1, \cdots, x_m, x_{m+1}, \cdots, x_{m+t}\}.$$

Now let $G \cong \mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ be the nth nilpotent product of some cyclic groups (m copies of $\mathbb{Z}$). We intend to obtain the structure of some outer commutator multipliers of G of the form

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G).$$

Using $(\star)$ we have

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{\gamma_{n+1}(F) \cap [\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}.$$

Now if $c_1 + c_2 + 1 \geq n$, then we have

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{[\gamma_{c_1+1}(F), \gamma_{c_2+1}(F)]}{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}.$$

Now consider the isomorphism

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \frac{[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z})}{\frac{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}}.$$

To determine the explicit structure of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$ it is enough to determine the structure of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z})$ and

$$\frac{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}.$$

But as we state in the following the authors [26, Theorem 2.8] gave the explicit structure of the first group.

Theorem 2.1. If $2c_2 - c_1 > 2n - 2$ and $c_1 \geq c_2$, then $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \ldots \overset{n}{*} \mathbb{Z})$ is the free abelian group with the following basis:

$$D = \{\ a[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)] \ \mid \ a \in A - C\ \},$$

where

$$A = \{[\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha,$$
$$c_1 + 1 \leq wt(\beta) \leq c_1 + n,\ c_2 + 1 \leq wt(\alpha) \leq c_2 + n\};$$

$$C = \{\ [\beta, \alpha] \mid \beta \text{ and } \alpha \text{ are basic commutators on } X \text{ such that } \beta > \alpha,$$
$$c_2 + n + 1 \leq wt(\beta)\ ,\ c_1 + 1 \leq wt(\alpha)\ ,\ wt(\beta) + wt(\alpha) \leq 2n + c_1 + c_2 + 1\}.$$

Now to determine the explicit structure of the second group we need the following lemma and some definitions and theorems from [13] which are as follows.

Lemma 2.2. With the above notations and assumptions we have

$$[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)] \cong [S,\ {}_{c_1}F, \gamma_{c_2+1}(F)] \pmod{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]}$$

and

$$[R,\ {}_{c_2}F, \gamma_{c_1+1}(F)] \cong [S,\ {}_{c_2}F, \gamma_{c_1+1}(F)] \pmod{[\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}$$

Proof. It is easy to see that for any normal subgroups of a group such as A, B, C and H if $A \cong B \pmod{C}$ then we have $[A,H] \cong [B,H] \pmod{[C,H]}$; now it is enough to show that $[R,F] \cong [S,F] \pmod{\gamma_{n+2}(F)}$ which is straightforward. Now induction completes the proof. □

Lemma 2.3. If $(p, r_1) = 1$ for all prime p less than or equal to $l-1$, then $[S,\ {}_{c+1-1}F]\gamma_{c+l}(F)/\gamma_{c+l}(F)$ is the free abelian group with a basis D_c, where

$$D_c = \{b^{r_j} |\ b \textit{ is a basic commutator of weight } c+i \textit{ on}$$

$$x_1, ..., x_m, ..., x_{m+j} \textit{ s.t. } x_{m+j} \textit{ appears in } b\ 1 \le j \le t \textit{ and } 1 \le i \le l-1\}.$$

Theorem 2.4. With the above notations and assumptions we have

$$[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)] \cong \langle D_1 \rangle \pmod{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}$$

in which $D_1 = \{[b,c] \mid b \in D_{c_1}$ and c is a basic commutator on X with $c_2+1 \le wt(c) \le c_2+n\}$, and

$$[R,\ {}_{c_2}F, \gamma_{c_1+1}(F)] \cong \langle D_2 \rangle \pmod{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}$$

in which $D_2 = \{[b,c] \mid b \in D_{c_2}$ and c is a basic commutator on X with $c_1+1 \le wt(c) \le c_1+n\}$.

Proof. Let $[a,b]$ be a generator of $[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)]$ so $a \in [R,\ {}_{c_1}F]$ and $b \in \gamma_{c_2+1}(F)$, using the above lemma and Hall's theorem we have $a = \prod a_i\alpha$ and $b = \prod b_j\beta$ in which $a_i \in D_{c_1}$, $\alpha \in \gamma_{c_1+n+1}(F)$, b_j's are basic commutators on X of weights $c_2+1, ..., c_2+n$ and $\beta \in \gamma_{c_2+n+1}(F)$. Now $[a,b]$ is a product of elements of the form $[a_i, b_j]^{f_{ij}}$, $[a_i, \beta]^{g_i}$, $[\alpha, b_j]^{h_j}$ and $[\alpha, \beta]^k$ in which $h_{ij}, g_i, h_j, k \in \gamma_{c_2+1}(F)$. It is enough to show that $[a_i, b_j, f_{ij}], [a_i, \beta], [\alpha, b_j]$ and $[\alpha, \beta]$ are all elements of $[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)]$ $[\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]$ and this can be done by a routine calculation. □

The immediate consequence of the latest theorem is that

$$\frac{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]} \cong \langle Y \rangle \quad (1)$$

in which $Y = \{[b,c]^{m_{i+1}}H \mid b \text{ and } c \text{ are basic commutators on } X \text{ with } c_1+1 \leq wt(b) \leq c_1+n, c_2+1 \leq wt(c) \leq c_2+n \text{ and } x_{i+1} \text{ appears in } [b,c]\}$.

Now considering Theorem 2.1 it is easy to see that every element of Y is a power of an element of $A-C$, so Y is linearly independent and that the group

$$\frac{[R,\ {}_{c_1}F, \gamma_{c_2+1}(F)][R,\ {}_{c_2}F, \gamma_{c_1+1}(F)]}{[\gamma_{c_1+n+1}(F), \gamma_{c_2+1}(F)][\gamma_{c_2+n+1}(F), \gamma_{c_1+1}(F)]}$$

is a free abelian group.

Now, we can state and prove the main result of this section.

Theorem 2.5. Let $G \cong \mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ be the nth nilpotent product of cyclic groups (m copies of $\mathbb{Z}$) where $r_{i+1} \mid r_i \ \ (1 \leq i \leq t-1)$, $c_1+c_2+1 \geq n$, $2c_2-c_1 > 2n-2$ and $(p, r_1) = 1$ for all prime less that or equal c_1+c_2+n, then

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(\mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}) \cong \mathbb{Z}^{(d_1)} \oplus \mathbb{Z}_{r_1}^{(d_2)} \oplus \cdots \oplus \mathbb{Z}_{r_t}^{(d_t)}$$

in which

(i) if $c_2+n < c_1+1$, then

$$d_1 = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m))(\sum_{c_2+1}^{c_1} \chi_i(m))$$

and

$$d_k = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m+k+1) - \sum_{i=c_1+1}^{c_1+n} \chi_i(m+k))(\sum_{i=c_2+1}^{c_2+n} \chi_i(m+k)) +$$

$$(\sum_{i=c_1+1}^{c_1+n} \chi_i(m+k))(\sum_{i=c_2+1}^{c_2+n} \chi_i(m+k+1) - \sum_{i=c_2+1}^{c_2+n} \chi_i(m+k))$$

(ii) if $c_2+n \geq c_1+1$, then

$$d_1 = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m))(\sum_{c_2+1}^{c_1} \chi_i(m)) + \chi_2(\sum_{i=c_1+1}^{c_2+n} \chi_i(m))$$

and

$$d_k = (\sum_{i=c_1+1}^{c_1+n} \chi_i(m+k+1) - \sum_{i=c_1+1}^{c_1+n} \chi_i(m+k))(\sum_{i=c_2+1}^{c_2+n} \chi_i(m+k)) +$$

$$(\sum_{i=c_1+1}^{c_1+n}\chi_i(m+k))(\sum_{i=c_2+1}^{c_2+n}\chi_i(m+k+1)-\sum_{i=c_2+1}^{c_2+n}\chi_i(m+k))+$$

$$\chi_2(\sum_{i=c_1+1}^{c_2+n}\chi_i(m+k+1)-\chi_2(\sum_{i=c_1+1}^{c_2+n}\chi_i(m+k)$$

Proof. First note that in case (i) if $c_2+n<c_1+1$, then $A-C=A$ and in case (ii) if $c_2+n\geq c_1+1$, then $|A-C|=(\sum_{i=c_1+1}^{c_1+n}\chi_i(m))(\sum_{c_2+1}^{c_1}\chi_i(m))+\chi_2(\sum_{i=c_1+1}^{c_2+n}\chi_i(m))$ [26, Corollary 2.5]. Now it is easy to see that the number of elements of $A-C$ which are not in Y is d_0 and the number of the elements of Y with power r_k is d_k. □

3 Capability

It was Hall [8] who initiated studying capable groups in order to classify p-groups. Several papers were devoted to capability and varietal generalization such as Burns and Ellis [2], Ellis [3], Moghaddam and Kayvanfar [23], and Magidin [12–15]. Recently Hokmabadi [10] classified varietal capable groups in the class of nilpotent products of cyclic groups with respect to the variety of polynilpotent groups under some conditions. In this section we intend to determine varietal capability of such groups with respect to the variety $[\mathfrak{N}_{c_1},\mathfrak{N}_{c_2}]$ with some conditions. To do this we need the following theorem and lemma.

Theorem 3.1. Let $H\cong \mathbb{Z}\overset{c+n}{*}\mathbb{Z}\overset{c+n}{*}\cdots\overset{c+n}{*}\mathbb{Z}\overset{c+n}{*}\mathbb{Z}_{r_1}\overset{c+n}{*}\mathbb{Z}_{r_2}\overset{c+n}{*}\cdots\overset{c+n}{*}\mathbb{Z}_{r_t}$ where $r_{i+1}\mid r_i\ \ (1\leq i\leq t-1)$ and all prime numbers smaller than $c+n$ are coprime to r_1, then

$$Z_c(H)=\begin{cases}\langle\gamma_{n+1}(H),x_1^{r_2}\rangle & m=0,\\ \langle\gamma_{n+1}(H),y_1^{r_1}\rangle & m=1,\\ \gamma_{n+1}(H) & m\geq 2.\end{cases}$$

Proof. See [10]. □

Lemma 3.2. Let $A_i=\langle a_i\mid a_i^{\alpha_i}\rangle$, $1\leq i\leq t$ and $G=A_1\overset{n}{*}\cdots\overset{n}{*}A_t$ with $n\geq 2$ and all prime number less than or equal to n are coprime to α_1. If $u\in G$ be an outer commutator on $a_1,\cdots,a_n$ such that only $a_{i_1},\cdots,a_{i_k}$ do appear in u then $u^N=1$ in which $N=(\alpha_{i_1},\cdots,\alpha_{i_k})$ (the greatest common divisor of $\alpha_{i_1},\cdots,\alpha_{i_k}$).

Proof. See [28]. □

Now let $G\cong \mathbb{Z}\overset{c+1}{*}\mathbb{Z}\overset{c+1}{*}\cdots\overset{c+1}{*}\mathbb{Z}\overset{c+1}{*}\mathbb{Z}_{r_1}\overset{c+1}{*}\mathbb{Z}_{r_2}\overset{c+1}{*}\cdots\overset{c+1}{*}\mathbb{Z}_{r_t}$ in which $c=c_1+c_2+1$. Similar to Theorem 3.1 we can prove

Theorem 3.3. Let $H \cong \mathbb{Z} \overset{c+n}{*} \mathbb{Z} \overset{c+n}{*} \cdots \overset{c+n}{*} \mathbb{Z} \overset{c+n}{*} \mathbb{Z}_{r_1} \overset{c+n}{*} \mathbb{Z}_{r_2} \overset{c+n}{*} \cdots \overset{c+n}{*} \mathbb{Z}_{r_t}$ where $r_{i+1} \mid r_i \ \ (1 \leq i \leq t-1)$ and all prime numbers smaller than $c+n$ are coprime to r_1, then

$$V^\star(H) = \begin{cases} \langle \gamma_{n+1}(H), x_1^{r_2} \rangle & m = 0, \\ \langle \gamma_{n+1}(H), y_1^{r_1} \rangle & m = 1, \\ \gamma_{n+1}(H) & m \geq 2. \end{cases} \tag{3.1}$$

Proof. Under the assumption of the theorem we have $V^\star(H) \subseteq Z_c(H)$. It is enough to show that $V^\star(H)$ contains the right hand side of (3.1). Clearly $\gamma_{n+1}(H) \subseteq V^\star(H)$. We show that $x_1^{r_2} \in V^\star(H)$. Let $h_i, 1 \leq i \leq c_1$ and $h'_j, 1 \leq j \leq c_2+1$ be arbitrary elements of the set $\{x_1, x_2, \cdots, x_t\}$ we have $[x_1^{r_2}, h_1, \ldots, h_{c_2}, [h'_1, \ldots, h'_{c_2+1}]] = [x_1, h_1, \ldots, h_{c_2}, [h'_1, \ldots, h'_{c_2+1}]]^{r_2} E_1^{f_1(r_2)} \ldots E_k^{f_k(r_2}$ in which E_i's are basic commutators on $\{x_1, x_2, \cdots, x_t\}$ and $w(E_1), \cdots, w(E_k) \geq c+1$ and $f_i(r_2) = \beta_1 \binom{r_2}{1} + \cdots + \beta_{\omega_i} \binom{r_2}{\omega_i}$ where $\omega_i = w(E_i) - c \leq n$. So we have $r_2 \mid f_i(r_2)$ and now by Lemma 3.2 $[x_1^{r_2}, h_1, \cdots, h_{c_2}, [h'_1, \cdots, h'_{c_2+1}]] = 1$ hence $x_1^{r_1} \in V^\star(H)$. The cases for which $m = 1$ or $m \geq 2$ have a similar proof. □

The following Theorem is the main result of this section.

Theorem 3.4. Let $G \cong \mathbb{Z} \overset{n}{*} \mathbb{Z} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z} \overset{n}{*} \mathbb{Z}_{r_1} \overset{n}{*} \mathbb{Z}_{r_2} \overset{n}{*} \cdots \overset{n}{*} \mathbb{Z}_{r_t}$ where $r_{i+1} \mid r_i$ $(1 \leq i \leq t-1)$ and all prime numbers smaller than $c+n$ are coprime to r_1 then G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if $m \geq 2$ or $m = 0$ and $r_1 = r_2$.

Proof. Let $H \cong \mathbb{Z} \overset{c+n}{*} \mathbb{Z} \overset{c+n}{*} \cdots \overset{c+n}{*} \mathbb{Z} \overset{c+n}{*} \mathbb{Z}_{r_1} \overset{c+n}{*} \mathbb{Z}_{r_2} \overset{c+n}{*} \cdots \overset{c+n}{*} \mathbb{Z}_{r_t}$ then under the assumption of theorem $V^\star(H) = \gamma_{n+1}(H)$ so $H/V^\star(H) \cong H/\gamma_{n+1}(H) \cong G$. □

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