

On varietal capability of direct products of groups and pairs of groups

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Abstract. In this paper, we give some conditions in which a direct product of groups is \mathcal{V} -capable if and only if each of its factors is \mathcal{V} -capable for some varieties \mathcal{V} . Moreover, we give some conditions in which a direct product of a finite family of pairs of groups is capable if and only if each of its factors is a capable pair of groups.

Keywords: Capable group; Direct product; Variety of groups; \mathcal{V} -capable group; Pair of groups; Capable pair of groups.

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1 Introduction

R. Baer [1] initiated an investigation of the question which conditions a group G must fulfill in order to be the group of inner automorphisms of a group E , that is ($G \cong E/Z(E)$). Following M. Hall and J. K. Senior [6], such a group G is called capable. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [5] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F.R. Beyl, U. Felgner and P. Schmid [2] proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G . This $Z^*(G)$ is characteristic in G and is the image of the center of every stem cover of G . Moreover, $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable [2, *Corollary 2.2*]. Hence G is capable if and only if $Z^*(G) = 1$ [2, *Corollary 2.3*]. They showed that the class of

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all capable groups is closed under the direct products [2, *Proposition 6.1*]. Also, they presented a condition in which the capability of a direct product of finitely many of groups implies the capability of each of the factors [2, *Proposition 6.2*]. Moreover, they proved that if N is a central subgroup of G , then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \rightarrow M(G/N)$ is monomorphic [2, *Theorem 4.2*].

Then, M.R.R. Moghadam and S. Kayvanfar [11] generalized the concept of capability to \mathcal{V} -capability for a group G . They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety \mathcal{V} defined by the set of laws V and a group G in order to establish a necessary and sufficient condition under which G can be \mathcal{V} -capable [11, *Corollary 2.4*]. They also showed that the class of all \mathcal{V} -capable groups is closed under the direct products [11, *Theorem 2.6*]. Moreover, they exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where N is a normal subgroup contained in the marginal subgroup of G with respect to the variety \mathcal{V} . Using this relationship, they gave a necessary and sufficient condition for a group G to be \mathcal{V} -capable [11, *Theorem 4.4*].

In this paper, in Section 2, we present some conditions in which the \mathcal{V} -capability of a direct product of finitely many groups implies the \mathcal{V} -capability of each of its factors.

In continue, we study on the capability of a pair of groups. The theory of capability of groups may be extended to the theory of pairs of groups. In fact capable pairs are defined in terms of J.-L. Loday's notion [8] of a relative central extensions. By a pair of groups we mean a group G and a normal subgroup N , denoted by (G, N) . If M is another group on which an action of G is given, the G -center of M is defined to be the subgroup

$$Z(M, G) = \{m \in M \mid m^g = m, \forall g \in G\}.$$

A relative central extension of the pair (G, N) consists of a group homomorphism $\sigma : M \rightarrow G$ together with an action of G on M such that

- i) $\sigma(M) = N$,
- ii) $\sigma(m^g) = g^{-1}\sigma(m)g$, for all $g \in G, m \in M$,
- iii) $m'^{\sigma(m)} = m^{-1}m'm$, for all $m, m' \in M$,
- iv) $\text{Ker}(\sigma) \subseteq Z(M, G)$.

Now a pair of groups (G, N) is said to be capable if it admits such a relative central extension with $\text{Ker}(\sigma) = Z(M, G)$.

In Section 3, we prove that the capability of the pair $(G_1 \times G_2, N_1 \times N_2)$ is equivalent to the capability of both pairs (G_1, N_1) and (G_2, N_2) in some conditions.

2 \mathcal{V} -Capability of a direct product of groups

Suppose that \mathcal{V} is a variety of groups defined by the set of laws V . A group G is said to be \mathcal{V} -capable if there exists a group E such that $G \cong E/V^*(E)$. If $\psi : E \rightarrow G$ is a surjective homomorphism with $\text{ker}\psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of G contained in $V^*(G)$. If \mathcal{V} is the variety of abelian groups, then the

subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case \mathcal{V} -capability is equal to capability [11].

Theorem 2.1. [11] (i) A group G is \mathcal{V} -capable if and only if $(V^*)^*(G) = 1$.
(ii) $(V^*)^*(\prod_{i \in I} G_i) \subseteq \prod_{i \in I} (V^*)^*(G_i)$.

As a consequence, if G_i 's are \mathcal{V} -capable groups, then $G = \prod_{i \in I} G_i$ is also \mathcal{V} -capable. In the above theorem, the equality does not hold in general (Example 2.3).

Theorem 2.2. [11] Let N be a normal subgroup contained in the marginal subgroup of G , $V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \rightarrow \mathcal{V}M(G/N)$ is a monomorphism.

In this section we verify the equation $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$ for some famous varieties.

In general, for an arbitrary variety of groups \mathcal{V} , and groups A and B , $\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B) \times T$, where T is an abelian group [10]. For some particular varieties, the group T is trivial with some conditions. For instance, some famous varieties as variety of abelian groups [10], variety of nilpotent groups [4], and some varieties of polynilpotent groups [7] have the property that: *for any two groups A and B with $(|A^{ab}|, |B^{ab}|) = 1$ the isomorphism $\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B)$ (*) holds.*

Now, suppose that \mathcal{V} is a variety, A and B are two groups with the property

$$\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B).$$

By Theorem 2.2, we have the following monomorphism

$$\mathcal{V}M(A) \times \mathcal{V}M(B) \hookrightarrow \mathcal{V}M\left(\frac{A}{(V^*)^*(A)}\right) \times \mathcal{V}M\left(\frac{B}{(V^*)^*(B)}\right).$$

Moreover, we have the following inclusion

$$\mathcal{V}M\left(\frac{A}{(V^*)^*(A)}\right) \times \mathcal{V}M\left(\frac{B}{(V^*)^*(B)}\right) \hookrightarrow \mathcal{V}M\left(\frac{A \times B}{(V^*)^*(A) \times (V^*)^*(B)}\right).$$

Finally, we get the monomorphism

$$\mathcal{V}M(A \times B) \hookrightarrow \mathcal{V}M\left(\frac{A \times B}{(V^*)^*(A) \times (V^*)^*(B)}\right).$$

Thus, by Theorem 2.2, we conclude that

$$(V^*)^*(A) \times (V^*)^*(B) \leq (V^*)^*(A \times B).$$

This note leads us to our main result.

Theorem 2.3. Let \mathcal{V} be a variety, A and B be two groups with $\mathcal{V}M(A \times B) \cong \mathcal{V}M(A) \times \mathcal{V}M(B)$, then $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$. Consequently $A \times B$ is \mathcal{V} -capable if and only if A and B are both \mathcal{V} -capable.

Remark 2.4. i) In the above theorem, the condition $\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B)$ is not necessary. For, the groups A and B in Example 2.3.(iii) satisfy the equation $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$ but $\mathcal{VM}(A \times B) \not\cong \mathcal{VM}(A) \times \mathcal{VM}(B)$.

ii) In some famous varieties as the variety of abelian groups and the variety of nilpotent groups, the isomorphism $\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B)$ holds, where $(|A^{ab}|, |B^{ab}|) = 1$ ([4, 12]). Thus, using Theorem 2.3, for a family of groups $\{A_i \mid 1 \leq i \leq n\}$ whose abelianizations have mutually coprime orders, $\prod_{i=1}^n A_i$ is capable (\mathcal{N}_c -capable) if and only if every A_i is capable (\mathcal{N}_c -capable). Also, we note that \mathcal{N}_c -capability of a group implies its \mathcal{N}_{c-1} -capability, for any $c \geq 2$.

iii) Note that in the varieties of abelian groups and nilpotent groups, for finitely generated groups A and B , $\mathcal{VM}(A \times B) \cong \mathcal{VM}(A) \times \mathcal{VM}(B)$ if and only if $|A^{ab}|$ and $|B^{ab}|$ are finite with $(|A^{ab}|, |B^{ab}|) = 1$ ([4, 12]).

Example 2.1. Suppose that G and H are two groups such that for any $g \in G^{ab}$ and $h \in H^{ab}$, $(|g|, |h|) = 1$. Hence, by Theorem 2.3, the capability of $G \times H$ is equivalent to the capability of each of groups G and H . For instance, i) for any prime numbers p and q ($p \neq q$), \mathbb{Z}_{p^∞} and \mathbb{Z}_{q^∞} which are two infinitely generated groups, ii) the groups G and H with $(\exp(G), \exp(H)) = 1$, satisfy the above conditions.

Corollary 2.5. Let $\{A_i \mid 1 \leq i \leq n\}$ be a family of groups whose abelianizations have mutually coprime orders. If $\prod_{i=1}^n A_i$ is nilpotent of class at most c_1 , then it is $\mathcal{N}_{c_1, \dots, c_s}$ -capable if and only if every A_i is $\mathcal{N}_{c_1, \dots, c_s}$ -capable.

Proof. First, for the variety of polynilpotent groups with above hypothesis, there exists the following isomorphism [7, Lemma 3.9]

$$\mathcal{N}_{c_1, \dots, c_s}(\prod_{i=1}^n A_i) \cong (\mathcal{N}_{c_s}M(\cdots \mathcal{N}_{c_2}M(\mathcal{N}_{c_1}M(\prod_{i=1}^n A_i))) \cdots).$$

Also we note to the property of variety of nilpotent groups that $\mathcal{N}_cM(\prod_{i=1}^n A_i) \cong \prod_{i=1}^n \mathcal{N}_cM(A_i)$, for above family of groups [4, Proposition 3]. Hence by Theorem 2.3 the result holds. \square

Example 2.2. If $\{A_i \mid 1 \leq i \leq n\}$ is a family of perfect groups, then $\prod_{i=1}^n A_i$ is \mathcal{V} -capable if and only if each A_i is \mathcal{V} -capable, where \mathcal{V} may be each of these three varieties, variety of abelian groups, variety of nilpotent groups, or variety of polynilpotent groups.

Note that Ellis gave a similar result for \mathcal{N}_c -capability of a direct product of groups [4, Theorem 2] with another method.

Example 2.3. i) Let $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, where $k \geq 3$, $n_{i+1} | n_i$ for all $1 \leq i \leq k-1$ and $n_1 = n_2 = n_3$. Then by [9, Lemma 3.4] G is $\mathcal{N}_{c_1, \dots, c_t}$ -capable if $t \geq 2$ and $c_1 = 1$; but by [9, Lemma 3.3] no one of its direct summands, \mathbb{Z}_{n_i} , $1 \leq i \leq k$, is $\mathcal{N}_{c_1, \dots, c_t}$ -capable. This shows that we cannot omit the condition of being mutually coprime orders for abelianizations of the family of groups $\{A_i \mid 1 \leq i \leq n\}$ in Corollary 2.5.

ii) Let $G \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, where $k \geq 2$, $n_{i+1} | n_i$ for all $1 \leq i \leq k-1$ and $n_1 = n_2$. Then by [9, Lemma 3.7] G is $\mathcal{N}_{c_1, \dots, c_t}$ -capable if $t = 1$ or $c_1 \geq 2$; but no one of its direct summands, \mathbb{Z}_{n_i} , $1 \leq i \leq k$, is $\mathcal{N}_{c_1, \dots, c_t}$ -capable. This example also shows that one cannot omit the condition of being mutually coprime orders for abelianizations of the family of groups $\{A_i \mid 1 \leq i \leq n\}$ in Remark 2.4(ii) and in Corollary 2.5, for the variety of abelian groups, variety of nilpotent groups, \mathcal{N}_c or the variety of polynilpotent groups $\mathcal{N}_{c_1, \dots, c_t}$ where $c_1 \geq 2$.

iii) Put $A \cong \mathbb{Z}_n \oplus \mathbb{Z}_n \cong B$. Then $(|A^{ab}|, |B^{ab}|) \neq 1$ and $A \oplus B$ is $\mathcal{N}_{c_1, \dots, c_t}$ -capable where $t = 1$ or $c_1 \geq 2$. Also its direct summands, A and B , are $\mathcal{N}_{c_1, \dots, c_t}$ -capable. This shows that the condition of being mutually coprime orders for abelianizations of the the family $\{A_i \mid 1 \leq i \leq n\}$, in Remark 2.4(ii) and in Corollary 2.5, is not necessary condition for transferring the varietal capability of a direct product to its factors.

3 Capability of a direct product of pairs of groups

In order to study the capability of a pair of groups (G, N) , Ellis [3] introduced a subgroup $Z_G^\wedge(N)$ with the property that the pair is capable if and only if $Z_G^\wedge(N) = 1$. To define this subgroup, we need to recall the definition of exterior product from [3] as follows.

Definition 3.1. Let N and P be arbitrary subgroups of G . The exterior product $P \wedge N$ is the group generated by symbols $p \wedge n$, for $p \in P$, $n \in N$, subject to the relations

$$1) pp' \wedge n = (p^{p'} \wedge n^p)(p \wedge n),$$

$$2) p \wedge nn' = (p \wedge n)(p^n \wedge n'^n),$$

$$3) x \wedge x = 1,$$

for $x \in P \cap N$, $n, n' \in N$, $p, p' \in P$.

For a group G and normal subgroups N and P , the exterior P-center of N is denoted by $Z_P^\wedge(N)$, and is defined to be

$$Z_P^\wedge(N) = \{n \in N \mid 1 = p \wedge n \in P \wedge N, \text{ for all } p \in P\}.$$

Ellis [3] proved that the pair (G, N) is capable if and only if $Z_G^\wedge(N) = 1$. In [12, Corollary 5.3] it is proved that if (G, N) is a pair of abelian groups, then $Z_G^\wedge(N) = Z^*(G) \cap N$. By a similar proof, we have the result without the condition of abelianness. Now, we can give the following result about the capability of the direct product of pairs of groups.

Theorem 3.2. Suppose that (G_i, N_i) ($i = 1, 2$) are pairs of groups with $(|(G_1)^{ab}|, |(G_2)^{ab}|) = 1$, then

$$Z_{G_1 \times G_2}^\wedge(N_1 \times N_2) = Z_{G_1}^\wedge(N_1) \times Z_{G_2}^\wedge(N_2).$$

Consequently the capability of $(G_1 \times G_2, N_1 \times N_2)$ is equivalent to the capability of both pairs (G_1, N_1) and (G_2, N_2) .

Proof. Since $(|(G_1)^{ab}|, |(G_2)^{ab}|) = 1$, we have $M(G_1 \times G_2) \cong M(G_1) \times M(G_2)$ and by Theorem 2.3, $Z^*(G_1 \times G_2) = Z^*(G_1) \times Z^*(G_2)$. Finally using the previous note, we conclude that

$$Z_{G_1 \times G_2}^\wedge(N_1 \times N_2) = Z^*(G_1 \times G_2) \cap (N_1 \times N_2)$$

$$\begin{aligned}
&= Z^*(G_1) \times Z^*(G_2) \cap (N_1 \times N_2) \\
&= (Z^*(G_1) \cap N_1) \times (Z^*(G_2) \cap N_2) \\
&= Z^{\wedge}_{G_1}(N_1) \times Z^{\wedge}_{G_2}(N_2).
\end{aligned}$$

This equation and [5, *Theorem 3*] complete the proof. \square

Remark 3.3. By induction we can also conclude the above theorem for a family of pairs of groups $\{(G_i, N_i) \mid 1 \leq i \leq n\}$, where $(G_i)^{ab}$'s have mutually coprime orders, that is, $(\prod_{i=1}^n G_i, \prod_{i=1}^n N_i)$ is a capable pair of groups if and only if every (G_i, N_i) is a capable pair of groups.

Example 3.1. Using [12, *Theorem 5.4*] $(\mathbb{Z}_6 \times \mathbb{Z}_6, \mathbb{Z}_3 \times \mathbb{Z}_3)$ is a capable pair of groups, but $(\mathbb{Z}_6, \mathbb{Z}_3)$ is not. This shows that the condition $(|(G_1)^{ab}|, |(G_2)^{ab}|) = 1$ in *Theorem 3.2* cannot be omitted.

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