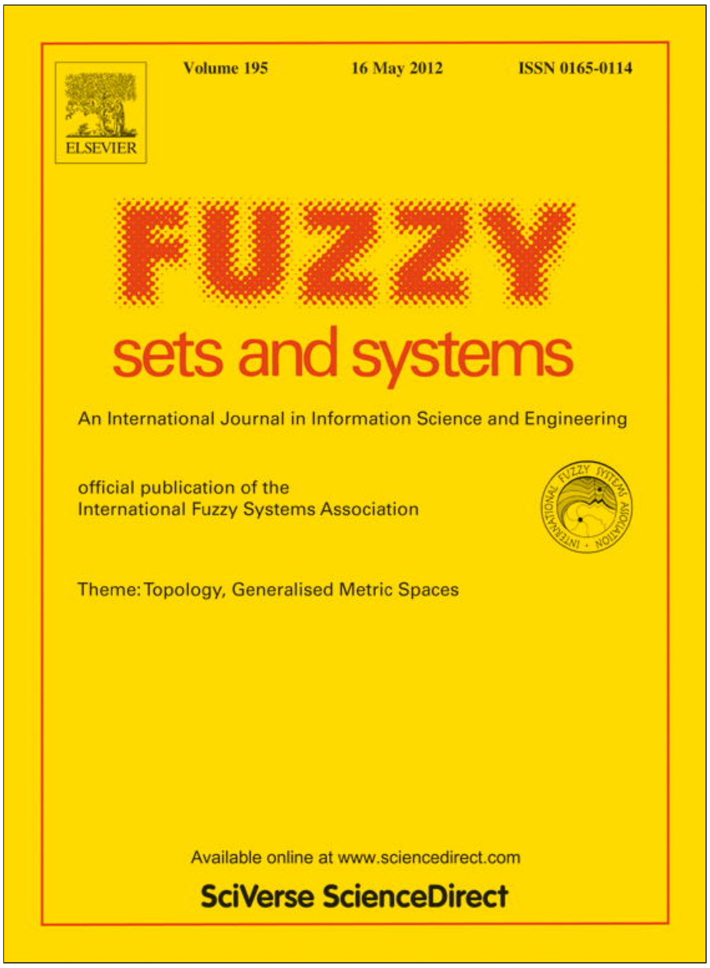
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Perturbation of generalized derivations in fuzzy Menger normed algebras

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Abstract

In this paper we introduce a notion for fuzzy Menger normed algebra. Then we will investigate continuity of algebraic operations in this space. We will show that the class of all fuzzy Menger normed algebras strictly contains the class of normed algebras. Finally, we will prove Hyers–Ulam–Rassias superstability of generalized derivation functional equation

$$f(ax+by+vw)=af(x)+bf(y)+vf(w)+g(v)w \quad (x,y,v,w\in X;\ 0\neq a,b\in\mathbb{C})$$

in complete fuzzy Menger normed algebras. A few applications of our results will be exhibited.

Keywords: Fuzzy Menger normed algebra; Hyers–Ulam–Rassias stability; Derivation

1. Introduction

In 1984, Katsaras [13] introduced the idea of fuzzy norm on a linear space. Following his pioneering work, several definitions for a fuzzy norm on a linear space have been introduced and discussed from different points of view (see e.g. [8,9,15]). In particular, Bag and Samanta in [5,6] introduced and studied an idea of a fuzzy norm on a linear space in such a manner that its corresponding fuzzy metric is of Kramosil and Michalek type [14]. They also give a comparative study of fuzzy norms on a linear space [7].

The concept of stability for a functional equation arising when we replace the functional equation by an inequality which acts as a perturbation of the equation. In 1940 Ulam [29] posed the first stability problem. In the next year, Hyers [12] gave the first partial affirmative answer to the question of Ulam. Subsequently, the result of Hyers was generalized for unbounded control functions by Aoki [3]. The concept of the Hyers–Ulam–Rassias stability was originated from Rassias' paper [26] for the stability of the linear mappings and its importance in the proof of further results in functional equations. A functional equation is called superstable if each of its approximate solution is an exact solution of the equation.

Let X be an algebra over complex numbers and $d : X \to X$ be an additive mapping. The function d is said to be a derivation if the functional equation $d(xy) = xd(y)+d(x)y$ holds for all $x, y \in X$. An additive mapping $f : X \to X$ is

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called a generalized derivation if there is a derivation $\delta : X \to X$ such that $f(xy) = xf(y) + \delta(x)y$ for each $x, y \in X$. A (generalized) derivation is called a linear (generalized) derivation if it is linear.

The first stability result concerning derivations between operator algebras was obtained by Semrl [27]. We refer the reader to [1,2,4,25] and the references therein for other related results concerning the stability of derivations.

The stability of functional equations in fuzzy normed spaces originated from [22]. Later several versions of fuzzy stability concerning Jensen, cubic, quadratic and quartic functional equations were investigated [16–24]. In [18], the author introduced a notion for "fuzzy Menger normed space" to extend some results in [19].

In this paper, we study fuzzy version of superstability of generalized derivations. In order to achieve this goal, we use an idea of Mihet [16] to introduce a notion for fuzzy Menger normed algebra (see [9,30,31] for some other definitions for the notion of a fuzzy algebra). Then we will investigate continuity of algebraic operations in a fuzzy normed algebra. In Section 3, we will prove Hers–Ulam–Rassias superstability of generalized derivations functional equation

$$f(ax + by + vw) = af(x) + bf(y) + vf(w) + g(v)w \quad (x, y, v, w \in X; 0 \neq a, b \in \mathbb{C})$$

in complete fuzzy Menger normed algebras. We also present some applications of our results in normed spaces.

2. Fuzzy Menger normed algebras

In this section, we will introduce a notion for fuzzy normed algebra. We will show that the algebraic operations in a fuzzy normed algebra are continuous. In order to achieve this goal, we need to recall some definitions.

Definition 2.1. A *triangular norm* (t-norm for short) [28] is a binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ which is commutative, associative and non-decreasing in each variable and has one as the unit element. A t-norm is called continuous if it is continuous with respect to the product topology on $[0, 1] \times [0, 1]$. The following are the basic examples of continuous t-norms:

(1) The minimum t-norm T_M, $T_M(a, b) = \min\{a, b\}$.
(2) The product t-norm T_P, $T_P(a, b) = a.b$.
(3) The Lukasiewicz t-norm, $T_L(a, b) = \max\{0, a + b - 1\}$.

For each $(x_1, \ldots, x_n) \in [0, 1]^n$, $n \geq 2$, we inductively define

$$T_{i=1}^3 x_i = T(x_1, x_2, x_3) = T(T(x_1, x_2), x_3), \ldots, T_{i=1}^n x_i = T(x_1, \ldots, x_n) = T(T(x_1, \ldots, x_{n-1}), x_n)$$

and for every sequence $\{x_n\}$ in $[0, 1]$, $T_{i=1}^\infty x_i = \lim_{n\to\infty} T_{i=1}^n x_i$.

Definition 2.2. A t-norm T is said to be Hadžić-type [10] if $\{T_{i=1}^n(x)\}$ is equicontinuous at the point $x = 1$. In the other words, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that $T_{i=1}^n x > 1 - \varepsilon$ whenever $x \in (1 - \delta, 1)$.

A trivial example of a t-norm of Hadžić-type is T_M. A few examples of non-trivial Hadžić-type t-norms are given in [11]. It follows from the definition that if T is of Hadžić-type and $\{x_n\} \subset [0, 1]$, then $\lim_{n\to\infty} T_{i=1}^n x_i = 1$ provided that $\lim_{n\to\infty} x_n = 1$.

Following Bag and Samanta [5,6], Mihet in [16] gave the following definition of a fuzzy Menger norm.

Definition 2.3. Let X be a complex linear space and T be a continuous t-norm. By a fuzzy Menger norm on X, we mean a fuzzy subset of $X \times [0, \infty)$ such that the following conditions hold for all $x, y \in X$ and scalars c, s, t:

(N1) $N(x, 0) = 0$ for each $x \neq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
(N3) $N(cx, t) = N(x, t/|c|)$, whenever $c \neq 0$;
(N4) $N(x + y, s + t) \geq T\{N(x, s), N(y, t)\}$ (the triangle inequality);
(N5) $\lim_{t\to\infty} N(x, t) = 1$.

In this case (X, N, T) is called a *fuzzy normed space*. It follows from (N2) and (N4) that $N(x, \cdot)$ is an increasing function for each $x \in X$. In fact, if $x \in X$ and $0 < s < t$, then

$$N(x, t) \geq T(N(x, s), N(0, t - s)) = N(x, s).$$

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy Menger normed linear space (X, N, T) is said to be convergent if there exists some $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the fuzzy limit of the sequence $\{x_n\}$.

A sequence $\{x_n\}$ in X is called Cauchy if for each $0 < \alpha < 1$ and $t > 0$ there exists some $n_0 \in \mathbb{N}$ such that $N(x_{n+p} - x_n, t) > \alpha$ for all $n > n_0$ and $p > 0$.

If each Cauchy sequence is convergent in (X, N, T), then the fuzzy Menger norm is said to be complete.

Remark 2.5. In [5, Theorem 2.1], the authors proved that if a fuzzy (Menger) normed space (X, N, T_M) satisfies the condition

(N6) $N(x, t) > 0$ for all $t > 0$ implies that $x = 0$,

then $\|x\|_\alpha = \inf\{t : N(x, t) \geq \alpha\}$, $\alpha \in (0, 1)$ defines an ascending family of norms on X. Moreover, if $\{x_n\}$ converges in (X, N, T_M), then $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$ for each $0 < \alpha < 1$.

However, the converse is not true in general. For example, if $X = \ell^\infty$ with the fuzzy (Menger) norm

$$N(x, t) = \begin{cases} 1, & t > \sup_n |x_n|, \\ 0.5, & \sup_n \left|\frac{x_n}{n}\right| < t \leq \sup_n |x_n|, \\ 0, & t \leq \sup_n \left|\frac{x_n}{n}\right| \end{cases}$$

for each $x = (x_1, x_2, \ldots) \in X$, then the sequence

$$e_1 = (1, 0, 0, \ldots), \quad e_2 = (0, 1, 0, \ldots), \ldots$$

converges to 0 with respect to $\|\cdot\|_{0.5}$. However, this sequence has no limit in (X, N, T_M) (see [6, Proposition 2. 1 and Example 2.1]).

Definition 2.6. Let X be an algebra, T, T' be continuous t-norms and (X, N, T) be a fuzzy Menger normed space. Let

$$N(xy, st) \geq T'(N(x, t), N(y, s)) \quad (x, y \in X, s, t \geq 0).$$

Then the quadruple (X, N, T, T') is called a fuzzy Menger normed algebra.

The following two examples show that the class of fuzzy Menger algebras strictly contains all normed algebras.

Example 2.7. Let $(X, \|\cdot\|)$ be an algebra normed space. Define

$$N(x, t) = \begin{cases} 0, & x \neq 0, t < 0, \\ \dfrac{t}{t + \|x\|}, & x \neq 0, t \geq 0, \\ 1, & x = 0. \end{cases}$$

In [5], it is shown that (X, N, T_M) is a fuzzy (Menger) normed space. An easy computation shows that $N(xy; st) \geq N(x; s).N(y; t)$ if and only if

$$\|xy\| \leq \|x\|\|y\| + s\|y\| + t\|x\| \quad (x, y \in X; s, t > 0).$$

It follows that (X, N, T_M, T_p) is a fuzzy normed algebra.

Example 2.8. Let X be the space of complex-valued continuous functions on the real line. Then X is not normable [32]. Define

$$N(f,t)=\begin{cases}0, & t\le 0,\\ \sup\left\{\dfrac{n}{n+1}:\|f\|_n\le t\right\}, & t>0,\end{cases}$$

where $\|\cdot\|_n$ denote the sup-norm on $[-n,n]$, $n\in\mathbb{N}$. By imitating the proof of Theorem 2.2 in [5], one can show that (X,N,T_M) is a fuzzy (Menger) normed space. An easy computation shows that $N(fg;st)\ge T_L\{N(f;s),N(g;t)\}$. Therefore (X,N,T_M,T_L) is a fuzzy normed algebra.

By Uryshon's lemma, for each $n\in\mathbb{N}$, there is a continuous function $f_n:\mathbb{R}\to[0,1]$ such that $f_n(n-\frac{1}{2})=1$ and $f_n(x)=0$ whenever $x\notin[n-\frac{1}{3},n+1]$. It is clear that $\{f_n\}$ has no limit in X with respect to the uniform convergence topology on X. However, this sequence converges to 0 in (X,N). To see this let $t>0$ and $0<\alpha<1$ be arbitrary. Choose $n_0\in\mathbb{N}$ such that $n_0/(1+n_0)>\alpha$. Then for each $n>n_0$, we have $\|f_n\|_{n_0}=0<t$, therefore $N(f_n-0,t)\ge n_0/(1+n_0)>\alpha$. This means that $\{f_n\}$ converges to zero in (X,N).

Definition 2.9. Let (X,N_X,T_X) and (Y,N_Y,T_Y) be two fuzzy Menger normed spaces. A function $f:X\to Y$ is said to be fuzzy continuous if $x_n\to x$ in (X,N_X,T_X) implies that $f(x_n)\to f(x)$ in (Y,N_Y,T_Y).

Lemma 2.10. *Let (X,N,T) be a fuzzy Menger normed space and let $\{x_n\}$ be a sequence X such that $x_n\to x$. Then for each $0<\alpha<1$ there are $M>0$ and $n_0\in\mathbb{N}$ such that*

$$N(x_n,M)\ge\alpha\quad for\ all\ n>n_0.$$

Proof. By (N5), there is some $M>0$ such that $N(x,M/2)>\alpha$. Since $x_n\to x$ and $T(\cdot,N(x,M)):[0,1]\to[0,1]$ is continuous, we can find some $n_0\in\mathbb{N}$ such that

$$N(x_n,M)\ge T\left(N\left(x_n-x,\frac{M}{2}\right),N\left(x,\frac{M}{2}\right)\right)\ge\alpha\quad(n\ge n_0).\qquad\square$$

Theorem 2.11. *Let (X,N,T,T') be a real fuzzy normed algebra. Then the algebraic operations are fuzzy continuous.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ converge to x and y respectively, $0<\alpha<1$ and $t>0$. By (N4) and the continuity of T, we can find some $n_0\in\mathbb{N}$ such that

$$N((x_n+y_n)-(x+y),t)\ge T\left(N\left(x_n-x,\frac{t}{2}\right),N\left(y_n-y,\frac{t}{2}\right)\right)>\alpha\quad(n\ge n_0).$$

This proves the continuity of addition in (X,N,T,T').

Since

$$x_ny_n-xy=x_n(y_n-y)+(x_n-x)y\quad(n\in\mathbb{N})$$

and "+" is continuous, it is enough to show that for each $t>0$,

$$\lim_n N(x_n(y_n-y),t)=1\quad\text{and}\quad\lim_n N((x_n-x)y,t)=1.\tag{2.1}$$

The inequalities

$$N(x_n(y_n-y),t)\ge T'\left(N(x_n,M),N\left(y_n-y,\frac{t}{M}\right)\right)$$

and

$$N((x_n-x)y,t)\ge T'\left(N\left(x_n-x,\frac{t}{M}\right),N(y,M)\right)$$

together with the continuity of T' and Lemma 2.10 proves (2.1). $\square$

3. Superstability of derivations

Throughout the rest of this paper, unless otherwise is explicitly stated, we will assume that (X, N, T, T') is a complete fuzzy normed algebra with unit, Z is a linear space, $N' : Z \times [0, \infty) \to [0, 1]$ is a fuzzy set and T'' is a continuous t-norm.

Theorem 3.1. *Let $f : X \to X$ and $\varphi, \psi : X^2 \to Z$ satisfy the inequality*

$$N(f(ax+by+vw) - af(x) - bf(y) - vf(w) - g(v)w, t) \geq T''(N'(\varphi(x,y),t), N'(\psi(v,w),t)), \tag{3.1}$$

for each $x, y, v, w \in X$ and non-zero scalars a and b with $s = a+b > 1$. If there are some $1 \leq \alpha < s$ and $0 < \beta, \gamma < s$ such that for each $x, y \in X$ and $t > 0$

$$N'(\varphi(sx, sy), \alpha t) \geq N'(\varphi(x,y), t) \tag{3.2}$$

and for each $x, y \in X$, and

$$N'(\psi(s^n x, s^m y), \beta^n \gamma^m t) \geq N'(\psi(x,y), t) \quad (n, m \geq 0). \tag{3.3}$$

Then f is a generalized derivation and g is a derivation.

Proof. Putting $x = y$ and $v = w = 0$ in (3.1), we have

$$N(f(sx) - sf(x), t) \geq T''(N'(\varphi(x,x),t), N'(\psi(0,0),t)) \quad (x \in X, t > 0).$$

Replacing x by $s^{n-1}x$ in the above inequality and using (3.2), we obtain

$$\begin{aligned} N\left(\frac{f(s^n x)}{s^n} - \frac{f(s^{n-1}x)}{s^{n-1}}, t\right) &\geq T''(N'(\varphi(s^{n-1}x, s^{n-1}x), s^n t), N'(\psi(0,0), s^n t)) \\ &\geq T''\left(N'\left(\varphi(x,x), \frac{s^n t}{\alpha^{n-1}}\right), N'(\psi(0,0), s^n t)\right) \quad (x \in X, t > 0). \end{aligned}$$

Replacing t by $\alpha^{n-1}t/s^n$ in the above inequality, we get to

$$N\left(\frac{f(s^n x)}{s^n} - \frac{f(s^{n-1}x)}{s^{n-1}}, \frac{\alpha^{n-1}t}{s^n}\right) \geq T''(N'(\varphi(x,x),t), N'(\psi(0,0), \alpha^{n-1}t)) \quad (x \in X, t > 0).$$

It follows that for each $m > n$, $x \in X$ and $t > 0$,

$$\begin{aligned} N\left(\frac{f(s^m x)}{s^m} - \frac{f(s^n x)}{s^n}, \sum_{i=n+1}^{m} \left(\frac{\alpha}{s}\right)^i \cdot \frac{t}{\alpha}\right) &= N\left(\sum_{i=n+1}^{m} \frac{f(s^i x)}{s^i} - \frac{f(s^{i-1}x)}{s^{i-1}}, \sum_{i=n+1}^{m} \left(\frac{\alpha}{s}\right)^i \frac{t}{\alpha}\right) \\ &\geq T_{i=n+1}^{m}\left(N\left(\frac{f(s^i x)}{s^i} - \frac{f(s^{i-1}x)}{s^{i-1}}, \left(\frac{\alpha}{s}\right)^i \frac{t}{\alpha}\right)\right) \\ &\geq T_{i=n+1}^{m} T''(N'(\varphi(x,x),t), N'(\psi(0,0), \alpha^{i-1}t)). \end{aligned}$$

Therefore

$$\begin{aligned} &N\left(\frac{f(s^m x)}{s^m} - \frac{f(s^n x)}{s^n}, t\right) \\ &\geq T_{k=n+1}^{m} T''\left(N'\left(\varphi(x,x), \frac{\alpha t}{\sum_{i=n+1}^{m}\left(\frac{\alpha}{s}\right)^i}\right), N'\left(\psi(0,0), \frac{\alpha^k t}{\sum_{i=n+1}^{m}\left(\frac{\alpha}{s}\right)^i}\right)\right) \end{aligned} \tag{3.4}$$

for each $x \in X$ and $t > 0$. Since $\sum(\alpha/s)^n$ converges and $\alpha > 1$,

$$T''\left(N'\left(\varphi(x,x), \frac{\alpha t}{\sum_{i=n+1}^{m}\left(\frac{\alpha}{s}\right)^i}\right), N'\left(\psi(0,0), \frac{\alpha^k t}{\sum_{i=n+1}^{m}\left(\frac{\alpha}{s}\right)^i}\right)\right)$$

tends to one as $n \to \infty$. Eq. (3.4) together with the fact that T is of Hadžić-type, (3.4) ensures that $f(s^n x)/s^n$ is a Cauchy sequence in X for each $x \in X$. By completeness of X, the limit

$$d(x) = \lim_{n\to\infty} \frac{f(s^n x)}{s^n} \tag{3.5}$$

exits. In (3.1), put $v = w = 0$ and replace x and y by $s^n x$ and $s^n y$ respectively to obtain

$$\begin{aligned} N\left(\frac{f(s^n(ax+by))}{s^n} - a\frac{f(s^n x)}{s^n} - b\frac{f(s^n y)}{s^n}, t\right) &\geq T''(N'(\varphi(s^n x, s^n y), s^n t), N'(\psi(0,0), s^n t)) \\ &\geq T''\left(N'\left(\varphi(x,y), \left(\frac{s}{\alpha}\right)^n t\right), N'(\psi(0,0), s^n t)\right) \end{aligned}$$

for each $x, y \in X$ and $t > 0$. Since the right hand side of the above inequality tends to one as n tends to infinity and multiplication is continuous, $d(ax+by) = ad(x) + bd(y)$ for each $x, y \in X$. By putting $x = 0$ in (3.5), we see that $d(0) = 0$. It follows that $d(ax) = ad(x)$ and $d(by) = bd(y)$ for each $x, y \in X$. Therefore

$$d(x+y) = d\left(a\frac{x}{a} + b\frac{y}{b}\right) = ad\left(\frac{x}{a}\right) + bd\left(\frac{y}{b}\right) = d(x) + d(y) \quad (x, y \in X).$$

Hence d is additive. Put $x = y = 0$ in (3.1) to obtain

$$N(f(vw) - vf(w) - g(v)w, t) \geq T''(N'(\varphi(0,0), t), N'(\psi(v,w), t)) \quad (v, w \in X, t > 0). \tag{3.6}$$

By replacing v and w respectively by $s^n v$ and $s^n w$ in the above inequality, by (3.3), we see that for each $v, w \in X$ and $t > 0$,

$$\begin{aligned} N\left(\frac{f(s^{2n} vw)}{s^{2n}} - v\frac{f(s^n w)}{s^n} - \frac{g(s^n v)}{s^n} w, t\right) &\geq T''(N'(\varphi(0,0), s^{2n} t), N'(\psi(s^n v, s^n w), s^{2n} t)) \\ &\geq T''\left(N'(\varphi(0,0), s^{2n} t), N'(\psi(v,w), \left(\frac{s}{\beta}\right)^n \left(\frac{s}{\gamma}\right)^n\right) \end{aligned}$$

Since the right hand side of the above inequality goes to one as $n \to \infty$, by the continuity of product in fuzzy Menger normed algebra (X, N, T, T'), it follows that

$$\lim_{n\to\infty} \frac{g(s^n v)}{s^n} w = d(vw) - vd(w) \quad (v, w \in X). \tag{3.7}$$

Put $w = e$ in the above identity to obtain

$$\delta(v) = \lim_{n\to\infty} \frac{g(s^n v)}{s^n} = d(v) - vd(e) \quad (v \in X).$$

For each $v, w \in X$, we have

$$\begin{aligned} \delta(vw) &= d(vw) - vwd(e) \\ &= vd(w) + \delta(v)w - vwd(e) \\ &= v[d(w) - wd(e)] + \delta(v)w \\ &= v\delta(w) + \delta(v)w \end{aligned}$$

and

$$\begin{aligned}\delta(v+w) &= d(v+w) - (v+w)d(e)\\ &= d(v) + d(w) - vd(e) - wd(e)\\ &= \delta(v) + \delta(w).\end{aligned}$$

Therefore δ is a derivation. By (3.7),

$$d(vw) = vd(w) + \delta(v)w \quad (v, w \in X). \tag{3.8}$$

It follows that d is a generalized derivation. Put $v = s^n e$ in (3.6), then we have

$$\begin{aligned}N\left(\frac{f(s^n ew)}{s^n} - ef(w) - \frac{g(s^n e)}{s^n}w, t\right) &\geq T''(N'(\varphi(0,0), s^n t), N'(\psi(s^n e, w), s^n t))\\ &\geq T''\left(N'(\varphi(0,0), s^n t), N'\left(\psi(e,w), \left(\frac{s}{\gamma}\right)^n t\right)\right)\end{aligned}$$

for each $w \in X$ and $t > 0$. Therefore the right hand side of the above inequality tends to one as $n \to \infty$. Thanks to continuity of multiplication, it follows that $d(w) = f(w) + \delta(e)w$ for each $w \in X$. Since δ is a derivation, $\delta(e) = 0$. Hence $f = d$ is a generalized derivation.

Replace w by $s^n w$ in (3.6) to obtain

$$\begin{aligned}N\left(\frac{f(s^n ve)}{s^n} - v\frac{f(s^n e)}{s^n} - g(v)e, t\right) &\geq T''(N'(\varphi(0,0), s^n t), N'(\psi(v, s^n e), s^n t))\\ &\geq T''\left(N'(\varphi(0,0), s^n t), N'\left(\psi(v,e), \left(\frac{s}{\gamma}\right)^n t\right)\right)\end{aligned}$$

for each $v \in X$ and $t > 0$. Since $s > \gamma$, the right hand side of the above inequality tends to one as $n \to \infty$. Therefore $d(v) = vd(e) + g(v)$ for each $v \in X$. On the other hand by (3.8), $d(v) = vd(e) + \delta(v)$ for each $v \in X$. It follows that $g = \delta$ is a derivation. □

Corollary 3.2. *Let $f, g : X \to X$ satisfy*

$$N(f(x+y+vw) - f(x) - f(y) - vf(w) - g(v)w, t) \geq T''(N'(z_0, t), N'(z_0, t))$$

for some $z_0 \in Z$; $x, y, v, w \in X$ and $t > 0$. Then f is a generalized derivation and g is a derivation.

Proof. Apply Theorem 3.1 for $a = b = \alpha = \beta = \gamma = 1$ and

$$\varphi(x, y) = z_0 = \psi(v, w) \quad (x, y, v, w \in X). \quad \square$$

Corollary 3.3. *Let X be a complete unital normed algebras over complex numbers. Let $f, g : X \to X$ for some $0 < p < 1$ satisfy the inequality*

$$\|f(ax+by+vw) - af(x) - bf(y) - vf(w) - g(v)w\| \leq \max\{\|x\|^p + \|y\|^p, \|v\|^p + \|w\|^p\} \tag{3.9}$$

for each $x, y, v, w \in X$ and non-zero scalars a and b with $1 < s = a + b$. Then f is a generalized derivation and g is a derivation.

Proof. Let N and N' be respectively the fuzzy norms on X and $\mathbb{R}$ defined in Example 2.7. Define $\varphi(x, y) = \psi(x, y) = \|x\|^p + \|y\|^p$. Then an easy computation shows that (3.9) is equivalent to (3.1) for $T'' = T_M$ and for $\alpha = \beta = \gamma = s^p$, we have

$$N'(\varphi(sx, sx), \alpha t) = N'(\varphi(x, x), t).$$

Moreover,

$$N'(\psi(s^n x, s^m y), \beta^n \gamma^m t) = \frac{s^{np} s^{mp} t}{s^{np} s^{mp} t + s^{np}\|x\|^p + s^{mp}\|y\|^p} = \frac{t}{t + \dfrac{s^{np}\|x\|^p + s^{mp}\|y\|^p}{s^{np} s^{mp}}}$$

$$\geq \frac{t}{t + \psi(x, y)} = N'(\psi(x, y)) \quad ((x, y) \in X^2, t > 0).$$

Hence the result follows from Theorem 3.1. □

4. Conclusion

We introduced a notion for fuzzy Menger normed algebra and investigated algebraic properties of this space. We have shown that the class of fuzzy Menger normed algebras contains the class of all normed algebras, however, the converse is not true in general. We also proved superstability of generalized derivations in fuzzy Menger normed algebras and applied our results in standard normed algebras. Our method in this paper may be extended to study some other Mathematical problems in fuzzy Menger normed algebras.

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References

[1] M. Amyari, F. Rahbarnia, Gh. Sadeghi, Some results on the stability of extended derivations, J. Math. Anal. Appl. 329 (2) (2007) 753–758.
[2] Ansari-Piri, E. Anjidani, Supersatability of generealized derivations, J. Inequality Appl. 329 (2) (2010) Art. no. 740156.
[3] T. Aoki, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Jpn. 2 (1950) 64–66.
[4] R. Badora, On approximate derivations, Math. Inequalities Appl. 9 (1) (2006) 167–173.
[5] T. Bag, S.K. Samanta, Finite dimensional fuzzy normed linear spaces, J. Fuzzy Math. 11 (3) (2003) 687–705.
[6] T. Bag, S.K. Samanta, Fuzzy bounded linear operators, Fuzzy Sets Syst. 151 (2005) 513–547.
[7] T. Bag, S.K. Samanta, A comparative study of fuzzy norms on a linear space, Fuzzy Sets Syst. 159 (2008) 670–684.
[8] C. Felbin, Finite dimensional fuzzy normed linear space, Fuzzy Sets Syst. 48 (1992) 239–248.
[9] W. Gähler, S. Gähler, Contributions to fuzzy analysis, Fuzzy Sets Syst. 105 (1999) 201–224.
[10] O. Hadžić, M. Budinčević, A class of t-norms in the fixed point theory on PM-spaces, Univ. u Novom Sadu Zb. Rad. Prirod. Mat. Fak. 9 (1979) 37–41.
[11] O. Hadžić, E. Pap, V. Radu, Generalized contraction mapping principles in probabilistic metric spaces, Acta Math. Hung. 101 (1–2) (2003) 131–148.
[12] D.H. Hyers, On the stability of the linear functional equation, Proc. Natl. Acad. Sci. USA 27 (4) (1941) 222–224.
[13] A.K. Katsaras, Fuzzy topological vector spaces II, Fuzzy Sets Syst. 12 (1984) 143–154.
[14] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, Kybernetica 11 (1975) 326–334.
[15] S.V. Krishna, K.K.M. Sarma, Separation of fuzzy normed linear spaces, Fuzzy Sets Syst. 63 (1994) 207–217.
[16] D. Mihet, The fixed point method for fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst. 160 (2009) 1663–1667.
[17] D. Mihet, On the stability of the additive Cauchy functional equation in random normed spaces, J. Math. Anal. Appl. 343 (2008) 567–572.
[18] D. Mihet, The stability of the additive Cauchy functional equation in non-Archimedean fuzzy normed spaces, Fuzzy Sets Syst. 161 (2010) 2206–2212.
[19] A.K. Mirmostafaee, A fixed point approach to almost quartic mappings in quasi fuzzy normed spaces, Fuzzy Sets Syst. 160 (2009) 1653–1662.
[20] A.K. Mirmostafaee, M.S. Moslehian, Stability of additive mappings in non-Archimedean fuzzy normed spaces, Fuzzy Sets Syst. 160 (2009) 1643–1652.
[21] A.K. Mirmostafaee, M. Mirzavaziri, M.S. Moslehian, Fuzzy stability of the Jensen functional equation, Fuzzy Sets Syst. 159 (2008) 730–738.
[22] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy versions of Hyers–Ulam–Rassias theorem, Fuzzy Sets Syst. 159 (2008) 720–729.
[23] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy almost quadratic functions, Results Math. 52 (2008) 161–177.
[24] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy approximately cubic mappings, Inf. Sci. 78 (19) (2008) 3791–3798.
[25] T. Miura, G. Hirasawa, S-E. Takahasi, A perturbation of ring derivations on Banach algebras, J. Math. Anal. Appl. 319 (2006) 522–530.
[26] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Am. Math. Soc. 72 (2) (1978) 297–300.
[27] P. Semrl, The functional equation of multiplicative derivation is superstable on standard operator algebras, Integr. Equat. Oper. Theory 18 (1) (1994) 118–122.

[28] B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, 1983.
[29] S.M. Ulam, Problems in Modern Mathematics, John-Wiley & Sons, New York, 1964 (chapter 6).
[30] R. Thakur, S.K. Samanta, Fuzzy Banach algebra, J. Fuzzy Math. 18 (3) (2010) 687–696.
[31] R. Thakur, S.K. Samanta, Fuzzy Banach algebra with Felbin's type fuzzy norm, J. Fuzzy Math. 18 (4) (2011) 943–954.
[32] B. Yood, On the non-existence of norms for some algebras of functions, Stud. Math. 111 (1) (1994) 97–101.