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T-duality of anomalous Chern–Simons couplings

Mohammad R. Garousi

Department of Physics, Ferdowsi University of Mashhad, P.O. Box 1436, Mashhad, Iran

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Abstract

It is known that the anomalous D_p-brane Chern–Simons couplings are not consistent with the standard rules of T-duality. Using compatibility of these couplings with the linear T-duality transformations, the B-field gauge transformations and the general coordinate transformations as guiding principles we find new couplings at order $O(\alpha'^2)$ for $\mathcal{C}^{(p-3)}$, $\mathcal{C}^{(p-1)}$, $\mathcal{C}^{(p+1)}$ and $\mathcal{C}^{(p+3)}$.

Keywords: T-duality; Chern–Simons couplings

1. Introduction

The dynamics of the D-branes of type II superstring theories is well-approximated by the effective world-volume field theory which consists of the Dirac–Born–Infeld (DBI) and the Chern–Simons (CS) actions. The DBI action describes the dynamics of the brane in the presence of NS–NS background fields. For constant fields, this action can be found by requiring its consistency with the nonlinear T-duality [1,2], *i.e.*,

$$S_{DBI} = -T_p \int d^{p+1}x\, e^{-\phi} \sqrt{-\det\big(G_{ab} + B_{ab} + 2\pi\alpha' f_{ab}\big)} \tag{1}$$

where G_{ab} and B_{ab} are the pull-back of the bulk fields $G_{\mu\nu}$ and $B_{\mu\nu}$ onto the world-volume of D-brane.[1] The curvature corrections to this action have been found in [3] by requiring the consistency of the effective action with the $O(\alpha'^2)$ terms of the corresponding disk-level scattering

E-mail address: garousi@ferdowsi.um.ac.ir.

[1] Our index convention is that the Greek letters $(\mu, \nu, \dots)$ are the indices of the space–time coordinates, the Latin letters $(a, d, c, \dots)$ are the world-volume indices and the letters $(i, j, k, \dots)$ are the normal bundle indices.

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amplitude [4,5]. The B-field corrections at this order have been found in [6] by requiring the consistency of the curvature couplings with the linear T-duality transformations.

On the other hand, the CS part describes the coupling of D-branes to the R–R potential. For constant fields it is given by [7,8]

$$S_{CS} = T_p \int_{M^{p+1}} e^B C \tag{2}$$

where M^{p+1} represents the world-volume of the D_p-brane, C is meant to represent a sum over all appropriate R–R forms and the multiplication rule is the wedge product. The abelian gauge field can be added to the action as $B \to B + 2\pi\alpha' f$. Curvature correction to this action has been found in [9–11] by requiring that the chiral anomaly on the world-volume of intersecting D-branes (I-brane) cancels the anomalous variation of the CS action. This correction is

$$S_{CS} = T_p \int_{M^{p+1}} \mathcal{C}\left(\frac{\mathcal{A}(4\pi^2\alpha' R_T)}{\mathcal{A}(4\pi^2\alpha' R_N)}\right)^{1/2} \tag{3}$$

where $\mathcal{C} = e^B C$ and $\mathcal{A}(R_{T,N})$ is the Dirac roof genus of the tangent and normal bundle curvatures, respectively,

$$\sqrt{\frac{\mathcal{A}(4\pi^2\alpha' R_T)}{\mathcal{A}(4\pi^2\alpha' R_N)}} = 1 + \frac{\pi^2\alpha'^2}{24}\left(\operatorname{tr} R_T^2 - \operatorname{tr} R_N^2\right) + \cdots \tag{4}$$

For totally-geodesic embeddings of the world-volume in the ambient space–time, $R_{T,N}$ are the pull-back curvature 2-forms of the tangent and normal bundles, respectively (see the appendix in Ref. [3] for more details).

It was shown in [16] that at order $O(\alpha'^2)$ the CS action (3) must include additional linear couplings to the NS–NS fields. These couplings were found by studying the S-matrix element of one R–R and one NS–NS vertex operator at order $O(\alpha'^2)$ [4]. In the string frame, they take the form [16][2]:

$$\begin{aligned} S_{CS} \supset \pi^2\alpha'^2 T_p \int d^{p+1}x\, \epsilon^{a_0\cdots a_p}\Bigg(& \frac{1}{2!(p-1)!}\left[F^{(p)}_{ia_2\cdots a_p,a} H_{a_0a_1}{}^{a,i} - F^{(p)}_{aa_2\cdots a_p,i} H_{a_0a_1}{}^{i,a}\right] \\ & + \frac{2}{p!}\left[\frac{1}{2!}F^{(p+2)}_{ia_1\cdots a_p j,a}\mathcal{R}^a{}_{a_0}{}^{ij} - \frac{1}{p+1}F^{(p+2)}_{a_0\cdots a_p j,i}\left(\hat{\mathcal{R}}^{ij} - \phi^{,ij}\right)\right] \\ & - \frac{1}{3!(p+1)!}F^{(p+4)}_{ia_0\cdots a_p jk,a} H^{ijk,a}\Bigg) \end{aligned} \tag{5}$$

[2] Using the on-shell relations, the standard definition of the curvature tensor $\hat{R}_{ij}$ has been changed in [16] to $\hat{R}_{ij} \equiv \frac{1}{2}(R_{ia}{}^a{}_j - R_{ik}{}^k{}_j)$. With this tensor the coupling $F^{(p+2)}_{a_0\cdots a_p j,i}\hat{R}^{ij}$ is then invariant under linear T-duality [16]. If one uses the standard definition $\hat{R}_{ij} \equiv R_{ia}{}^a{}_j$, then the second term in the second line of (5) can be written at the linear order as $F^{(p+2)}_{a_0\cdots a_p}{}^{j,i}(h_{ij,aa} + h_{aa,ij} - h_{ia,aj} - h_{ja,ai} - 2\phi_{,ij})/2(p+1)$ where h is the metric perturbation. Under T-duality along the world-volume direction y, the R–R factor $F^{(p+2)}_{a_0\cdots a_p}{}^{j,i}/(p+1)$ which includes the Killing index y, transforms to $F^{(p+1)}_{a_0\cdots a_{p-1}}{}^{j,i}$. The latter, however, does not include the Killing index. Hence, the indices i, j in the T-dual theory do not include the Killing index y. Using this observation, one can easily verify that the metric/dilaton factor $(h_{ij,aa} + h_{aa,ij} - h_{ia,aj} - h_{ja,ai} - 2\phi_{,ij})$ is invariant under the linear T-duality. Hence, the second term in the second line of (5) is invariant under the T-duality.

where $\mathcal{R}$ is the linearized Riemann curvature tensor of the background metric, $F^{(n)} = dC^{(n-1)}$, and commas are used to denote partial differentiation. Since these couplings have been found by the S-matrix method, there is an on-shell ambiguity in defining these terms [17,18]. The above couplings are consistent with the T-duality transformations at a linearized level and are invariant under the B-field gauge transformations. In particular, the sum of the second term in the first line and the last two terms in the second line form a T-duality invariant set of terms, and the remaining terms form another T-duality invariant set. We call each of these a T-dual multiplet.

One may extend (5) to the nonlinear couplings by replacing C with $\mathcal{C} = e^B C$ and by replacing the ordinary derivatives with their covariant counter parts. In fact the first replacement is required for consistency of the above couplings with the nonlinear T-duality transformations [4]. When the R–R potential carries one transverse index, this replacement produces the following couplings for $C^{(p-3)}$:

$$\frac{\pi^2\alpha'^2 T_p}{2!(p-4)!}\int d^{p+1}x\,\epsilon^{a_0a_1\cdots a_p}\left(\frac{1}{2!}C^{(p-3)}_{ia_2\cdots a_{p-3},a_{p-2}}B_{a_{p-1}a_p}\right.$$
$$\left.-\frac{1}{3!}C^{(p-3)}_{a_2\cdots a_{p-3}i}H_{a_{p-2}a_{p-1}a_p}\right)_{,a}H_{a_0a_1}{}^{a,i}$$

The first term breaks the B-field gauge symmetry. However, it can be restored by the standard replacement of $B_{a_{p-1}a_p}$ with $(B_{a_{p-1}a_p} + 2\pi\alpha' f_{a_{p-1}a_p})$. It has been shown in [12] that the S-matrix element of one R–R potential and two B-field vertex operators reproduce exactly the above couplings. When the R–R potential carries only the world-volume indices, the above replacement does not restore the gauge symmetry in many terms. The non-gauge invariant terms, however, are invariant under the linear T-duality at the level of two B-fields, so it is consistent with the linear T-duality to remove them. On the other hand, the S-matrix calculations produce only the gauge invariant couplings [26].

It has been pointed out in [13] that the anomalous CS couplings (3) must be incomplete for non-constant B-field as they are not compatible with the T-duality. T-duality exchanges the components of the metric and the B-field whereas the couplings (3) involve only the metric through the curvature terms. A systematic approach for including the B-field in a theory might be provided by the 'double field theory' formalism in which the fields depend both on the usual space–time coordinates and on the winding coordinates [14]. In this paper, however, we use the method that was used in [13] to find the Myers terms in the non-abelian CS action at order $O(\alpha'^0)$. That is, we add new couplings to the CS action at order $O(\alpha'^2)$ to make it compatible with T-duality. The T-dual multiplet that we find includes the R–R potentials $C^{(p-3)}$, $C^{(p-1)}$ and $C^{(p+1)}$. These couplings have been also found in [15]. They are, however, neither covariant nor invariant under the B-field gauge transformations.

The disk-level S-matrix element of one R–R potential $C^{(p-3)}$ and two B-field vertex operators produces not only the $C^{(p-3)}$ component of the above T-dual multiplet, but also produces some other contact terms as well as massless poles at order $O(\alpha'^2)$ [26]. Consistency of the amplitude with linear T-duality then requires one to extend the latter contributions to contact-term and the massless-pole T-dual multiplets. More generally, one may extend the S-matrix element to a set of S-matrix elements which are invariant under the linear T-duality transformations. We call this set the S-matrix T-dual multiplet.

Having both the contact-term as well as the massless-pole T-dual multiplets at order $O(\alpha'^2)$, it raises the question of how they come together to produce explicit covariant/gauge-invariant results. One may expect that these multiplets can be combined separately to become covariant/gauge-invariant. However, as we will show the S-matrix calculation indicates that some of the terms in a contact-term multiplet combine with the massless-pole multiplets to produce the covariant and gauge-invariant results. We will show that such terms must be proportional to the Mandelstam variables. This phenomenon does not appear in the T-dual multiplets in (5) because the S-matrix element of two closed string vertex operators at order $O(\alpha'^2)$ has only contact terms [4].

The outline of the paper is as follows: We begin in Section 2 by reviewing the T-duality transformations and the method for finding the T-dual completion of a coupling. In Section 3.1, we show that the standard CS coupling (3) is not consistent with the linear T-duality transformations and add new couplings at order $O(\alpha'^2)$ to find its corresponding T-dual multiplet. The $C^{(p-3)}$ component of this CS multiplet, however, is not invariant under the B-field gauge transformations. In Sections 3.2, by adding another T-dual multiplet, we write the $C^{(p-3)}$ component of the combined multiplet in a T-dual and gauge-invariant form (see Eq. (24)). In Section 3.3, we argue that the contact terms in an S-matrix T-dual multiplet which are proportional to the Mandelstam variables, may combine with the massless poles to produce covariant and gauge-invariant results. Since we are not considering the massless poles of the S-matrix multiplet in this paper, we will not attempt to make such contact terms to be covariant/gauge-invariant. Adding three contact-term T-dual multiplets to the CS multiplet, we then write the $C^{(p-1)}$ component of these multiplets in a covariant and gauge-invariant form (see Eq. (34)). In Section 3.4, by adding one more T-dual multiplet to the list, we write the $C^{(p+1)}$ components in a covariant and gauge-invariant form (see Eq. (36)). Finally, we show in Section 3.5 that the $C^{(p+3)}$ components of the above multiplets are covariant and gauge-invariant (see Eq. (37)).

2. T-duality

The full set of nonlinear T-duality transformations for massless R–R and NS–NS fields have been found in [19–23]. The nonlinear T-duality transformations of the fields C and B are such that the expression $\mathcal{C} = e^B C$ transforms linearly under T-duality [24]. When the T-duality transformation acts along the Killing coordinate y, the massless NS–NS fields and $\mathcal{C}$ become

$$\begin{aligned}
e^{2\widetilde{\phi}} &= \frac{e^{2\phi}}{G_{yy}} \\
\widetilde{G}_{yy} &= \frac{1}{G_{yy}} \\
\widetilde{G}_{\mu y} &= \frac{B_{\mu y}}{G_{yy}} \\
\widetilde{G}_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\mu y}G_{\nu y} - B_{\mu y}B_{\nu y}}{G_{yy}} \\
\widetilde{B}_{\mu y} &= \frac{G_{\mu y}}{G_{yy}} \\
\widetilde{B}_{\mu\nu} &= B_{\mu\nu} - \frac{B_{\mu y}G_{\nu y} - G_{\mu y}B_{\nu y}}{G_{yy}} \\
\widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu y} &= \mathcal{C}^{(n-1)}_{\mu\cdots\nu} \\
\widetilde{\mathcal{C}}^{(n)}_{\mu\cdots\nu} &= \mathcal{C}^{(n+1)}_{\mu\cdots\nu y}
\end{aligned} \tag{6}$$

where $\mu, \nu \neq y$. In above transformation the metric is given in the string frame. If y is identified on a circle of radius R, *i.e.*, $y \sim y + 2\pi R$, then after T-duality the radius becomes $\tilde{R} = \alpha'/R$. The string coupling is also shifted as $\tilde{g} = g\sqrt{\alpha'}/R$. We would like to study the consistency of the CS couplings (3) with the linear T-duality transformations. Assuming that the NS–NS fields are small perturbations around the flat space, the above transformations take the following linear form:

$$\tilde{\phi} = \phi - \frac{1}{2}h_{yy}, \qquad \tilde{h}_{yy} = -h_{yy}, \qquad \tilde{h}_{\mu y} = B_{\mu y}, \qquad \tilde{B}_{\mu y} = h_{\mu y}, \qquad \tilde{h}_{\mu\nu} = h_{\mu\nu}$$
$$\tilde{B}_{\mu\nu} = B_{\mu\nu}, \qquad \widetilde{C}^{(n)}_{\mu\cdots\nu y} = C^{(n-1)}_{\mu\cdots\nu}, \qquad \widetilde{C}^{(n)}_{\mu\cdots\nu} = C^{(n+1)}_{\mu\cdots\nu y} \tag{7}$$

The strategy to find T-duality invariant couplings is given in [6]. Let us review it here. Suppose we are implementing T-duality along a world-volume direction y of a D_p-brane. First, we separate the world-volume indices along and orthogonal to y direction and then apply the T-duality transformations. The orthogonal indices are the complete world-volume indices of the T-dual D_{p-1}-brane. However, y in the T-dual theory, which is a normal bundle index, is not complete. On the other hand, the normal bundle indices of the original theory are not complete in the T-dual D_{p-1}-brane. They do not include the y index. In a T-duality invariant theory, y must be combined with the incomplete normal bundle indices to make them complete. If a theory is not invariant under the T-duality, one should then add new terms to it to have the complete indices in the T-dual theory. In this way one makes the theory to be T-duality invariant by adding new couplings.

One may also implement T-duality along a transverse direction y of a D_p-brane. In this case, we separate the transverse indices along and orthogonal to y direction and then apply the T-duality transformations. The latter indices are complete in the dual D_{p+1}-brane. However, the complete world-volume indices of the original D_p-brane are not complete in the dual D_{p+1}-brane. They must include the y index to be complete. In a T-duality invariant theory, y, which is a world-volume index in the dual theory, must be combined with the incomplete world-volume indices of the dual D_{p+1}-brane to become complete.

Let us apply the above method to the DBI action. Expansion of the DBI action (1) produces the following terms at order $O(\alpha'^0)$:

$$S_{DBI} = -T_p \int d^{p+1}x \left[1 - \phi + \frac{1}{2}h_a{}^a + \frac{1}{8}\left(h_a{}^a\right)^2 - \frac{1}{4}h_a{}^b h_b{}^a - \frac{1}{4}B_a{}^b B_b{}^a \right.$$
$$\left. + \frac{1}{2}\phi^2 - \frac{1}{2}\phi h_a{}^a + \cdots\right]$$

where we have considered perturbations around flat space. The metric takes the form $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a small perturbation. We want to implement T-duality along a world-volume direction. So we write the linear terms above in the following form:

$$-\phi + \frac{1}{2}h_a{}^a = -\phi + \frac{1}{2}h_{\tilde{a}}{}^{\tilde{a}} + \frac{1}{2}h_{yy}$$

where the world-volume index $\tilde{a}$ does not include y. Under the linear T-duality transformations (7), it transforms to $-\phi + \frac{1}{2}h_{\tilde{a}}{}^{\tilde{a}}$. Since there is no incomplete index, one concludes that the linear terms in the DBI action are invariant under the linear T-duality transformations. Doing the same steps, one finds that the quadratic terms transform under the linear T-duality transformations as

$$\frac{1}{8}\big(h_{\tilde{a}}{}^{\tilde{a}}\big)^2-\frac{1}{4}(h_{yy})^2-\frac{1}{4}h_{\tilde{a}}{}^{\tilde{b}}h_{\tilde{b}}{}^{\tilde{a}}-\frac{1}{4}B_{\tilde{a}}{}^{\tilde{b}}B_{\tilde{b}}{}^{\tilde{a}}+\frac{1}{2}h_{\tilde{a}}{}^{y}h_{y}{}^{\tilde{a}}+\frac{1}{2}B_{\tilde{a}}{}^{y}B_{y}{}^{\tilde{a}}$$
$$+\frac{1}{2}\phi^2-\frac{1}{2}h_{\tilde{a}}{}^{\tilde{a}}\phi \tag{8}$$

This expression includes terms with the y index. However, one should not conclude that the quadratic terms are not invariant under the T-duality transformations. One has to add the non-linear T-duality transformations of the linear terms $-\phi+h_a{}^a/2$, which include the following quadratic terms:

$$\frac{1}{4}(h_{yy})^2-\frac{1}{2}h_{\tilde{a}}{}^{y}h_{y}{}^{\tilde{a}}-\frac{1}{2}B_{\tilde{a}}{}^{y}B_{y}{}^{\tilde{a}}$$

to the above couplings. This will cancel the terms in (8) which have y index. Hence, according to our expectations, the quadratic order terms in the DBI action are invariant under the T-duality transformations.

3. New couplings

It is known that the anomalous CS couplings of D-branes to space–time curvature are incomplete, as they are inconsistent with T-duality. We will construct a form of the couplings which are consistent with the linear T-duality. We are interested in the $O(\alpha'^2)$ terms in (4). The world-volume curvature R_T and the field strength R_N are related to the pull-back of the space–time Riemann tensor and the second fundamental form though the Gauss–Codazzi equation:

$$(R_T)_{abcd}=R_{abcd}+\delta_{ij}\big(\Omega^i_{ac}\Omega^j_{bd}-\Omega^i_{ad}\Omega^j_{bc}\big)$$
$$(R_N)_{ab}{}^{ij}=-R^{ij}{}_{ab}+G^{cd}\big(\Omega^i_{ac}\Omega^j_{bd}-\Omega^j_{ac}\Omega^i_{bd}\big)$$

where Ω is the second fundamental form (see the appendix in [3]). For totally-geodesic embedding, Ω is zero. In the static gauge, that we are going to use in this paper, the second fundamental form is non-zero. Hence, at order $O(\alpha'^2)$ there are three different terms: Terms with two Riemann tensors, terms with one Riemann tensor and two fundamental forms, and terms with four fundamental forms. At the linearized level, the Riemann curvature tensor is the second derivative of the fluctuation of the space–time metric and the second fundamental form is the second derivative of the massless transverse scalar fields on the D-brane. In this paper we are interested in studying the T-duality transformation of the two Riemann curvature terms. Hence, we consider the following CS couplings in (3):

$$\frac{T_p}{2!2!(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abcd}C^{(p-3)}_{a_0\cdots a_{p-4}}\big[R_{ab}{}^{ef}R_{cdfe}-R_{ab}{}^{kl}R_{cdlk}\big] \tag{9}$$

where we have employed the static gauge. That is, first we have used the space–time diffeomorphisms to define the D$_p$-brane world-volume as $x^i=0$, where $i=p+1,\dots,9$, and then with the world-volume diffeomorphisms, we matched the internal coordinates with the remaining space–time coordinates on the surface: $\sigma^a=x^a$ for $a=0,1,\dots,p$. We have also ignored the pull-back operations, *i.e.*, we work only with the restriction of the Riemann tensor to the appropriate subspace.

To find the T-dual completion of the above couplings at the linearized level, we will consider perturbation around flat space where the metric takes the form $G_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, where $h_{\mu\nu}$ is a small perturbation. We denote the Riemann tensor to linear order in h by $\mathcal{R}_{\mu\nu\rho\lambda}$. This linear Riemann tensor is

$$\mathcal{R}_{\mu\nu\rho\lambda} = \frac{1}{2}(h_{\mu\lambda,\nu\rho} + h_{\nu\rho,\mu\lambda} - h_{\mu\rho,\nu\lambda} - h_{\nu\lambda,\mu\rho}) \tag{10}$$

The coupling (9) at the linearized level is then

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\, \epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\big[h_a{}^f{}_{,b}{}^e(h_{ce,df} - h_{cf,de}) - h_a{}^l{}_{,b}{}^k(h_{ck,dl} - h_{cl,dk})\big] \tag{11}$$

The indices that are contracted with the volume form are totally antisymmetric so we do not use the antisymmetric notation for them. The above couplings have been verified by the S-matrix element of one R–R and two graviton vertex operators in [25]. We will examine the expression (11) under the linear T-duality transformations (7), and find its corresponding T-dual multiplet. We call this multiplet, which has the Chern–Simons couplings in its first component, the Chern–Simons multiplet.

3.1. Chern–Simons multiplet

We begin by implementing T-duality along a world-volume direction of D_p-brane, which is denoted by y. From the contraction with the world-volume form, one of the indices $a_0, \ldots, a_{p-4}$ of the R–R potential[3] or the indices a, c of the metric fluctuation in (11) must include y. So there are two cases to consider: First when the R–R potential $\mathcal{C}^{(p-3)}$ carries the y index and second when the metric carries the y index. In the former case, we write (11) as

$$\frac{T_p}{2!(p-4)!}\int d^{p+1}x\, \epsilon^{a_0\cdots a_{p-4}yabcc}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-5}y}\big[h_a{}^f{}_{,b}{}^e(h_{ce,df} - h_{cf,de}) - h_a{}^l{}_{,b}{}^k(h_{ck,dl} - h_{cl,dk})\big] \tag{12}$$

The indices e and f include the Killing coordinate y which is a world-volume coordinate. However, in the T-dual theory, y is a transverse coordinate. To be able to use the T-duality transformation rules (7), we separate y from e, f. Hence, we write the above equation as

$$\frac{T_p}{2!(p-4)!}\int d^{p+1}x\, \epsilon^{a_0\cdots a_{p-4}yabcc}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-5}y}\big[h_a{}^{\tilde f}{}_{,b}{}^{\tilde e}(h_{c\tilde e,d\tilde f} - h_{c\tilde f,d\tilde e}) - h_a{}^y{}_{,b}{}^{\tilde e}h_{cy,d\tilde e} - h_a{}^l{}_{,b}{}^k(h_{ck,dl} - h_{cl,dk})\big]$$

where the "tilde" over the world-volume indices e, f means they do not include the Killing direction y. Now, the above equation transforms under (7) to the following couplings of D_{p-1}-brane:

$$\frac{T_{p-1}}{2!(p-4)!}\int d^px\, \epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-4)}_{a_0\cdots a_{p-5}}\big[h_a{}^f{}_{,b}{}^e(h_{ce,df} - h_{cf,de}) - B_a{}^y{}_{,b}{}^eB_{cy,de} - h_a{}^{\tilde l}{}_{,b}{}^{\tilde k}(h_{c\tilde k,d\tilde l} - h_{c\tilde l,d\tilde k})\big] \tag{13}$$

where we have used the fact that $T_p \sim 1/g_s$ and the relation $2\pi\sqrt{\alpha'}T_p = T_{p-1}$. In the above equation the "tilde" over the transverse indices k, l means they do not include the Killing direction y which is now a direction normal to the D_{p-1}-brane. The contracted indices of the second

[3] In the literature, the R–R potential is C. However, in this paper we always work with $\mathcal{C} = e^BC$ and call it R–R potential.

and third terms are not complete, *i.e.*, the second term has y which does not include all other transverse coordinates, and the last term has the index $\tilde{l}$ which does not include the transverse coordinate y. This indicates that the original action (11) is not consistent with the linear T-duality.

To remedy this failure, one has to add some new couplings. These couplings must be such that when they combine with the couplings (11), the indices in the combination must remain complete after T-duality. Consider then the following couplings on the world-volume of the D_p-brane:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\big[-B_a{}^k{}_{,b}{}^e B_{ck,de}+B_a{}^e{}_{,b}{}^k B_{ce,dk}\big] \tag{14}$$

Doing the same steps as we have done for the couplings (11), one finds that the above couplings transforms to the following couplings of D_{p-1}-brane:

$$\frac{T_{p-1}}{2!(p-4)!}\int d^p x\,\epsilon^{a_0\cdots a_{p-5}abcd}\mathcal{C}^{(p-4)}_{a_0\cdots a_{p-5}}\big[-B_a{}^{\tilde{k}}{}_{,b}{}^e B_{c\tilde{k},d\tilde{e}}+B_a{}^e{}_{,b}{}^k B_{ce,dk}+h_a{}^y{}_{,b}{}^k h_{cy,dk}\big] \tag{15}$$

In this equation also the index $\tilde{k}$ in the first and the index y in the last terms are not complete. This indicates that the coupling (14) is not consistent with the T-duality either. However, the sum of the first term above and the second term of (13), and the sum of the last terms above and the last term of (13) have complete indices. Hence, the combination of actions (11) and (14) are consistent with T-duality when y is an index on the R–R potential. That is, the following couplings of D_p-brane:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\big[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de})-B_a{}^k{}_{,b}{}^e B_{ck,de}$$
$$-h_a{}^l{}_{,b}{}^k(h_{ck,dl}-h_{cl,dk})+B_a{}^e{}_{,b}{}^k B_{ce,dk}\big] \tag{16}$$

are consistent with the linear T-duality transformations (7) when the R–R potential carries the Killing index.

In order to proceed further, one observes that in the actions (16), two indices a and c, which are carried by the metric/B-field terms, contract with the volume form. When performing T-duality along a particular world-volume direction, either one of these or one of the indices on the R–R potential must equal the T-dual coordinate y. We have already shown that the case, in which the index y is carried by the R–R field, is consistent with T-duality. Now we will check the second case where index y is carried by the metric/B-field terms. The strategy is to choose one of the two indices to perform the T-duality and infer what extra terms must be included for the consistency. The resulting terms will have one remaining index. So we repeat this procedure to arrive at an action in which the metric/B-field terms have no index contracted with the volume form.

There are two ways for the metric/B-field terms in (16) to carry the Killing coordinate y, *i.e.*, either a or c carries the index y. One can write the D_p-brane couplings (16) as

$$\frac{T_p}{(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abyd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\big[h_a{}^f{}_{,b}{}^e(h_{ye,df}-h_{yf,de})-B_a{}^k{}_{,b}{}^e B_{yk,de}$$
$$-h_a{}^l{}_{,b}{}^k(h_{yk,dl}-h_{yl,dk})+B_a{}^e{}_{,b}{}^k B_{ye,dk}\big]$$

Mimicking the steps which are used to get (12), one finds that the transformation of the above couplings under T-duality (7) gives the following couplings for D_{p-1}-brane:

$$\frac{T_{p-1}}{(p-3)!}\int d^px\,\epsilon^{a_0\cdots a_{p-4}abd}\mathcal{C}^{(p-2)}_{a_0\cdots a_{p-4}}{}^{y}\big[-h_a{}^f{}_{,b}{}^e(B_{ye,df}-B_{yf,de})+B_a{}^k{}_{,b}{}^e h_{yk,de}$$
$$+h_a{}^l{}_{,b}{}^k(B_{yk,dl}-B_{yl,dk})-B_a{}^e{}_{,b}{}^k h_{ye,dk}\big] \tag{17}$$

In this case the world-volume indices e, f and the transverse indices k,l are all complete. However, the y index is not a complete index. Inspired by the above couplings, one can guess that for the D_p-brane, the couplings should be following:

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-3}abd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\big[-h_a{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_a{}^k{}_{,b}{}^e h_{ik,de}$$
$$+h_a{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})-B_a{}^e{}_{,b}{}^k h_{ie,dk}\big] \tag{18}$$

One can easily verify that the above couplings are invariant under the linear T-duality transformations (7) when the world-volume Killing coordinate y is carried by the R–R potential, *i.e.*, the R–R potential $\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-4}y}{}^{i}$ transforms to $\mathcal{C}^{(p-2)}_{a_0\cdots a_{p-4}}{}^{i}$ in which the transverse index i does not include y, and the couplings for $i=y$ are given by (17).

Finally, one observes that there is one possibility for the metric/B-field terms in (18) to carry the Killing coordinate y, *i.e.*, a carries the index y. One can write it as

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-3}ybd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\big[-h_y{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_y{}^k{}_{,b}{}^e h_{ik,de}$$
$$+h_y{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})-B_y{}^e{}_{,b}{}^k h_{ie,dk}\big]$$

Under T-duality it transforms to the following couplings for D_{p-1}-brane:

$$\frac{T_{p-1}}{(p-2)!}\int d^px\,\epsilon^{a_0\cdots a_{p-3}bd}\mathcal{C}^{(p)}_{a_0\cdots a_{p-3}}{}^{iy}\big[B_y{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_y{}^k{}_{,b}{}^e h_{ik,de}$$
$$-B_y{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})+h_y{}^e{}_{,b}{}^k h_{ie,dk}\big] \tag{19}$$

where again, the world-volume indices e, f and the transverse indices k,l are all complete, but y is not. Eq. (19) suggests the following couplings for the D_p-brane:

$$\frac{T_p}{2!(p-1)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-2}bd}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\big[B_j{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_j{}^k{}_{,b}{}^e h_{ik,de}$$
$$-B_j{}^l{}_{,b}{}^k(B_{ik,dl}-B_{il,dk})+h_j{}^e{}_{,b}{}^k h_{ie,dk}\big] \tag{20}$$

One can again verify that the above couplings are invariant under T-duality when y is carried by the R–R potential, *i.e.*, The R–R potential $\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-3}y}{}^{ij}$ transforms to $\mathcal{C}^{(p)}_{a_0\cdots a_{p-3}}{}^{ij}$ in which the transverse indices i,j do not include y, and the couplings for $i=y$ or $j=y$ are given by (19).

There is no index in the B-field/metric in (20) that contracts with the volume form. Hence, the combination of couplings (16), (18) and (20) forms a complete T-dual multiplet, *i.e.*, the CS multiplet. This multiplet is

$$T_p\int d^{p+1}x\bigg(\frac{\epsilon^{a_0\cdots a_{p-4}abcd}}{2!(p-3)!}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}\big[h_a{}^f{}_{,b}{}^e(h_{ce,df}-h_{cf,de})-B_a{}^k{}_{,b}{}^e B_{ck,de}\big]$$
$$+\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\big[-h_a{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})+B_a{}^k{}_{,b}{}^e h_{ik,de}\big]$$
$$+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{2!(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\big[B_j{}^f{}_{,b}{}^e(B_{ie,df}-B_{if,de})-h_j{}^k{}_{,b}{}^e h_{ik,de}\big]\bigg)-(\cdots) \tag{21}$$

where dots refer to the similar expressions as above with the replacement of the world-volume indices (e, f) by the transverse indices (k, l) and (e, k) by (k, e). We call the $\mathcal{C}^{(p-3)}$ couplings the first component of the multiplet, the $\mathcal{C}^{(p-1)}$ couplings are called the second component and so on. The above couplings have been also found in [15] and verified with some of the contact terms of the S-matrix element of one R–R and two NS–NS vertex operators. A more details study of the S-matrix element [26], however, reveals that the string amplitude has more contact terms than those considered in [15].

3.2. $\mathcal{C}^{(p-3)}$ couplings

One can easily check that the first component of the CS multiplet (21) is not invariant under the B-field gauge transformations. To write it in terms of field strength H, we add another T-dual multiplet to (21). Since the gravity couplings to $\mathcal{C}^{(p-3)}$ are those given by (21) [25], the first component of the new T-dual multiplet must include only the B-field. This happens when the indices of the R–R potential and the B-fields contract either with the volume form or with the derivative of these fields. Consider the following couplings for $\mathcal{C}^{(p-3)}$:

$$\frac{T_p}{2!(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}(B_{ak,be}-B_{ae,bk})B_{cd}{}^{,ek} \tag{22}$$

As indices e and k appear in derivatives, it is easy to verify that this coupling is invariant under linear T-duality transformations (7) when the Killing coordinate y is carried by the R–R potential. When y is carried by the B-field, it is not invariant under T-duality. In those cases one has to add more terms involving the higher R–R forms to arrive at a complete T-dual multiplet. Applying the steps that are used in the previous section, one finds the following T-dual multiplet corresponding to (22)

$$\begin{aligned}T_p\int d^{p+1}x\bigg(&\frac{\epsilon^{a_0\cdots a_{p-4}abcd}}{2!(p-3)!}\mathcal{C}^{(p-3)}_{a_0\cdots a_{p-4}}B_{ak,be}B_{cd}{}^{,ek}\\&+\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{2!(p-2)!}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\big[h_{ik,be}B_{ad}{}^{,ek}-2B_{ak,be}h_{id}{}^{,ek}\big]\\&+\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\bigg[-h_{ik,be}h_{jd}{}^{,ek}+\frac{1}{2}B_{dk,be}B_{ij}{}^{,ek}\bigg]\\&+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{2!p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}h_{ie,bk}B_{jn}{}^{,ek}\bigg)-(\cdots)\end{aligned} \tag{23}$$

where dots refer to the expressions similar to the one in the first bracket with indices (e, k) replaced by (k, e).

Now, the first components of the CS multiplet (21) and the above multiplet can be written in terms of H, *i.e.*,

$$\begin{aligned}\frac{T_p}{2!2!(p-3)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-4}abcd}\mathcal{C}^{p-3}_{a_0\cdots a_{p-4}}\Big[&\mathcal{R}_{ab}{}^{ef}\mathcal{R}_{cdfe}-\mathcal{R}_{ab}{}^{kl}\mathcal{R}_{cdlk}\\&-\frac{1}{2}H_{abk,e}H_{cd}{}^{k,e}+\frac{1}{2}H_{abe,k}H_{cd}{}^{e,k}\Big]\end{aligned} \tag{24}$$

where $H_{\mu\nu\rho}=B_{\mu\nu,\rho}+B_{\rho\mu,\nu}+B_{\nu\rho,\mu}$. The terms in the second line are reproduced by the S-matrix calculation [26]. Unlike the gravity couplings in the first line, the B-field couplings are not invariant under the R–R gauge transformation.

One may then expect that there are some other T-dual multiplets that should be included in the action (24). As we have pointed out above, their first component must include only the B-field. The presence of such couplings can be fixed by the S-matrix calculation. In fact the couplings (24) as well as the following couplings are produced by the S-matrix element of one R–R and two NS–NS vertex operators [26]:

$$\frac{T_p}{(p-3)!}\int d^{p+1}x\,\epsilon^{a_0a_1\cdots a_p}C^{(p-3)}_{a_4\cdots a_p}\bigg(\frac{1}{2!2!}H^{ea_0a_1}{}_{,ef}H^{fa_2a_3}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,ek}H^{kea_3}$$
$$+\frac{1}{2!2!}H^{a_0a_1e,k}{}_{e}H^{a_2a_3}{}_{k}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,e}H^{efa_3}{}_{,f}+\frac{1}{3!}H^{a_0a_1a_2}{}_{,k}H^{kea_3}{}_{,e}\bigg) \tag{25}$$

The S-matrix element also produces some massless open-string/closed-string poles at order $O(\alpha'^2)$. The open string poles are reproduced in field theory by the DBI action (1) and the following couplings [26]:

$$\frac{T_p}{(p-3)!}\int d^{p+1}x\,\epsilon^{a_0a_1\cdots a_p}\bigg(\frac{1}{2!2!}C^{(p-3)}_{a_4\cdots a_p,k}\big(2H_{a_0a_1}{}^{e,k}{}_{e}-H_{a_0a_1}{}^{k,e}{}_{e}\big)\big(B_{a_2a_3}+2\pi\alpha' f_{a_2a_3}\big)$$
$$-C^{(p-3)}_{a_4\cdots a_p}\bigg[\frac{1}{3!}H^{a_0a_1a_2,e}{}_{ef}\big(B^{fa_3}+2\pi\alpha' f^{fa_3}\big)$$
$$+\frac{1}{2!2!}H^{a_0a_1f,e}{}_{e}\big(B^{a_2a_3}+2\pi\alpha' f^{a_2a_3}\big)_{,f}\bigg]\bigg) \tag{26}$$

The closed string poles, on the other hand, can be reproduced by the bulk supergravity and the D-brane couplings (5). It is shown in [26] that even though the contact terms and the massless poles are not separately invariant under the R–R gauge transformations, their combination satisfies this symmetry.

3.3. $C^{(p-1)}$ couplings

Before making the other components to be covariant/gauge-invariant, let us digress to discuss a subtle point in finding the field theory couplings from the corresponding string theory S-matrix elements. It has been shown in [26] that the S-matrix element of one R–R potential $C^{(p+5)}$ and two gravitons is zero, and the S-matrix element of one $C^{(p+5)}$ and two B-fields is non-zero. When writing the latter amplitude in terms of field strength H, one finds that it has only massless closed string poles at order $O(\alpha'^2)$ (see Appendix B in [26]). On the other hand, using the steps that are applied in Section 3.1, one finds that the T-dual multiplet corresponding to the couplings (24) has no $C^{(p+5)}$ component. However, for the couplings (25), one find following component in the T-dual multiplet:

$$C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn}B^{ij,e}{}_{ef}B^{mn,k}+C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn}B^{ij,e}{}_{ek}B^{mn,k} \tag{27}$$

which arises from the contact terms of the S-matrix element. Similarly for the couplings (26), T-dual multiplet has following component:

$$C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn,k}B^{ij,e}{}_{ek}B^{mn}-C^{(p+5)}_{a_0\cdots a_p}{}^{ijmn}B^{ij,e}{}_{ef}B^{mn,f} \tag{28}$$

which arises from the massless open string poles of the S-matrix element. The S-matrix element also produces some massless closed string poles.

The above results indicate that the contact terms (27) and (28) must be canceled by the massless closed string poles. To see this explicitly, we apply the same steps as we have used in Section 3.1 to calculate the $C^{(p+5)}$ component of the S-matrix element of one $C^{(p-3)}$ and two B-fields [26]. The $C^{(p+5)}$ component of the amplitude (35) in [26] is

$$\mathcal{A} \sim p_2 \cdot V \cdot p_2 \left[p_3 \cdot V \cdot p_3 \mathcal{J}_3 - \frac{1}{2} p_2 \cdot V \cdot p_3 \mathcal{J}_1 + \frac{1}{2} p_2 \cdot N \cdot p_3 \mathcal{J}_2 - p_1 \cdot N \cdot p_3 \mathcal{I}_7 \right] + \frac{1}{4} p_1 \cdot N \cdot p_2 [p_1 \cdot N \cdot p_3 \mathcal{I}_1 - p_2 \cdot N \cdot p_3 \mathcal{I}_2 + p_2 \cdot \cdot p_3 \mathcal{I}_3] + (2 \leftrightarrow 3) \tag{29}$$

The amplitude also has the overall factor of the polarization of the external states, *i.e.*, $\epsilon^{a_0\cdots a_p} \varepsilon_{1a_0\cdots a_p ijmn} \varepsilon_2^{ij} \varepsilon_3^{mn}$. In the above equation, $\mathcal{J}$s and $\mathcal{I}$s are functions of the Mandelstam variables. We refer the interested reader to [26] for the notations. At low energy, the terms in the first line produce the contact terms (27) and (28) and some massless closed string poles, whereas the terms in the second line produce only massless closed string poles. Using the identities between $\mathcal{J}$s and $\mathcal{I}$s, *i.e.*, Eq. (33) in [26], one finds that the terms in the first line add up to zero, as anticipated above. On the other hand, the terms in the second line combine with some other S-matrix T-dual multiplets to produce the result for the S-matrix element of one $C^{(p+5)}$ and two B-fields.

This phenomenon may happen only when two derivatives in a coupling contract with each other or, in momentum space, when a contact term is proportional to a Mandelstam variable, *e.g.*, the couplings in (27) and (28). The reason is that the identities between $\mathcal{J}$s and $\mathcal{I}$s which arise from the requirement that the S-matrix element must satisfy the Ward identities corresponding to the NS–NS massless fields, have the structure of $\sum_i (M_i \mathcal{J}_i + N_i \mathcal{I}_i) = 0$ where M_i and N_i are at least the linear order of some Mandelstam variables. To clarify this point, suppose the S-matrix element has the structure $\sum_i (\cdots)\mathcal{J}_i$ where $(\cdots)$ refers to the polarization tensors and the four momenta that are produced by performing the correlation between the vertex operators [12]. Upon imposing the Ward identity corresponding to one of the NS–NS states, *i.e.*, $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + p_\mu \zeta_\nu \pm p_\nu \zeta_\mu$ one finds a relation like $\sum_i M_i \mathcal{J}_i = 0$.

These identities make the covariant/gauge-invariant form of the couplings, in which two derivatives contract with each other, ambiguous. This ambiguity, however, is resolved when one considers both the contact terms as well as the massless poles at order $O(\alpha'^2)$. Hence, we will not discuss the covariant/gauge-invariant form of such couplings, *e.g.*, we will not consider the $C^{(p-1)}$, $C^{(p+1)}$ and $C^{(p+3)}$ components of (23). Note that the first component of this multiplet is combined with the first component of the CS multiplet to produce the gauge invariant result (24).

We now try to make the other components of the CS multiplet to be covariant/gauge-invariant. One can easily verify that the structure of the couplings in (25) and (26) is different from the structure of the couplings in the CS multiplet (21). In particular, the couplings (21) are antisymmetric under $(e, f) \to (k, e)$ and $(e, k) \to (k, l)$ whereas the couplings (25) and (26) do not have such antisymmetric property. So the couplings (21) cannot be combined with the T-dual multiplets corresponding to the couplings (25) and (26) to produce the covariant/gauge-invariant results. Therefore, there must be other T-dual multiplets to make the CS multiplet covariant/gauge-invariant. The first component of these multiplets should be $\mathcal{C}^{(p-1)}$. The strategy for finding these multiplets is to find its first component by requiring that when they are combined with the corresponding coupling in (21), they become covariant/gauge-invariant. Then using the same steps as in Section 3.1, one finds all the other components of the T-dual multiplets.

There are two multiplets for making the first term in the second line of the CS multiplet (21) to be invariant under the B-field gauge transformations. The first multiplet which has only two components, is given by the following expression:

$$\alpha T_p \int d^{p+1}x \left(\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!} \mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^i \left[-h_a{}^f{}_{,b}{}^e B_{ef,id} - B_a{}^k{}_{,b}{}^e h_{ek,id} \right] + \frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!} \mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij} \left[B_j{}^f{}_{,b}{}^e B_{ef,id} + h_j{}^k{}_{,b}{}^e h_{ek,id} \right] \right) - (\cdots) \tag{30}$$

where again dots refer to the similar terms as above with $(e, f) \to (k, l)$ and $(e, k) \to (k, e)$. The coefficient α is a constant which we will determine shortly. Note that the first term in the first line above is the coupling that is needed to make the corresponding coupling in (21) gauge invariant. The other term in the first line is needed for T-duality. One can easily check that the sum of the first term above for $\alpha = 1$ and the first term in the second line of (21) can be written in terms of H, *i.e.*, $\mathcal{R}_{ab}{}^{ef} H_{ief,d}$. However, the sum of the first term in the second line above for $\alpha = 1$ and the first term in the third line of (21) cannot be written in a gauge invariant form.

The other multiplet is:

$$\begin{aligned} \beta T_p \int d^{p+1}x \Bigg(& \frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!} \mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^i \left[-\left(h_a{}^f{}_{,b}{}^e - h_a{}^e{}_{,b}{}^f\right) B_{ed,if} - B_a{}^k{}_{,b}{}^e h_{dk,ie} \right] \\ & + \frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!} \mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij} \left[\left(B_j{}^f{}_{,b}{}^e - B_j{}^e{}_{,b}{}^f\right) B_{ed,if} - B_d{}^k{}_{,b}{}^e B_{jk,ie} \right. \\ & \left. - \left(h_b{}^f{}_{,d}{}^e - h_b{}^e{}_{,d}{}^f\right) h_{ej,if} + h_j{}^k{}_{,b}{}^e h_{dk,ie} \right] \\ & + \frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!} \mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn} \left[\left(B_j{}^f{}_{,b}{}^e - B_j{}^e{}_{,b}{}^f\right) h_{en,if} + h_n{}^k{}_{,b}{}^e B_{jk,ie} \right] \Bigg) - (\cdots) \end{aligned} \tag{31}$$

The coefficient β is a constant. One can check that the sum of the first term above for $\beta = 1$ and the first term in the second line of (21) can also be written in terms of H, *i.e.*, $\mathcal{R}_{ab}{}^{ef} H_{ide,f}$. The sum of the first term in the second line above for $\beta = 1$ and the first term in the third line of (21) cannot be written in a gauge invariant form. To remedy these failures, we will consider both multiplets with

$$\alpha = \beta = 1/2 \tag{32}$$

We will see in the next section that the above choice of the constants makes it possible to write the first term in the third line of (21) in a gauge invariant form.

The last term in the first line of (31) is proportional to the Mandelstam variable $p_2 \cdot V \cdot p_3$ in the momentum space. So as argued before, we are not interested in making it covariant/gauge-invariant. However, the last term in the first line of (30) is not proportional to a Mandelstam variable, so there must be other T-dual multiplets to make it covariant/gauge-invariant. One can write it in covariant form by adding the terms $B_a{}^k{}_{,b}{}^e(h_{id,ek} - h_{ik,de} - h_{de,ik})$. The first two terms are again proportional to the Mandelstam variable $p_2 \cdot V \cdot p_3$ and are not relevant for our discussion here. The last term is the $C^{(p-1)}$ component of the following multiplet:

$$\alpha T_p \int d^{p+1}x \left(\frac{\epsilon^{a_0\cdots a_{p-3}abd}}{(p-2)!} \mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^i \left[B_a{}^k{}_{,b}{}^e h_{de,ik} \right] + \frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!} \mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij} \left[-h_j{}^k{}_{,b}{}^e h_{de,ik} \right] + \frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!} \mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn} \left[h_j{}^k{}_{,b}{}^e B_{ne,ik} \right] \right) - (\cdots) \tag{33}$$

Now, the $\mathcal{C}^{(p-1)}$ component of the above multiplet and the multiplets (31), (30) and (21) add up to the following covariant/gauge-invariant results:

$$\frac{T_p}{(p-2)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-3}abd}\mathcal{C}^{(p-1)}_{a_0\cdots a_{p-3}}{}^{i}\bigg[-\frac{1}{2}\mathcal{R}_{ab}{}^{ef}\bigg(H_{ied,f}+\frac{1}{2}H_{ief,d}\bigg)-\mathcal{R}_{iekd}H_{ab}{}^{k,e}$$
$$+\frac{1}{2}\mathcal{R}_{ab}{}^{kl}\bigg(H_{ikd,l}+\frac{1}{2}H_{ikl,d}\bigg)+\mathcal{R}_{iked}H_{ab}{}^{e,k}\bigg] \tag{34}$$

where we have added/removed some terms which are proportional to the Mandelstam variables. They are related to the massless-pole T-dual multiplets and we are not concerned about them in this paper.

3.4. $\mathcal{C}^{(p+1)}$ couplings

Adding the multiplets (30) and (31) to the CS multiplet (21), one finds that the B-field terms in the $\mathcal{C}^{(p+1)}$ component of the CS multiplet can be written in terms of field strength H, provided one adds one more multiplet, *i.e.*,

$$\frac{1}{2}T_p\int d^{p+1}x\bigg(\frac{\epsilon^{a_0\cdots a_{p-2}bd}}{(p-1)!}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\big[B^{fe}{}_{,jb}B_{ed,if}-h^{ek}{}_{,jb}h_{kd,ie}\big]$$
$$+\frac{\epsilon^{a_0\cdots a_{p-1}b}}{p!}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}\big[B^{fe}{}_{,jb}h_{en,if}-h^{ek}{}_{,jb}B_{kn,ie}\big]\bigg)-(\cdots) \tag{35}$$

Now, the $\mathcal{C}^{(p+1)}$ components of the multiplets (21), (30), (31), (33), and (35) add up to the following covariant/gauge-invariant result:

$$\frac{T_p}{(p-1)!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-2}bd}\mathcal{C}^{(p+1)}_{a_0\cdots a_{p-2}}{}^{ij}\bigg[\frac{1}{2}H_j{}^{fe}{}_{,b}H_{ied,f}+\frac{1}{4}\mathcal{R}_{bd}{}^{ef}\mathcal{R}_{ijfe}+\mathcal{R}^{e}{}_{jb}{}^{k}\mathcal{R}_{eidk}$$
$$-\frac{1}{2}H_j{}^{lk}{}_{,b}H_{ikd,l}-\frac{1}{4}\mathcal{R}_{bd}{}^{kl}\mathcal{R}_{ijlk}-\mathcal{R}^{k}{}_{jb}{}^{e}\mathcal{R}_{kide}\bigg] \tag{36}$$

where again we have added/removed some terms which are proportional to the Mandelstam variables. One may also add $H_{bd}{}^{k,e}H_{ijk,e}-H_{bd}{}^{e,k}H_{ije,k}$ to the above bracket. Since this contact term is proportional to the Mandelstam variables, our present calculation, which does not consider the massless poles, cannot confirm the presence of this coupling.

3.5. $\mathcal{C}^{(p+3)}$ couplings

The CS multiplet (21) does not have a $C^{(p+3)}$ component. However, the multiplets (31), (33) and (35), which have made the CS multiplet covariant/gauge-invariant have such a component. They combined to the following covariant/gauge invariant result:

$$\frac{T_p}{p!}\int d^{p+1}x\,\epsilon^{a_0\cdots a_{p-1}b}\mathcal{C}^{(p+3)}_{a_0\cdots a_{p-1}}{}^{ijn}\bigg[\frac{1}{4}H_j{}^{fe}{}_{,b}\mathcal{R}_{nife}-\frac{1}{4}H_{ni}{}^{e,k}\mathcal{R}_{jebk}$$
$$-\frac{1}{4}H_j{}^{kl}{}_{,b}\mathcal{R}_{nikl}+\frac{1}{4}H_{ni}{}^{k,e}\mathcal{R}_{jkbe}\bigg] \tag{37}$$

There is no coupling for $C^{(p+5)}$ which is consistent with the S-matrix calculation [26].

Therefore the couplings (24), (34), (36) and (37) are the T-dual multiplet corresponding to the CS multiplet (21) which are covariant and are invariant under the B-field gauge transformations. The T-duality of the multiplet, however, is off by some contact terms which are proportional to the Mandelstam variables. As we argued in Section 3.3, they are related to the massless poles. This ends our construction of making the CS multiplet (21) covariant/gauge-invariant by adding new T-dual multiplets. One can use the same technique as we have done in this paper to find the covariant/gauge-invariant T-dual multiplets corresponding to the couplings (25) and (26), and then confirm the results with the S-matrix calculation.

Our studies indicate that the object that must be invariant under the T-duality is the S-matrix element, not the low energy field theory of the D-brane. For the cases where the S-matrix element has only contact terms at a given order of α', the field theory is invariant under T-duality at that order. In other cases, the combination of the D-brane contact terms and the massless poles arising from the bulk and the brane actions, is invariant under T-duality. The same thing is true for the R–R gauge transformation of the D-brane action at order $O(\alpha'^2)$. One can check that the couplings (24) and (25) are not invariant under the R–R gauge transformations. However, taking the transformation of the massless poles at order $O(\alpha'^2)$ into account, one recovers the R–R gauge symmetry [26].

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