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A NEW APPROACH FOR SOLVING OF LINEAR TIME VARYING CONTROL SYSTEMS

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Abstract. This paper is concerned with the solution of Linear Time Varying [LTV] control systems. The concept of a solution for LTV systems is defined on the basis of finding the fundamental matrix corresponding to LTV control systems. There are some numerical methods such as Euler method and Taylor method for obtaining approximate solution of LTV system [LTVs], each of them has some limitations. In the recent years, other kinds of constructive approaches for the solution of LTVs are presented limited to the particular cases of it. In this paper, we introduced a new approach that we call it AVK approach to obtain a global optimal approximation of the fundamental matrix of LTVs, by introducing a problem in calculus of variations corresponding to our LTVs problem. A global optimal approximate solution (general solution of LTV systems) by using linear programming is considered.

1. Introduction

Linear Time Varying [LTV] systems are of great importance because they are very frequently used to represent the dynamical behavior of the physical systems encountered in engineering practice [6]. We are going to make use of a new approach to obtain an approximation of the fundamental matrix of linear time varying system [LTVs] to obtain a general solution to the initial-value problem for LTV control systems. Until now, a number of constructive theories for the solution of LTV systems have been available [1], [4], [8], [11], [12]. Some limitations of previous methods are: Euler method [2] has not acceptable accuracy; Taylor method [2], [9] involves computation of coefficients high order derivatives so it cannot be used for LTVs with non-smooth coefficients; in Runge-Kutta method [5], [7], [10] error

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control is possible when the coefficients of system are smooth, and otherwise it does not guarantee solution to be accurate and other kinds of constructive approaches for the solution of LTVs are presented limited to the particular cases of it [5]. AVK approach substitutes the LTVs with an equivalent problem in the field of calculus of variations. Using this approach, we found an approximate optimal solution of the new problem by solving a Linear Programming [LP] problem and obtained an approximate solution for the LTVs with controlled error, even when the coefficients of system are non-smooth.

2. Preliminaries

Definition 2.1. Let (T_1, T_2) be an open interval (which may be all of $\mathbb{R}$ or a set $\{t : t > T_1\}$,etc).Let U(t) be a piecewise continuous function on (T_1, T_2) to $\mathbb{R}^n$ and let A(t) be a continuous function from (T_1, T_2) into the set $M(n, n)$ of all $n \times n$ matrix then the system of equations

$$\dot{X}(t) = A(t)X(t) + U(t) \tag{2.1}$$

Or,equivalently

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(t)x_j(t) + u_i(t) \tag{2.2}$$

For $i = 1, ..., n$, where $A(t) = [a_{ij}(t)]$ that $a_{ij}(t)$ are the entries of the matrix $A(t)$ and $u_i(t)$ is the component of the vector $U(t)$ [we assume some or all of the entries functions $a_{ij}(t)$ or $u_i(t)$ may be non-smooth functions], is called a linear system with forcing function $U(t)$. The system of equations

$$\dot{X}(t) = A(t)X(t)$$

Is often called the homogeneous or unforced part of the LTVs, if $A(t)$ is a constant matrix then we say that the linear system is time-invariant.

Theorem 2.2. *Let $\mathfrak{s}$ denote the set of all solutions of the homogeneous LTVs (2.1), in other words,*

$$\mathfrak{s} = \{X(\cdot) : \dot{X}(t) = A(t)X(t) \text{ for } t \in (T_1, T_2)\}$$

Where $X(\cdot)$ is a function defined on (T_1, T_2) as follow:

$$X(\cdot) : (T_1, T_2) \to \mathbb{R}^n$$

Then $\mathfrak{s}$ is in $\mathcal{C}^1(T_1, T_2)$, the continuous differentiable functions on (T_1, T_2), and a basis $\{X^1(t), ..., X^n(t)\}$ of $\mathfrak{s}$ may be obtained by letting $X^j(\cdot) \in \mathbb{R}^n$ be the (unique) element of $\mathfrak{s}$ which satisfies the condition

$$
(2.3) \qquad \begin{cases} \dot{X}^j(t) = A(t)X^j(t) \\ X^j(t_0) = e_j \end{cases}
$$

Where $e_j = (0, ..., 1, 0, ..., 0)^T$ *is a vector whose jth component value is 1 and the other components are zero [3].*

Definition 2.3. Let $\Phi(t, t_0)$ be the $n \times n$ matrix function whose jth column is the vector function $X^j(t)$, with $X^j(t_0) = e_j$ that $j = 1, ..., n$. In other words, the columns of $\Phi(t, t_0)$ as defined below, are the solutions of the homogeneous part of the LTVs (2.1) satisfying the initial condition $X^j(t_0) = e_j$. We say that $\Phi(t, t_0)$ is the fundamental, or transition matrix of the system (2.1). Then

$$
(2.4) \qquad \Phi(t, t_0) = \begin{pmatrix} x_1^1(t) & \dots & x_1^n(t) \\ \vdots & \ddots & \vdots \\ x_n^1(t) & \dots & x_n^n(t) \end{pmatrix}
$$

Where $x_i^j(t)$ is the ith component of $X^j(t)$, and we note that

$$
(2.5) \qquad \Phi(t_0, t_0) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} = I
$$

Where I is the identity matrix.

3. A NEW APPROACH TO APPROXIMATE SOLUTION OF ODE PROBLEMS

We want to obtain an approximate solution of the following ODE problem:

$$
(3.1) \qquad \begin{cases} \dot{X}^*(t) = A(t)X^*(t) & t \in [a, b] \\ X^*(a) = X_0^* \end{cases}
$$

Definition 3.1. First, consider LTVs (2.1), we define the following functional $E(X^*(\cdot))$ on the space of continuous functions $\mathcal{C}[a, b]$ that we may call it the total error functional of ODE problem (3.1) in L_1 space:

$$
(3.2) \qquad E(X^*(\cdot)) = \int_a^b \left\| \dot{X}^*(t) - A(t)X^*(t) \right\|_{l_1} dt
$$

Where $A(t) = [a_{ij}(t)]$, $a_{ij}(t)$ are in general measurable functions on $[a, b]$ and the integral is the Lebesgue integral,

$$X^*(t) = \begin{pmatrix} x_1^1(t) & \dots & x_1^n(t) \\ \vdots & \ddots & \vdots \\ x_n^1(t) & \dots & x_n^n(t) \end{pmatrix}, \quad \|(x_i^j(t))_{n\times n}\|_{l_1} = \sum_{i,j=1}^{n} |x_i^j(t)| \tag{3.3}$$

Theorem 3.2. *If $h(t)$ is a real nonlinear continuous function on $[a,b]$ and non-negative, $h(t) \geq 0$, then the necessary and sufficient condition for $\int_a^b h(t)dt = 0$ is h(t)=0 on $[a,b]$ [6].*

Proof. Let assume $\int_a^b h(t)dt = 0$ but $h(t) \neq 0$ at a point t_1 on $[a,b]$, by continuity of $h(t)$ on $[a,b]$ there exists some neighborhood of t_1 such that $h(t) > 0$ for all $t_1 \in (t_1 - \epsilon, t_1 + \epsilon)$ that ϵ is a positive number. Therefore $\int_a^b h(t)dt \geq \int_{t_1-\epsilon}^{t_1+\epsilon} h(t)dt > 0$ i.e. $\int_a^b h(t)dt > 0$, which is a contradiction to our assumption. Thus $h(t)$ must be zero on $[a,b]$. On the other hand, if $h(t) = 0$ on $[a,b]$ then obviously $\int_a^b h(t)dt = 0$ [6].

□

Theorem 3.3. *The necessary and sufficient condition for $X^*(t) = (X^1, ..., X^n)$ be a solution of the following problem*

$$\begin{cases} \dot{X}^*(t) = A(t)X^*(t) & t \in [a,b] \\ X^*(a) = I \end{cases}$$

is

$$E(X^*(\cdot)) = 0$$

Where $A(t)$ is a known continuous function on $[a,b]$ and I is the identity matrix.

Proof. We define $h(t)$ in theorem 3.2 as follows: $h(t) = \left\|\dot{X}^*(t) - A(t)X^*(t)\right\|_{l_1}$, $t \in [a,b]$ and Since $\|\cdot\|_{l_1}$ is a norm function on $\mathbb{R}^n$ and is non-negative and $A(t)$ is also a continuous function, then $h(t)$ is a continuous non-negative function on $[a,b]$ then by theorem 3.2 we conclude $\int_a^b \left\|\dot{X}^*(t) - A(t)X^*(t)\right\|_{l_1} dt = 0$ is equivalent to $\left\|\dot{X}^*(t) - A(t)X^*(t)\right\|_{l_1} = 0$ for all $t \in [a,b]$ or $\dot{X}^* = A(t)X(t)^*$. □

Remark 3.4. Without the loss of generality, we may assume $a = 0$ and $b = 1$ by applying the following bijective function:

$$[a,b] \overset{f(x)}{\Longleftrightarrow} [0,1] \qquad f(x) = \frac{x-a}{x-b} \qquad \forall x \in [a,b]$$

Thus, the interval $[a,b]$ is converted to $[0,1]$ equivalently.

4. A NEW APPROACH FOR OBTAINING FUNDAMENTAL MATRIX WE CALL IT AVK APPROACH

Let our problem be the following problem in calculus of variations:

$$\underset{X^*}{\mathit{Minimize}}\ E(X^*(\cdot)) = \int_0^1 \left\|\dot{X}^*(t) - A(t)X^*(t)\right\|_{l_1} dt \qquad (4.1)$$
$$s.t. \qquad X^*(0) = I \quad (\mathit{identity\ matrix})$$

Or, equivalently

$$\underset{x_i^j}{\mathit{Minimize}}\ E(X^*(\cdot)) = \int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{n} \left|\dot{x}_i^j(t) - \sum_{k=1}^{n} a_{ik}(t)x_k^j(t)\right| dt \qquad (4.2)$$
$$s.t. \qquad x^j(0) = e_j \qquad j = 1, ..., n$$

Where $\{e_1, ..., e_n\}$ is the natural basis of the linear space $\mathbb{R}^n$.

We assume the optimal solution of problem (4.2) to be $X^*(t)$, then according to theorem 3.2 and theorem 3.3, $X^*(t)$ is a solution of the following LTVs

$$\begin{cases} \dot{X}(t) = A(t)X(t) & t \in [0, 1] \\ X(0) = I \end{cases}$$

if and only if: $E(X^*(\cdot)) = 0$ or,

$$\begin{cases} \dot{x}_i^{*j}(t) = \sum_{k=1}^{n} a_{ik}(t)x_k^{*j}(t) & i, j = 1, ..., n \\ x^{*j}(0) = e_j \end{cases}$$

So, with respect to theorem 2.2 and definition 2.3, $X^*(t)$ is the fundamental matrix for LTVs (2.1) whose jth column is the vector x^{*j}, then in general, for solving LTVs (2.1) we may solve the minimization problem (4.2) and we may write $\Phi(t, 0) = X^*(t)$. References [4], [7] show using this fundamental matrix, the solution of problem

$$\begin{cases} \dot{X}(t) = A(t)X(t) + U(t) & t \in [0, 1] \\ X(0) = X_0 \end{cases}$$

is

$$X(t) = \Phi(t, 0)\left(X_0 + \int_0^1 \Phi^{-1}(\tau, 0)U(\tau)d\tau\right) \qquad (4.3)$$

or,we have

$$X(t) = X^*(t)\left(X_0 + \int_0^1 X^{*-1}(\tau)U(\tau)d\tau\right) \qquad (4.4)$$

Since the analytic solution for the fundamental matrix generally is unknown, so we try to obtain an approximate solution of problem (4.2) which is an approximation of the fundamental matrix, we partition the interval [0,1] to m equal subintervals, where m is an arbitrary positive integer, by using this partition from (4.2) we have:

$$\underset{x_i^j}{Minimize}\ E(X^*(\cdot)) = \sum_{l=1}^{m}\int_{\frac{l-1}{m}}^{\frac{l}{m}}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\dot{x}_i^{*j}(t) - \sum_{k=1}^{n} a_{ik}(t)x_k^{*j}(t)\right|dt \tag{4.5}$$
$$s.t. \qquad x_i^{*j}(0) = e_j^i$$

Here, x_i^{*j} denotes an element of the ith row and the jth column of the fundamental matrix.

Theorem 4.1. *If* $\mathcal{S} = \sum_{n=1}^{\infty} a_n$ *for* $a_n \geq 0$ $(n \in \mathbb{N})$, *then* $\mathcal{S} = sup\{\sum_{k=1}^{n} a_k : n \in \mathbb{N}\}$.

Proof. Suppose that $\mathcal{S}_n = \sum_{k=1}^{n} a_k$. Since $\mathcal{S}_n$ is an increasing sequence (i.e. $\mathcal{S}_n \leq \mathcal{S}_{n+1}$ $\forall n \in \mathbb{N}$) and $\lim_{n\to\infty}\mathcal{S}_n = \mathcal{S}$ we have $\mathcal{S}_n \leq \mathcal{S}$. In other hand $\lim_{n\to\infty}\mathcal{S}_n = \mathcal{S}$, so there exist some $m \in \mathbb{N}$ such that for every $n \geq m$ we have $\mathcal{S} - \epsilon \leq \mathcal{S}_n$; therefore $\mathcal{S} = sup\{\sum_{k=1}^{n} a_k : n \in \mathbb{N}\}$. □

By theorem 4.1 the bigger m for the partition cause convergence of the approximate solution to the exact solution. Let $h = \frac{1}{m}$, we have:

$$\dot{x}^*(t_l) \simeq \frac{x^*(t_l + h) - x^*(t_l - h)}{2h} \tag{4.6}$$
$$t_l = \frac{l}{m} \qquad l = 1, ..., m-1$$

The bigger m for the approximate value the nearer exact derivative of $x^*(t_l)$.So we substitute the above fraction instead of $\dot{x}^*(t)$ in problem (4.5) and we have the following optimization problem which is an approximation of the problem (4.5):

$$\underset{x_i^{*j}}{Minimize}\ E(X^*(\cdot)) = \sum_{l=1}^{m}\int_{\frac{l-1}{m}}^{\frac{l}{m}}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{m}{2}\big(x_i^{*j}(t+h) - x_i^{*j}(t-h)\big) - \sum_{k=1}^{n} a_{ik}(t)x_k^{*j}(t)\right|dt$$
$$s.t. \qquad x_i^{*j}(0) = e_j^i \tag{4.7}$$

For the beginning and the end of the interval, respectively $t = 0, t = 1$ we use these $\frac{x^*(t+h)-x^*(t)}{h}$, $\frac{x^*(t-h)-x^*(t)}{-h}$ as an approximation for $\dot{x}^*(t)$.
We know the approximate value of $\int_a^{a+\epsilon} h(t)dt$ is $\frac{h(a)+h(a+\epsilon)}{2}\epsilon$ where ϵ is a positive number. Now, we approximate

$$\int_{\frac{l-1}{m}}^{\frac{l}{m}}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{m}{2}\big(x_i^{*j}(t+h) - x_i^{*j}(t-h)\big) - \sum_{k=1}^{n} a_{ik}(t)x_k^{*j}(t)\right|dt \tag{4.8}$$

to

$$\frac{1}{2m}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x_i^{*j}(\frac{l}{m})-x_i^{*j}(\frac{l-2}{m})\big)-\sum_{k=1}^{n}a_{ik}(\frac{l-1}{m})x_k^{*j}(\frac{l-1}{m})\Big|$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x_i^{*j}(\frac{l+1}{m})-x_i^{*j}(\frac{l-1}{m})\big)-\sum_{k=1}^{n}a_{ik}(\frac{l}{m})x_k^{*j}(\frac{l}{m})\Big|\Big\} \tag{4.9}$$

And finally, the approximate minimization problem (4.7) is transformed to the following problem:

$$\underset{x_i^{*j}}{Minimize}\ \sum_{l=1}^{m}\frac{1}{2m}\Big\{\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x_i^{*j}(t_l)-x_i^{*j}(t_{l-2})\big)-\sum_{k=1}^{n}a_{ik}(t_{l-1})x_k^{*j}(t_{l-1})\Big|$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x_i^{*j}(t_{l+1})-x_i^{*j}(t_{l-1})\big)-\sum_{k=1}^{n}a_{ik}(t_l)x_k^{*j}(t_l)\Big|\Big\}$$
$$s.t. \qquad x_i^{*j}(0)=e_j^i \tag{4.10}$$

Where, $e_j^i=\delta_j^i$ in which δ_j^i is Kronecker delta, $\delta_j^i=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$.

For simplification we define $x^*_{ijl}\triangleq x_i^{*j}(t_l), a_{ikl}\triangleq a_{ik}(t_l)$.

Thus, we simplify obtain the discretized problem (4.10) in the following form:

$$\underset{x^*_{ijl}}{Minimize}\ \sum_{l=1}^{m}\Big(\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x^*_{ijl}-x^*_{ijl-2}\big)\big)-\sum_{k=1}^{n}a_{ikl-1}x^*_{kjl-1})\Big|$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\Big|\frac{m}{2}\big(x^*_{ijl+1}-x^*_{ijl-1}\big)\big)-\sum_{k=1}^{n}a_{ikl}x^*_{kjl})\Big|\Big)$$
$$s.t. \qquad x^*_{ij0}=\delta_j^i \tag{4.11}$$

In the problem (4.11) the factor $\frac{1}{2m}$ is omitted because of having no effect on the solution of it. Now, the problem (4.11) is a nonlinear programming (NLP) problem. By solving this problem where x^*_{ijl} as unknown real numbers, we obtain an approximation of the functions $x_i^{*j}(t)$, $t\in[0,1]$ so we obtain the value of functions $x_i^{*j}(t_l)$ for all $l=0,...,m$. We may transform the NLP problem (4.11) to an LP problem by below assumption:

$$|a|=v+u\ ,a=v-u\ ,v\geq 0\ ,u\geq 0$$

Where a is any real number. Now, the NLP problem (4.11) is transformed to the LP problem (4.12):

$$Minimize \sum_{l=1}^{m}\sum_{k=1}^{n}\sum_{i=1}^{n} v_{ikl}+u_{ikl}+r_{ikl}+s_{ikl}$$
$$s.t.$$
$$\frac{m}{2}(x^*_{ijl}-x^*_{ijl-2})-\sum_{k=1}^{n}a_{ikl-1}x^*_{kjl-1}=v_{ikl}-u_{ikl}$$
$$\frac{m}{2}(x^*_{ijl+1}-x^*_{ijl-1})-\sum_{k=1}^{n}a_{ikl}x^*_{kjl}=r_{ikl}-s_{ikl}$$
$$x^*_{ij0}=\delta^i_j \quad v_{ikl},u_{ikl},r_{ikl},s_{ikl}\geq 0 \tag{4.12}$$

5. CONCLUSION

In this paper, we have introduced a new approach to obtain an approximate solution(general solution) of LTV control systems, for an LTVs, while the total error to be controlled. The interests of such an approach are: Simplicity: obtaining the global optimal approximate solution by solving an LP or NLP, High Performance: solving a new LTVs instead of a set of LTV systems, even when coefficients of systems are non-smooth, Computation: by selecting the refinement of domain's partition in discretizing, the solution will converge to the analytical solution and Flexibility: finding the global optimal approximate solution (general solution) for a Nonlinear Time Varying system by solving of an NLP.

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