

SOME EQUIVALENCE CLASSES OF OPERATORS ON $\mathcal{B}(\mathcal{H})$

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ABSTRACT. Let $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ be the algebra of all linear operators on $\mathcal{B}(\mathcal{H})$ and \mathcal{P} be a property on $\mathcal{B}(\mathcal{H})$. For $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, we say that $\phi_1 \sim_{\mathcal{P}} \phi_2$, whenever $\phi_1(T)$ has property \mathcal{P} , if and only if $\phi_2(T)$ has this property. In particular, if \mathcal{I} is the identity map on $\mathcal{B}(\mathcal{H})$, then $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves property \mathcal{P} in both directions. Each property \mathcal{P} produces an equivalence relation on $\mathcal{L}(\mathcal{B}(\mathcal{H}))$. We study the relation between equivalence classes with respect to different properties such as being Fredholm, semi-Fredholm, compact, finite rank, generalized invertible, or having a specific semi-index.

1. Introduction

Let \mathcal{H} be an infinite-dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} . We denote by $\mathcal{F}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$ the ideals of all finite rank and compact operators in $\mathcal{B}(\mathcal{H})$, respectively. The Calkin algebra of \mathcal{H} is the quotient algebra $\mathcal{C}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a Fredholm operator if $Im(T)$, the range of T , is closed and both its kernel and co-kernel are finite-dimensional. We recall that $T \in \mathcal{B}(\mathcal{H})$ is called upper (resp. lower) semi-Fredholm if $Im(T)$ is closed and its kernel (resp.

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co-kernel) is finite-dimensional. An operator which is either upper semi-Fredholm or lower semi-Fredholm is called a semi-Fredholm operator. We denote by $\mathcal{UF}(\mathcal{H})$, $\mathcal{LF}(\mathcal{H})$, $\mathcal{SF}(\mathcal{H})$, and $\mathcal{FR}(\mathcal{H})$ the sets of upper semi-Fredholm, lower semi-Fredholm, semi-Fredholm and Fredholm operators, respectively. By Atkinson's Theorem, [4, Theorem 1.4.16], if \mathcal{H} is an infinite-dimensional Hilbert space, then $U \in \mathcal{B}(\mathcal{H})$ is Fredholm if and only if $U + K(\mathcal{H})$ is invertible in the Calkin algebra $\mathcal{C}(\mathcal{H})$. The reader is referred to [4, 6] for more on Fredholm operators. Let $A \in \mathcal{B}(\mathcal{H})$. If there exists $B \in \mathcal{B}(\mathcal{H})$ such that $ABA = A$, then A is called generalized invertible and B is said to be a generalized inverse of A . Note that $A \in \mathcal{B}(\mathcal{H})$ is generalized invertible if and only if $Im(A)$ is closed [5]. The set of generalized invertible elements of $\mathcal{B}(\mathcal{H})$ is denoted by $\mathcal{G}(\mathcal{H})$.

The nullity (resp. defect) of an operator $T \in \mathcal{B}(\mathcal{H})$ is defined to be $dim(Ker(T))$ (resp. $dim(coker(T))$), denoted by $nul(T)$ (resp. $def(T)$). Now, we define the function $s-index : \mathcal{B}(\mathcal{H}) \rightarrow \{0, \infty\} \cup \mathbb{N}$ as follows:

$$s-index(T) = \begin{cases} \infty & T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{SF}(\mathcal{H}), \\ 0 & T \in \mathcal{FR}(\mathcal{H}), \\ nul(T) & T \in \mathcal{UF}(\mathcal{H}) \setminus \mathcal{FR}(\mathcal{H}), \\ def(T) & T \in \mathcal{LF}(\mathcal{H}) \setminus \mathcal{FR}(\mathcal{H}). \end{cases}$$

The number $s-index(T)$ is called the semi-index of T . Note that for a Fredholm operator T , in general, $s-index(T)$ does not coincide with the classical index of T which is defined by $nul(T) - def(T)$.

Let $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ be the set of all linear mappings on $\mathcal{B}(\mathcal{H})$. Recall that $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ is said to be surjective up to finite rank operators if $\mathcal{B}(\mathcal{H}) = Im(\phi) + \mathcal{F}(\mathcal{H})$, and ϕ is said to be surjective up to compact operators if $\mathcal{B}(\mathcal{H}) = Im(\phi) + \mathcal{K}(\mathcal{H})$. Obviously, if ϕ is surjective up to finite rank operators, then it is surjective up to compact operators and each surjective linear map satisfies both of these properties.

Let \mathcal{P} be a property on $\mathcal{B}(\mathcal{H})$. For $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, we say that $\phi_1 \sim_{\mathcal{P}} \phi_2$, whenever $\phi_1(T)$ has property \mathcal{P} , if and only if $\phi_2(T)$ has this property. It is easy to see that each property \mathcal{P} produces an equivalence relation on $\mathcal{L}(\mathcal{B}(\mathcal{H}))$. Throughout this paper, we use the following notations for some specific properties:

- (i) “ f ” is the property of “being finite-rank”;
- (ii) “ k ” is the property of “being compact”;
- (iii) “ fr ” is the property of “being Fredholm”;
- (iv) “ sf ” is the property of “being semi-Fredholm”;

- (v) “ g ” is the property of “being generalized invertible”;
 (vi) “ si ” is the property of “having a specific semi-index”.

Let \mathcal{I} denote the identity operator of $\mathcal{L}(\mathcal{B}(\mathcal{H}))$ and $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Then, $\phi \sim_{\mathcal{P}} \mathcal{I}$ means that ϕ preserves the property \mathcal{P} in both directions, that is, $\phi(T)$ has property \mathcal{P} if and only if T has this property. Mbekhta and Šemrl in [3] study those ϕ which satisfy $\phi \sim_g \mathcal{I}$ and $\phi \sim_{sf} \mathcal{I}$. In general, if ψ is a linear operator on $\mathcal{B}(\mathcal{H})$, which preserves property \mathcal{P} in both directions, then $\psi\phi \sim_{\mathcal{P}} \phi$, for all $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$. Also, if v is a linear operator on $\mathcal{B}(\mathcal{H})$, which does not preserve property \mathcal{P} in both directions, then for each surjective linear operator ϕ , $v\phi \not\sim_{\mathcal{P}} \phi$.

In the next section, we study the equivalence classes with respect to the above properties. We show that for surjective up to finite rank mappings ϕ_1 and ϕ_2 in $\mathcal{L}(\mathcal{B}(\mathcal{H}))$, $\phi_1 \sim_g \phi_2$ implies that $\phi_1 \sim_{sf} \phi_2$ and $\phi_1 \sim_f \phi_2$. Also, if ϕ_1, ϕ_2 are linear mappings on $\mathcal{B}(\mathcal{H})$, which are surjective up to compact operators, and $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$, then $\phi_1 \sim_k \phi_2$. It is also proved that $\phi_1 \sim_{si} \phi_2$ implies $\phi_1 \sim_{sf} \phi_2$. We give some examples to illustrate that some of the reverse implications do not hold, in general. We also prove that for surjective linear operators $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, $\phi_1 \sim_{si} \phi_2$ and $\phi_1 \sim_f \phi_2$ imply that $Ker(\phi_1) = Ker(\phi_2)$, and we give an example to show that the converse is not true, in general. Finally, it is proved that if ϕ_1, ϕ_2 are bijections such that $\phi_1 \sim_{sf} \phi_2$, then $\phi_1\phi_2^{-1}$ induces a map $\tilde{\psi} : \mathcal{C}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$, which is either an automorphism or an anti-automorphism, multiplied by an invertible element $A \in \mathcal{C}(\mathcal{H})$.

2. The Results

In the following lemma, (i) \Leftrightarrow (ii) comes from [2, Lemma 2.2] and (i) \Leftrightarrow (iii) comes from [3, Lemma 2.2].

Lemma 2.1. *Let $K \in \mathcal{B}(\mathcal{H})$. Then, the following are equivalent.*

- (i) K is compact.
- (ii) for every $B \in \mathcal{FR}(\mathcal{H})$, we have $B + K \in \mathcal{FR}(\mathcal{H})$.
- (iii) for every $B \in \mathcal{SF}(\mathcal{H})$, we have $B + K \in \mathcal{SF}(\mathcal{H})$.

Take $C = \{T \in \mathcal{B}(\mathcal{H}) \mid \text{for every operator } A \in \mathcal{B}(\mathcal{H}) \text{ with } Im(A) \text{ not closed, there exists } \lambda \in \mathbb{C} \text{ such that } A + \lambda T \neq 0 \text{ and } Im(A + \lambda T) \text{ is closed}\}$. It is proved in [1, Lemma 3.1] that $C = \mathcal{SF}(\mathcal{H})$.

We recall that if $T \in \mathcal{G}(\mathcal{H})$, then, for each finite rank operator F , we have $T + F \in \mathcal{G}(\mathcal{H})$.

Theorem 2.2. Let $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be linear mappings. Then,

$$(i) \quad \phi_1 \sim_{si} \phi_2 \Rightarrow \phi_1 \sim_{sf} \phi_2;$$

if ϕ_1, ϕ_2 are surjective up to finite rank operators, then

$$(ii) \quad \phi_1 \sim_g \phi_2 \Rightarrow \phi_1 \sim_{sf} \phi_2;$$

$$(iii) \quad \phi_1 \sim_g \phi_2 \Rightarrow \phi_1 \sim_f \phi_2;$$

if ϕ_1, ϕ_2 are surjective up to compact operators, then

$$(iv) \quad \phi_1 \sim_{fr} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2;$$

$$(v) \quad \phi_1 \sim_{sf} \phi_2 \Rightarrow \phi_1 \sim_k \phi_2.$$

Proof. (i) It is trivial by the definition of semi-index.

(ii) Suppose that $\phi_1(T)$ is a semi-Fredholm operator. We show that $\phi_2(T) \in C$. Suppose that $B \in \mathcal{B}(\mathcal{H})$ is not generalized invertible, or equivalently $Im(B)$ is not closed. Since ϕ_2 is surjective up to finite rank operators, there exist $A \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ such that $\phi_2(A) = B + F$. Since F is finite rank, $Im(\phi_2(A))$ is not closed and it follows that the range of $\phi_1(A)$ is not closed. Since $\phi_1(T)$ is semi-Fredholm, we have $\phi_1(T) \in C$. Thus, there exists $\alpha \in \mathbb{C}$ such that $Im(\phi_1(\alpha T + A))$ is closed. It follows that the range of $\phi_2(\alpha T + A) = \alpha\phi_2(T) + B + F$ is also closed, which implies that $Im(\alpha\phi_2(T) + B)$ is closed. Note that $\alpha\phi_2(T) + B \neq 0$. Otherwise, $Im(\alpha\phi_2(T)) = Im(B)$ is not closed, which contradicts $\phi_1 \sim_g \phi_2$. Therefore, $\phi_2(T) \in C$.

(iii) Let $\phi_1 \sim_g \phi_2$. Suppose that $\phi_1(T) \in \mathcal{F}(\mathcal{H})$, but $\phi_2(T)$ is not finite-rank. Therefore, the range of $\phi_1(T)$ is closed, but it is not semi-Fredholm. By the fact that $\phi_1 \sim_g \phi_2$, $Im(\phi_2(T))$ is closed. Also, by (ii), $\phi_2(T)$ is not semi-Fredholm. Take $S = \phi_2(T)$. Then, both $Ker(S)$ and $Im(S)^\perp$ are infinite-dimensional and we can define a bounded linear bijection $S' : Ker(S) \rightarrow Im(S)^\perp$. Extend S' on \mathcal{H} by $S'(x) = 0$, for all $x \in Ker(S)^\perp$, and denote this extension by S' as well. Since S is not finite rank, S' is not a semi-Fredholm operator on \mathcal{H} . Now, take $\tilde{T} \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}(\mathcal{H})$ such that $\phi_2(\tilde{T}) = S' + F$. We have $S + S'$ is a bijective bounded linear operator on \mathcal{H} , and hence it is Fredholm. Therefore, $\phi_2(T + \tilde{T}) = S + S' + F \in \mathcal{FR}(\mathcal{H})$. On the other hand, $\phi_1(T + \tilde{T})$ is not semi-Fredholm. Otherwise, $\phi_1(\tilde{T})$ must be semi-Fredholm and it follows by (ii) that $S' + F = \phi_2(\tilde{T})$ is semi-Fredholm, which is not correct. Thus, $\phi_1 \not\sim_{sf} \phi_2$, a contradiction with (ii), and so $\phi_1 \sim_f \phi_2$.

(iv) Suppose that $\phi_1(T)$ is compact. Let S be an arbitrary Fredholm operator. Since ϕ_2 is surjective up to compact operators, there exist $A \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$ such that $\phi_2(A) = S + K$. Obviously, $\phi_2(A) \in$

$\mathcal{FR}(\mathcal{H})$, and since $\phi_1 \sim_{fr} \phi_2$, we have $\phi_1(A)$ is Fredholm. On the other hand, $\phi_1(T)$ is compact and so by Lemma 2.1, $\phi_1(T + A) \in \mathcal{FR}(\mathcal{H})$. Thus, $\phi_2(T + A)$ is Fredholm, and it follows that $\phi_2(T) + S = \phi_2(T + A) - K$ is also a Fredholm operator and Lemma 2.1 implies that $\phi_2(T)$ is compact.

(v) The proof is similar to the one given in (iv). \square

In what follows we give some examples to show that in Theorem 2.2 some of the reverse implications do not hold, in general.

Example 2.3. Suppose that $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is a surjective linear map and $S \in \mathcal{B}(\mathcal{H})$ is a lower semi-Fredholm operator such that $s\text{-index}(S) = 1$. Since ϕ is surjective, there exists $T \in \mathcal{B}(\mathcal{H})$ such that $\phi(T) = S$. Now, if $A \in \mathcal{B}(\mathcal{H})$ is a Fredholm operator with $\text{def}(A) = 2$, then $\phi \sim_{sf} L_A \phi$, but $\phi \not\sim_{si} L_A \phi$, since $s\text{-index}(A\phi(T)) \geq 2 > 1$. Here $L_A : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is the left multiplier operator defined by $L_A(S) = AS$.

Example 2.4. We show that, in general, $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$ does not imply $\phi_1 \sim_g \phi_2$. Let S be a surjective bounded linear map on \mathcal{H} such that $\dim(\text{Ker}(S)) = 1$. Note that $S \in \mathcal{FR}(\mathcal{H})$. Let $P : \mathcal{H} \rightarrow \text{Ker}(S)$ be the projection of \mathcal{H} onto $\text{Ker}(S)$. Take $\phi_0 = L_S$. Since S has a bounded right inverse, ϕ_0 is surjective. Extend $\{P\}$ to a vector space basis $\{T_\alpha\}$ for $\mathcal{B}(\mathcal{H})$. Suppose that $K \in \mathcal{K}(\mathcal{H})$ has a non-closed range. Define a linear map $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{K}(\mathcal{H})$ by

$$\lambda(T_\alpha) = \begin{cases} K & T_\alpha = P \\ 0 & T_\alpha \neq P. \end{cases}$$

Now, define $\phi_1 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ by $\phi_0(T) + \lambda(T)$. Then, by Lemma 2.1, $\phi_1 \sim_{sf} \phi_0$. Also, ϕ_1 is surjective. To see this, take $T \in \mathcal{B}(\mathcal{H})$. There exists $U \in \mathcal{B}(\mathcal{H})$ such that $\phi_0(U) = T$. Since $\{T_\alpha\}$ is a vector space basis for $\mathcal{B}(\mathcal{H})$, there exist $\beta_1, \dots, \beta_n \in \mathbb{C}$ and $T_{\alpha_1}, \dots, T_{\alpha_n} \in \{T_\alpha\}$ such that $U = \sum_{i=1}^n \beta_i T_{\alpha_i}$. If for each $1 \leq j \leq n$, $T_{\alpha_j} \neq P$, then $\phi_1(U) = \phi_0(U) = T$. Otherwise, if for some $1 \leq j \leq n$, $T_{\alpha_j} = P$, then take $U' = U - \beta_j T_{\alpha_j}$ and we have $\phi_0(U) = \phi_0(U')$, and therefore, $\phi_1(U') = \phi_0(U') + \lambda(U') = T$.

Finally, $\phi_1(P) = K$, which is not generalized invertible since $\text{Im}(K)$ is not closed, while $\phi_0(P) = 0$ is generalized invertible and this shows $\phi_1 \not\sim_g \phi_0$.

We do not know any example of two linear mappings $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, which are surjective up to compact operators, $\phi_1 \sim_k \phi_2$ but $\phi_1 \not\sim_{fr} \phi_2$ or $\phi_1 \not\sim_{sf} \phi_2$. As seen in the above examples, the multiplier operator L_T

for a suitable T plays an important role. But, here we show that if ϕ_1 and ϕ_2 satisfy the above mentioned conditions, they can not be related by $\phi_2 = L_A R_B \phi_1 + \lambda$, where λ is a linear mapping from $\mathcal{B}(\mathcal{H})$ to $\mathcal{K}(\mathcal{H})$. Here, R_B denotes the right multiplier operator $T \mapsto TB$ on $\mathcal{B}(\mathcal{H})$.

Proposition 2.5. *Let $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ be surjective up to compact operators and $\phi_2 = L_A R_B \phi_1 + \lambda$, where, $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{K}(\mathcal{H})$ is a linear mapping. If $\phi_1 \sim_k \phi_2$, then A and B are Fredholm operators, and hence $\phi_1 \sim_{fr} \phi_2$ and $\phi_1 \sim_{sf} \phi_2$.*

Proof. Let $\phi_2 = L_A R_B \phi_1 + \lambda$. For $i = 1, 2$, consider $\tau_i : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ defined by $\tau_i(T) = \pi \circ \phi_i(T)$, where, $\pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ is the canonical quotient map. It is easy to check that $\tau_2(T) = a\tau_1(T)b$, for all $T \in \mathcal{B}(\mathcal{H})$, where, $a = \pi(A), b = \pi(B)$.

The condition that ϕ_1, ϕ_2 are surjective up to compact operators implies that τ_1, τ_2 are surjective. The condition $\phi_1 \sim_k \phi_2$ says that $\tau_1(T) = 0$ if and only if $\tau_2(T) = 0$ if and only if $a\tau_1(T)b = 0$. Since τ_1 is onto, this in turn says that with $x \in \mathcal{C}(\mathcal{H})$, $axb = 0$ if and only if $x = 0$.

Now, τ_2 is onto, and so $azb = \pi(I)$, for some $z \in \mathcal{C}(\mathcal{H})$. Thus, there exists $Z \in \mathcal{B}(\mathcal{H})$ such that AZB is a Fredholm operator, which shows that A and B are semi-Fredholm. If A were not Fredholm, then (since $a = \pi(A)$ is right invertible), we must have $nul(A) = \infty$. Let $P \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection of \mathcal{H} onto $Ker(A)$. Then, $p = \pi(P) \neq 0$, but $apb = \pi(APB) = \pi(0B) = 0$, which is a contradiction. Thus, A is Fredholm. Finally, since AZB is a Fredholm operator, we have that $B^*Z^*A^*$ is also a Fredholm operator. The same argument implies that B^* , and hence B is a Fredholm operator. \square

In the sequel, we explore the consequences when both ϕ_1 and ϕ_2 are in certain equivalence classes.

Theorem 2.6. *If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ are surjective linear maps such that $\phi_1 \sim_{si} \phi_2$ and $\phi_1 \sim_f \phi_2$, then $Ker(\phi_1) = Ker(\phi_2)$.*

Proof. Let $T \in Ker(\phi_1)$. Since $\phi_1 \sim_f \phi_2$, $\phi_2(T)$ is finite-rank, then it is not a semi-Fredholm operator. It follows that $null(\phi_2(T)) = \infty = def(\phi_2(T))$. Now, we can write $Ker(\phi_2(T)) = \mathcal{M} \oplus \mathcal{N}$, where \mathcal{M} and \mathcal{N} are infinite-dimensional closed subspaces of $Ker(\phi_2(T))$. Define a bounded linear bijection $T' : \mathcal{N} \rightarrow Im(\phi_2(T))^\perp$. Extend T' on \mathcal{H} by $T'(x) = 0$, for all $x \in (Ker(\phi_2(T)))^\perp \oplus \mathcal{M}$ and denote this extension by T' as well. Clearly, T' is a semi-Fredholm operator and

$s\text{-index}(T') = \text{rank}(\phi_2(T))$. We have $\phi_2(T) + T'$ is surjective and $\mathcal{M} \subseteq \text{Ker}(\phi_2(T) + T')$. Thus, it is a semi-Fredholm operator with $s\text{-index}(\phi_2(T) + T') = 0$. Since ϕ_2 is surjective, there exists $S \in \mathcal{B}(\mathcal{H})$ such that $\phi_2(S) = T'$. Therefore, $\text{rank}(\phi_2(T)) = s\text{-index}(\phi_2(S)) = s\text{-index}(\phi_1(S)) = s\text{-index}(\phi_1(S + T)) = s\text{-index}(\phi_2(S + T)) = 0$. It follows that $\text{Ker}(\phi_1) \subseteq \text{Ker}(\phi_2)$. The reverse inclusion follows similarly and we have the result. \square

Corollary 2.7. *If ϕ is a surjective linear map on $\mathcal{B}(\mathcal{H})$ that preserves finite rank operators and semi-index property in both directions, then ϕ is injective.*

As a consequence, by Theorem 2.2 (iii), if ϕ is a surjective linear map on $\mathcal{B}(\mathcal{H})$ that preserves generalized invertible operators and semi-index property in both directions, then it is injective.

Remark 2.8. (i) *In general, if surjective linear maps $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ have the same kernels, then it may happen that $\phi_1 \approx_f \phi_2$, and hence the converse of Theorem 2.6 does not hold. To see this, take $e \in \mathcal{H}$ with $\|e\| = 1$. It is clear that I and $e \otimes e$ are linearly independent. We can extend $\{I, e \otimes e\}$ to a basis for the vector space $\mathcal{B}(\mathcal{H})$. Now, define $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ such that $\phi(I) = e \otimes e$, $\phi(e \otimes e) = I$ and $\phi(T) = T$, for every T in the basis with $T \neq I, T \neq e \otimes e$. Extend ϕ to a bijection on $\mathcal{B}(\mathcal{H})$, by linearity. Hence, $\text{Ker}(\phi) = 0 = \text{Ker}(\mathcal{I})$, but we do not have $\phi \sim_f \mathcal{I}$, since $\phi(I) = e \otimes e$. It also follows that $\phi \approx_g \mathcal{I}$, since otherwise by Theorem 2.2 (iii), we must have $\phi \sim_f \mathcal{I}$.*

(ii) *The condition $\phi_1 \sim_{si} \phi_2$ in Theorem 2.6 can not be omitted. Suppose that $S \in \mathcal{B}(\mathcal{H})$ is surjective and $\dim(\text{Ker}(S)) = 1$. Thus, $\mathcal{I} \sim_f L_S$, $\mathcal{I} \sim_{si} L_S$, and clearly $\text{Ker}(\mathcal{I}) \neq \text{Ker}(L_S)$. We do not know any example of surjective linear operators ϕ_1, ϕ_2 on $\mathcal{B}(\mathcal{H})$ such that $\phi_1 \sim_{si} \phi_2$, but $\phi \not\sim_f \phi_2$. So, at this point we do not know whether the case $\phi_1 \sim_{si} \phi_2$, $\phi_1 \not\sim_f \phi_2$ happens or not.*

Now, we consider the case $\text{Ker}(\phi_1) = \{0\} = \text{Ker}(\phi_2)$.

Proposition 2.9. *Let \mathcal{P} be a property on $\mathcal{B}(\mathcal{H})$. If $\phi_1, \phi_2 : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ are bijective linear maps such that $\phi_1 \sim_{\mathcal{P}} \phi_2$, then $\phi_1 \phi_2^{-1}$ preserves property \mathcal{P} in both directions.*

Proof. Let $\mathcal{M}_{\mathcal{P}} = \{T \in \mathcal{B}(\mathcal{H}) : T \text{ has property } \mathcal{P}\}$. Then, $\phi_1^{-1}(\mathcal{M}_{\mathcal{P}}) = \phi_2^{-1}(\mathcal{M}_{\mathcal{P}})$, and hence $\mathcal{M}_{\mathcal{P}} = \phi_1 \phi_2^{-1}(\mathcal{M}_{\mathcal{P}})$. It follows that $\phi_1 \phi_2^{-1}$ preserves \mathcal{P} in both directions. \square

The following theorem was proved by Mbekhta and Šemrl [3, Theorem 1.2].

Theorem 2.10. *Let \mathcal{H} be an infinite-dimensional separable Hilbert space and $\phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be a linear map preserving semi-Fredholm operators in both directions. Suppose that ϕ is surjective up to compact operators. Then,*

$$\phi(\mathcal{K}(\mathcal{H})) \subseteq \mathcal{K}(\mathcal{H}),$$

and the induced map $\tilde{\phi} : \mathcal{C}(\mathcal{H}) \rightarrow \mathcal{C}(\mathcal{H})$ is either an automorphism, or an anti-automorphism multiplied by an invertible element $a \in \mathcal{C}(\mathcal{H})$.

Corollary 2.11. *Suppose that ϕ_1, ϕ_2 are bijective linear maps on $\mathcal{B}(\mathcal{H})$ such that $\phi_1 \sim_{sf} \phi_2$. Take $\psi = \phi_1 \phi_2^{-1}$. Then, $\psi(\mathcal{K}(\mathcal{H})) = \mathcal{K}(\mathcal{H})$ and the induced map $\tilde{\psi}$ on $\mathcal{C}(\mathcal{H})$ is an automorphism or an anti-automorphism multiplied by an invertible element $a \in \mathcal{C}(\mathcal{H})$.*

Note that, by Theorem 2.2 (ii), we have the same result for $\phi_1 \sim_g \phi_2$. Now, a question comes to mind: *Is it possible to identify the equivalence class of ϕ with respect to a property \mathcal{P} ?*

Remark 2.12. (i) *Let $\tau : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ be the linear map which takes T to its transpose with respect to a given basis for \mathcal{H} . If $\phi \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$, then it is easy to see that $\tau\phi \sim_g \phi$, and hence $\tau\phi \sim_{sf} \phi$, $\tau\phi \sim_k \phi$, and $\tau\phi \sim_f \phi$. Also, $\tau\phi \sim_{fr} \phi$ and $\tau\phi \sim_{si} \phi$.*

(ii) *Let A and B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{K}(\mathcal{H})$ be a linear map. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are related as $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$, then it is easy to see that $\phi_1 \sim_{sf} \phi_2$, $\phi_1 \sim_{fr} \phi_2$.*

(iii) *Let A and B be Fredholm operators and $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$ be a linear map. If $\phi_1, \phi_2 \in \mathcal{L}(\mathcal{B}(\mathcal{H}))$ are such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$, then $\phi_1 \sim_g \phi_2$.*

(iv) *Let $A, B \in \mathcal{B}(\mathcal{H})$ be invertible operators. If $\phi_2 = L_A R_B \phi_1$ or $\phi_2 = L_A R_B \tau \phi_1$, then it is easy to see that $\phi_1 \sim_{si} \phi_2$.*

Question 2.13. *Let $\phi_1 \sim_g \phi_2$. Are there $A, B \in \mathcal{FR}(\mathcal{H})$ and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{F}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$?*

Question 2.14. *Let $\phi_1 \sim_{sf} \phi_2$ or $\phi_1 \sim_{fr} \phi_2$. Are there $A, B \in \mathcal{FR}(\mathcal{H})$ and a linear map $\lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{K}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1 + \lambda$ or $\phi_2 = L_A R_B \tau \phi_1 + \lambda$?*

Question 2.15. Let $\phi_1 \sim_{si} \phi_2$. Are there invertible operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $\phi_2 = L_A R_B \phi_1$ or $\phi_2 = L_A R_B \tau \phi_1$?

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