Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 312 (2007) 1–8
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Some properties on the Schur multiplier of a pair of groups

Mohammad Reza R. Moghaddam [a], Ali Reza Salemkar [b,*], Kazem Chiti [a]

[a] *Faculty of Mathematical Sciences, Ferdowsi University of Mashhad, Iran*
[b] *Faculty of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran*

Received 18 February 2003
Available online 9 February 2007
Communicated by Richard Lyons

Abstract

J.L. Loday (1978) introduced the concept of relative central extension and G. Ellis (1998) gave the notion of covering pair of a pair of groups (G, N). In this paper under some condition we show the existence of covering pair for the pair of groups (G, N) and also show that every relative central extension is a homomorphic image of a covering pair of (G, N). Finally, we present some inequalities for the Schur multiplier of a pair of finite groups (G, N).

Keywords: Schur-multiplier; Relative central extension; Covering pair; Pair of groups

1. Introduction

Let G be a group with a normal subgroup N, then (G, N) is said to be a *pair* of *groups*. In 1978, J.L. Loday [5] introduced the concept of *relative central extension* of the pair (G, N) to be a group homomorphism $\delta : M \to G$, together with an action

$$M \times G \to M,$$
$$(m, g) \mapsto m^g$$

* Corresponding author.
E-mail addresses: mrrm5@yahoo.ca, moghadam@math.um.ac.ir (M.R.R. Moghaddam), salemkar@sbu.ac.ir (A.R. Salemkar).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2006.11.047

satisfying the following conditions:

(i) $\delta(M) = N$;
(ii) $\delta(m^g) = g^{-1}\delta(m)g$, for all $g \in G$, $m \in M$;
(iii) $m^{\delta(m_1)} = m_1^{-1} m m_1$, for all $m, m_1 \in M$;
(iv) G acts trivially on $\ker\delta$.

Clearly, the inclusion map $i : N \to G$ is a simple example of a relative central extension of the pair (G, N).

If $\delta : M \to G$ is a relative central extension of the pair (G, N), we define the *G-commutator subgroup* of M to be the subgroup $[M, G]$ generated by the *G-commutators*

$$[m, g] = m^{-1} m^g,$$

for all $g \in G$, $m \in M$. Also we define the *G-central* of M to be the central subgroup

$$Z_G(M) = \{ m \in M \mid m^g = m, \text{ for all } g \in G \}.$$

In particular, if N is equal to G then the central extension $\delta : M \to G$ together with the action

$$\begin{aligned} &M \times G \to M, \\ &(m, g) \mapsto m^g = m^{\delta(m_1)} \end{aligned} \tag{$*$}$$

where $g = \delta(m_1)$, for some $m_1 \in M$, gives the following central extension

$$1 \to \ker\delta \to M \to G \to 1.$$

In this case, $[M, G] = M'$ and $Z_G(M) = Z(M)$ are the commutator and the central subgroups of M, respectively.

Let G be a group with a free presentation $1 \to R \to F \to G \to 1$, then by I. Schur [9] the *Schur multiplier* of G, denoted by $M(G)$, is isomorphic with

$$\frac{R \cap F'}{[R, F]}.$$

It is a routine exercise to check that $M(G)$ is an abelian group and independent of the choice of the free presentation of G.

Now, consider the pair of groups (G, N) such that N has a complement in G, then G. Ellis in [1] defined the *Schur multiplier* of the *pair* (G, N) to be

$$M(G, N) = \ker\bigl(\mu : M(G) \to M(G/N)\bigr),$$

where μ is the natural epimorphism. Hence if S is a normal subgroup of F such that $N \cong S/R$, then

$$M(G, N) \cong \frac{R \cap [S, F]}{[R, F]}.$$

So in the rest of the paper, we always assume that all the pairs (G, N) under the consideration, N has a complement in G.

Let N^* be any other group, then a relative central extension $\delta : N^* \to G$ of the pair (G, N) is called a *covering pair* if there exists a subgroup A of N^* such that

(i) $A \subseteq Z_G(N^*) \cap [N^*, G]$;
(ii) $A \cong M(G, N)$;
(iii) $N \cong N^*/A$.

One observes that the covering pair $\delta : M \to G$ of the pair (G, G) together with the action defined as in $(*)$ is the usual covering group M of G introduced by I. Schur [8]. He also showed that every finite group G admits a covering group. G. Ellis in [1] generalized this result to a pair of finite groups (G, N), and proved that any pair of finite groups (G, N) admits at least one covering pair $\delta : N^* \to G$. In the next section, under some condition, we prove a similar result for the pair (G, N), where G is an arbitrary group and N is a finite normal subgroup of G (Theorem 2.3). We also show that if $\delta : M \to G$ is a relative central extension of the pair (G, N) such that $\ker \delta \subseteq [M, N]$, then there exists a covering pair $N^* \to G$ such that M is a homomorphic image of N^* (Theorem 2.6), which is a wide generalization of Yamazaki [10], when $N = G$.

In final section, we extend the works of M.R. Jones [3,4] to the pair of groups (G, N) (see Theorem 3.2).

2. Covering pair of groups

In the following lemma we present a relative central extension of a pair of groups (G, N), which is useful in our further investigations.

Lemma 2.1. *Let (G, N) be a pair of groups, with N being finite. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S a normal subgroup of F such that $N \cong S/R$. Then the mapping $\delta : S/[R, F] \to F/R$ which is given by $\delta(s[R, F]) = sR$, is a relative central extension of the pair (G, N).*

Proof. We define the action of $G = F/R$ on $S/[R, F]$ as follows:

$$\frac{S}{[R, F]} \times \frac{F}{R} \to \frac{S}{[R, F]},$$
$$\big(s[R, F], fR\big) \mapsto \big(s[R, F]\big)^{fR} = s^f[R, F].$$

Then for each $s_1, s_2 \in S$, $f \in F$, $r \in R$, we have

$$\delta\big(\big(s_1[R, F]\big)^{fR}\big) = \delta\big(s_1^f[R, F]\big) = s_1^f R = \delta\big(s_1[R, F]\big)^{fR},$$
$$\big(s_1[R, F]\big)^{\delta(s_2[R,F])} = \delta\big(s_1[R, F]\big)^{s_2 R} = s_1^{s_2}[R, F] = \big(s_1[R, F]\big)^{s_2[R,F]}, \quad \text{and}$$
$$\big(r[R, F]\big)^{fR} = r^f[R, F] = r[R, F].$$

Also $\delta(S/[R, F]) = S/R \cong N$, which proves the assertion. □

In the following we give a sufficient condition for the existence of covering pair of (G, N). In particular, if $N = G$ then the result of I. Schur [9] is obtained.

Proposition 2.2. *Let (G, N) be a pair of groups, $1 \to R \to F \to G \to 1$ a free presentation of G and S a normal subgroup of F such that $N \cong S/R$. If T is a normal subgroup of F such that*

$$\frac{R}{[R,F]} = \frac{R \cap [S,F]}{[R,F]} \times \frac{T}{[R,F]},$$

then the mapping $\delta : S/T \to F/R$ is given by $\delta(sT) = sR$ is a covering pair of (G, N).

Proof. Clearly δ together with the action

$$\frac{S}{T} \times \frac{F}{R} \to \frac{S}{T},$$
$$(sT, fR) \mapsto (sT)^{fR} = s^f T$$

is a relative central extension of the pair (G, N). We set $N^* = S/T$ and $A = R/T$, then $N \cong N^*/A$, $A \cong M(G, N)$ and $A \subseteq Z_G(N^*)$. Moreover

$$\begin{aligned} A = \frac{R}{T} \subseteq \frac{[S,F]T}{T} &= \big\langle (sT)^{-1} s^f T \bigm| s \in S,\ f \in F \big\rangle \\ &= \big\langle [sT, fR] \bigm| s \in S,\ f \in F \big\rangle \\ &= [N^*, G]. \end{aligned}$$

Therefore $\delta : N^* = S/T \to F/R$ is a covering pair of (G, N). □

The following main result gives the existence of covering pair of a given pair (G, N), in which N is a finite normal subgroup of G. In special cases, when G is a finite group or $N = G$ then the results of G. Ellis [1] and I. Schur [8] are obtained, respectively.

Theorem 2.3. *Let (G, N) be a pair of groups, with N being finite. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S a normal subgroup of F with $N \cong R/S$ such that for all $\bar{s} \in S/[R,F]$ and a positive integer n, $\bar{s}^n \in [S,F]/[R,F]$ implies that $\bar{s} \in [S,F]/[R,F]$. Then (G, N) admits a covering pair.*

Proof. By Lemma 2.1, $\delta : S/[R,F] \to F/R$ given by $\delta(\bar{s}) = sR$ is a relative central extension of the pair (G, N). Now using the assumptions, $R/(R \cap [S,F])$ is isomorphic with a subgroup of the free abelian group

$$\frac{\frac{S}{[R,F]}}{[\frac{S}{[R,F]}, \frac{F}{R}]}.$$

Then the following exact sequence splits

$$1 \to \frac{R \cap [S,F]}{[R,F]} \to \frac{R}{[R,F]} \to \frac{R}{R \cap [S,F]} \to 1.$$

Hence

$$\frac{R}{[R,F]} \cong \frac{R \cap [S,F]}{[R,F]} \times \frac{T}{[R,F]},$$

where $T/[R,F] \cong R/(R \cap [S,F])$. Therefore by Proposition 2.2,

$$\sigma : \frac{S}{T} \to \frac{F}{R}, \qquad sT \mapsto sR$$

is a covering pair of (G,N). $\square$

The above theorem has the following corollary, which is of interest in its own.

Corollary 2.4. *By the assumptions of the above theorem, if the Schur multiplier of G/N is trivial. Then the pair (G,N) admits at least one covering pair.*

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of G and S a normal subgroup of F such that $N \cong R/S$. Now the triviality of $M(G/N)$ and the fact that

$$\frac{S}{S \cap F'} \leqslant \frac{F}{F'},$$

imply that $S/[S,F] \cong S/(S \cap F')$ is a free abelian group. Therefore by Theorem 2.3, the pair (G,N) admits a covering pair. $\square$

The following lemma is needed for the next result and its proof is straightforward.

Lemma 2.5. *Let $1 \to R \to F \to G \to 1$ a free presentation of G and S a normal subgroup of F such that $S/R \cong N$. If $\sigma : M \to G$ is a relative central extension of the pair (G,N), then there exists an epimorphism $\beta : S/[R,F] \to M$ such that the following diagram is commutative*:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \frac{R}{[R,F]} & \longrightarrow & \frac{S}{[R,F]} & \xrightarrow{\delta} & N & \longrightarrow & 1 \\
 & & \downarrow \beta_1 & & \downarrow \beta & & \Vert & & \\
1 & \longrightarrow & \ker\sigma & \longrightarrow & M & \xrightarrow{\sigma} & N & \longrightarrow & 1
\end{array}$$

where δ is the relative central extension defined as in Lemma 2.1 and β_1 is the restriction of β.

As a final result in this section we show that, under some conditions, a relative central extension of a pair (G,N) is a homomorphic image of a covering pair of (G,N).

Theorem 2.6. *Let (G,N) satisfy the assumptions of Theorem 2.3. If $\sigma : M \to G$ is a relative central extension of (G,N) then M is a homomorphic image of the domain of a covering pair.*

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G, and S be a normal subgroup of F such that $N \cong S/R$. By Lemma 2.5, there exists an epimorphism $\beta : S/[R, F] \to M$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \frac{R}{[R,F]} & \longrightarrow & \frac{S}{[R,F]} & \xrightarrow{\delta} & N & \longrightarrow & 1 \\ & & \downarrow \beta_1 & & \downarrow \beta & & \| & & \\ 1 & \longrightarrow & \ker\sigma & \longrightarrow & M & \xrightarrow{\sigma} & N & \longrightarrow & 1 \end{array}$$

where δ is the relative central extension defined as in Lemma 2.1 and β_1 is the restriction of β. Put $\ker\beta_1 = \ker\beta = T/[R, F]$, where T is a normal subgroup of R. For any $r \in R$, $\beta(r[R, F]) \in \ker\sigma \subseteq [M, N]$. But $[M, N] = \langle m^{-1}m^{\delta(m_1)} \mid m, m_1 \in M\rangle = \beta([\frac{S}{[R,F]}, \frac{S}{[R,F]}])$, hence

$$\beta\big(r[R, F]\big) = \beta\big(\big[s_1, s_1'\big] \dots \big[s_l, s_l'\big][R, F]\big),$$

for some $s_i, s_i' \in S$, $1 \leqslant i \leqslant l$. So $r^{-1}[s_1, s_1'] \dots [s_l, s_l']t \in [R, F] \subseteq R \cap [S, F]$, for some $t \in T$. Therefore $r \in (R \cap [S, F])T$ and then $R = (R \cap [S, F])T$. Since $T/(T \cap [S, F])$ is isomorphic with a subgroup of the free abelian group

$$\frac{\frac{S}{[R,F]}}{[\frac{S}{[R,F]}, \frac{F}{R}]},$$

then the following exact sequence splits.

$$1 \to \frac{T \cap [S, F]}{[R, F]} \to \frac{T}{[R, F]} \to \frac{T}{T \cap [S, F]} \to 1.$$

Hence

$$\frac{T}{[R, F]} \cong \frac{T \cap [S, F]}{[R, F]} \times \frac{K}{[R, F]},$$

where $K/[R, F] \cong T/(T \cap [S, F])$. Now we have $K \cap (R \cap [S, F]) = [R, F]$ and $K(R \cap [S, F]) = R$, which implies that

$$\frac{R}{[R, F]} \cong \frac{R \cap [S, F]}{[R, F]} \times \frac{K}{[R, F]}.$$

Thus by Lemma 2.2, $\eta : S/K \to G$ is a covering pair of (G, N). Moreover $(S/K)/(T/K) \cong M$, which completes the proof. □

3. Some inequalities for the Schur-multiplier of a pair (G, N)

In this final section we give some inequalities for a pair of finite groups (G, N). The following lemma plays a fundamental role in proving our main theorem.

Lemma 3.1. *Let $1 \to R \to F \to G \to 1$ be a free presentation of the group G, S and T are normal subgroups of the free group F such that $T \subseteq S$, $S/R \cong N$ and $T/R \cong K$, then the following sequence is exact.*

$$1 \to \frac{R \cap [T,F]}{[R,F]} \to M(G,N) \xrightarrow{\alpha} M\left(\frac{G}{K}, \frac{N}{K}\right) \to \frac{K \cap [N,G]}{[K,G]} \to 1. \tag{1}$$

Proof. By the definition

$$M(G,N) = \frac{R \cap [S,F]}{[R,F]}, \qquad M\left(\frac{G}{K}, \frac{N}{K}\right) = \frac{T \cap [S,F]}{[T,F]}$$

and

$$\frac{K \cap [N,G]}{[K,G]} \cong \frac{(T \cap [S,F])R}{[T,F]R}.$$

Clearly the following sequence, with obvious natural homomorphisms, is exact

$$1 \to \frac{R \cap [T,F]}{[R,F]} \to \frac{R \cap [S,F]}{[R,F]} \to \frac{T \cap [S,F]}{[T,F]} \to \frac{(T \cap [S,F])R}{[T,F]R} \to 1,$$

which gives the result. □

In the above lemma, one notes that if $M(G,N)$ is trivial then $M(\frac{G}{K}, \frac{N}{K}) \cong \frac{K \cap [N,G]}{[K,G]}$. Also, if $1 \to A \to G \to H \to 1$ is a central extension of H such that $A \subseteq [N,G]$ and $M(\frac{G}{A}, \frac{N}{A})$ is triviality, then the exactness of (1) yields that G and H are isomorphic.

Now using the above lemma, we are able to prove our main theorem of this section, which is a wide generalization of M.R. Jones [3], when $N = G$.

Theorem 3.2. *Using the assumptions and notations of the above lemma we have*

(i) $|K \cap [N,G]||M(G,N)| = |M(\frac{G}{K}, \frac{N}{K})||\frac{[T,F]}{[R,F]}|$;
(ii) $d(M(G,N)) \leqslant d(M(\frac{G}{K}, \frac{N}{K})) + d(\frac{[T,F]}{[R,F]})$;
(iii) $e(M(G,N))$ *divides* $e(M(\frac{G}{K}, \frac{N}{K}))e(\frac{[T,F]}{[R,F]})$,

where $e(X)$ and $d(X)$ are the exponent and the minimal number of generators of the group X, respectively.

Proof. By Lemma 3.1,

$$\left|M(G,N)\right| = |L|\left|\frac{R \cap [T,F]}{[R,F]}\right| \quad \text{and} \quad \frac{M(\frac{G}{K}, \frac{N}{K})}{L} \cong \frac{K \cap [N,G]}{[K,G]},$$

where L is $\mathrm{Im}(\alpha)$ in Lemma 3.1. Hence

$$\left|K \cap [N,G]\right|\left|M(G,N)\right| = \left|M\left(\frac{G}{K}, \frac{N}{K}\right)\right|\left|[K,G]\right|\left|\frac{R \cap [T,F]}{[R,F]}\right|.$$

But $[K,G]\cong[T,F]/(R\cap[T,F])$, and

$$\frac{\frac{[T,F]}{[R,F]}}{\frac{R\cap[T,F]}{[R,F]}}\cong\frac{[T,F]}{R\cap[T,F]}.$$

This implies part (i). The proofs of parts (ii) and (iii) are obtained similarly. □

The following lemma of M.R. Jones [2] is needed, in order to state an interesting corollary to the above theorem.

Lemma 3.3. *(See M.R. Jones [2].) Let G be a finite group with a free presentation $G\cong F/R$. Let K be a central subgroup of G with $K\cong T/R$. Then $[T,F]/[R,F]T'$ is a homomorphic image of $(G/K)\otimes K$.*

Finally, we obtain the following corollary which is a generalization of M.R. Jones [4].

Corollary 3.4. *Let (G,N) be a pair of finite groups and K a normal subgroup of G such that $K\subseteq Z(G)\cap N$. Then*

(i) $|M(G,N)||[N,G]\cap K|$ *divides* $|M(\frac{G}{K},\frac{N}{K})||M(K)||\frac{G}{K}\otimes K|$;
(ii) $d(M(G,N))\leqslant d(M(\frac{G}{K},\frac{N}{K}))+d(M(K))+d(\frac{G}{K}\otimes K)$;
(iii) $e(M(G,N))$ *divides* $e(M(\frac{G}{K},\frac{N}{K}))e(M(K))e(\frac{G}{K}\otimes K)$.

Proof. Using the notations of Lemma 3.1, we have

$$\frac{([T,F]/[R,F])}{([R,F]T'/[R,F])}\cong\frac{[T,F]}{[R,F]T'}\quad\text{and}\quad\frac{[T,F]T'}{[R,F]}\cong\frac{M(K)}{(T'\cap[R,F])/[R,T]}.$$

Now, the results follow by Theorem 3.2 and Lemma 3.3. □

Finally, we remark that if $N=G$ the first author proves the above results in [6] for arbitrary variety of groups and in [7] we have shown these with respect to two varieties of groups.

References

[1] G. Ellis, The Schur multiplier of a pair of groups, Appl. Categ. Structures 6 (3) (1997) 355–371.
[2] M.R. Jones, Multipliers of p-groups, Math. Z. 127 (1972) 165–166.
[3] M.R. Jones, Some inequalities for the multiplicator of a finite group, Proc. Amer. Math. Soc. 39 (1973) 450–456.
[4] M.R. Jones, Some inequalities for the multiplicator of a finite group II, Proc. Amer. Math. Soc. 45 (1974) 167–172.
[5] J.L. Loday, Cohomologie et group de Steinberg relatif, J. Algebra 54 (1978) 178–202.
[6] M.R.R. Moghaddam, Some inequalities for the Baer-invariant of a finite group, Bull. Iranian Math. Soc. 9(1) (Serial No. 16) (1979) 5–10.
[7] M.R.R. Moghaddam, A.R. Salemkar, M. Taheri, The Baer-invariants with respect to two varieties of groups, Algebra Colloq. 8 (2) (2001) 145–151.
[8] I. Schur, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 127 (1904) 20–50.
[9] I. Schur, Untersuchungen über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, J. Reine Angew. Math. 132 (1907) 85–137.
[10] K. Yamazaki, Projective representations and ring extensions, J. Fac. Sci. Univ. Tokyo Sect. I 10 (1964) 147–195.