

S-duality of S-matrix

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ABSTRACT: Recently, it has been observed that the kinematic factor of the disk-level S-matrix element of two RR two-forms and the disk-level S-matrix element of two B-fields on the world volume of D₃-brane are compatible with the standard rules of S-duality. Inspired by this observation, we speculate that the S-matrix elements on the world volume of D₃-brane are invariant under the SL(2, ℤ) transformation. Compatibility with S-duality requires the S-matrix elements on the world volume of D₁-brane and D₅-brane to be extended to the SL(2, ℤ)-covariant form. In particular, this implies the S-matrix elements on the world volume of F₁-string and NS₅-brane at strong coupling to be related by S-duality to the disk-level S-matrix elements of D₁-string and D₅-brane, respectively. The contact terms of these S-matrix elements at order $O(\alpha'^0)$ produce a Born-Infeld and Chern-Simons type effective action for both F₁-string and NS₅-brane. They are consistent with the SL(2, ℝ)-covariant action appears in the literature.

KEYWORDS: p-branes, Superstrings and Heterotic Strings, D-branes, String Duality

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1 Introduction

It is known that the type II superstring theory is invariant under T-duality [1–5] and S-duality [5–11]. Compatibility of a given solution of equations of motion with these dualities can be used to generate new solutions [12–15]. In this paper, we would like to apply this compatibility to the other on-shell quantities, i.e., the S-matrix elements.

The S-matrix elements of any nonlinear gauge theory must satisfy the Ward identity. This is the linearized gauge transformations in the momentum space. For example, the S-matrix elements of a gravity theory must be invariant under the replacement $\varepsilon_{\mu\nu} \rightarrow \varepsilon_{\mu\nu} + p_\mu \zeta_\nu + p_\nu \zeta_\mu$ where $\varepsilon_{\mu\nu}$ is the polarization of external graviton, ζ_μ is an arbitrary vector and $p \cdot \zeta = 0$. We expect similar Ward identities for the global S and T dualities in string theory. Since there is no derivative in the duality transformations, one should replace a polarization tensor with its dual tensor, which is related to the original one by the linear duality transformations. This replacement may produce new S-matrix elements which are related to the original one by the linear dualities. On the other hand, we expect the S-matrix elements to be invariant/covariant under the nonlinear duality transformation on the background fields in the S-matrix element. The T-duality holds order by order in string loop expansion [5]. This indicates that a given S-matrix element at any loop order is invariant/covariant under the T-duality transformation on the background fields. The invariance of the S-matrix element under linear T-duality transformation on the quantum states then can be used to extend the S-matrix element to a family of S-matrix elements. We call such S-matrix elements the T-dual S-matrix multiplet. Let us examine this idea for the simple example of disk level S-matrix element of two gravitons. In general, the disk-level two-point function of closed strings is given by [16, 17]

$$A(D_p; \varepsilon_1, \varepsilon_2) \sim T_p \alpha'^2 K(D_p; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-t/4)\Gamma(-s)}{\Gamma(1-t/4-s)} \delta^{p+1}(p_1^a + p_2^a) \quad (1.1)$$

where $\varepsilon_1, \varepsilon_2$ are the polarization of external states. In this amplitude, the Mandelstam variable $s = -\alpha'(p_1)_a (p_1)_b \eta^{ab}$ is the momentum flowing along the world volume of brane,

and $t = -\alpha'(p_1 + p_2)^2$ is the momentum transfer in the transverse directions on the brane. The background metric $\eta_{\mu\nu}$ is the string frame metric.¹ The s -channel describes the open string excitation of the D-brane, and the t -channel describes the closed string couplings to the D-brane. The background has no axion, hence, the tension of the intermediate string in the string frame is $T_{F_1} = 1/(2\pi\alpha')$. In above equation, T_p is the tension of external D_p -brane in the string frame, i.e.,

$$T_p = \frac{1}{g_s(2\pi)^p(\alpha')^{(p+1)/2}} \tag{1.2}$$

where $g_s = e^{\phi_0}$ is the closed string coupling. We have normalized this amplitude and all other amplitudes in this paper by requiring them to be consistent with T-duality. We will not fix, however, the numeric factor of the amplitudes.

The kinematic factor in (1.1) for two gravitons is a lengthy expression in terms of the graviton polarizations [16]. Using on-shell relations, this factor can be written in terms of the linearized curvature tensor of the external states in the momentum space as [18]

$$K(D_p; h_1, h_2) = R_{1abcd}R_2^{abcd} - 2\hat{R}_{1ab}\hat{R}_2^{ab} - R_{1abij}R_2^{abij} + 2\hat{R}_{1ij}\hat{R}_2^{ij} \tag{1.3}$$

The indices are raised and lowered by the flat metrics η_{ab} and η_{ij} . In above equation $\hat{R}_{1ab} = \eta^{cd}R_{1cadb}$ and $\hat{R}_{1ij} = \eta^{cd}R_{1cidj}$. The linearized curvature tensor corresponding to the first graviton is

$$R_{1\mu\nu\rho\lambda} = \frac{1}{2}(h_{1\mu\lambda,\nu\rho} + h_{1\nu\rho,\mu\lambda} - h_{1\mu\rho,\nu\lambda} - h_{1\nu\lambda,\mu\rho})$$

where the metric in the curvature tensor is $\eta_{\mu\nu} + h_{1\mu\nu}$ and $h_{1\mu\nu}$ is the graviton polarization tensor. The commas denote partial differentiation in the momentum space.

To apply the T-duality on the 2-point function, one should first use the nonlinear T-duality on the background, and then use linear T-duality on the quantum states. If one implements T-duality along a world volume direction of D_p -brane, then the background fields transform under the nonlinear T-duality to

$$\begin{aligned} T_p\delta^{p+1}(p_1^a + p_2^a) &\longrightarrow T_{p-1}\delta^p(p_1^a + p_2^a) \\ s, t &\longrightarrow s, t \end{aligned} \tag{1.4}$$

where we have used the assumption implicit in the T-duality that fields are independent of the Killing coordinate, e.g., $\delta^{p+1}(p_1^a + p_2^a) = \delta^p(p_1^a + p_2^a)\delta(0)$ where $\delta(0) = 2\pi R$ and R is the radius of the compact direction on which the T-duality is applied. So the D_p -brane of type IIA/IIB transforms to the D_{p-1} -brane of type IIB/IIA. The quantum fluctuations h_1, h_2 , in the kinematic factor (1.3) transform to the following expression under the linear T-duality [19]:

$$K(D_p; B_1, B_2) = \frac{1}{6}H_{1ijk,a}H_2^{ijk,a} + \frac{1}{3}H_{1abc,i}H_2^{abc,i} - \frac{1}{2}H_{1bci,a}H_2^{bci,a} \tag{1.5}$$

¹Our index convention is that the Greek letters (μ, ν, \dots) are the indices of the space-time coordinates, the Latin letters (a, d, c, \dots) are the world-volume indices and the letters (i, j, k, \dots) are the normal bundle indices.

where B_1, B_2 are the polarization tensors of the antisymmetric B-field and H_1, H_2 are their field strengths. Using the above transformations, one observes that the T-duality transformation of the disk-level 2-point function of gravitons is given by the amplitude (1.1) in which the kinematic factor is given by the above expression. It has been shown in [19] that this result is in fact the disk-level 2-point function of B-field. So the disk-level 2-point function of graviton and the the disk-level 2-point function of B-field come together as a T-dual multiplet.

Another example that shows the S-matrix elements are invariant under T-duality is the disk-level S-matrix element of one RR and one NSNS vertex operators which is given by (1.1) with appropriate kinematic factor [16]. This factor has been studied in details in [20, 31] for various RR forms and NSNS states. It is given by

$$\begin{aligned}
 K(D_p; \varepsilon_1, \varepsilon_2) = \epsilon^{a_0 \dots a_p} & \left(\frac{1}{2!(p-1)!} [F_1^{(p)}{}_{ia_2 \dots a_p, a} H_{2a_0 a_1}{}^{a, i} - F_1^{(p)}{}_{aa_2 \dots a_p, i} H_{2a_0 a_1}{}^{i, a}] \right. \\
 & + \frac{2}{p!} \left[\frac{1}{2!} F_1^{(p+2)}{}_{ia_1 \dots a_p j, a} (R_2{}^a{}_{a_0}{}^{ij} - \frac{1}{p+1} F_1^{(p+2)}{}_{a_0 \dots a_p j, i} (\hat{R}_2{}^{ij} - \phi_2{}^{, ij})) \right] \\
 & \left. - \frac{1}{3!(p+1)!} F_1^{(p+4)}{}_{ia_0 \dots a_p j k, a} H_2{}^{ijk, a} \right) \tag{1.6}
 \end{aligned}$$

where $F_1^{(n)}$ is the field strength polarization tensor of the RR potential, and ϕ_2 is the polarization of the dilaton which is one. The sum of the second term in the first line and the last two terms in the second line form a T-dual multiplet, and the remaining terms form another T-dual multiplet. Hence, the 2-point functions are invariant under T-duality.

On the other hand, the S-duality holds order by order in α' and is nonperturbative in the string loop expansion [5]. This indicates that a given S-matrix element at tree-level is *not* invariant/covariant under the nonlinear S-duality transformations on the background fields. Hence, one must include the loops and the nonperturbative effects [21] to make it invariant. This is unlike the T-duality, in which the tree-level is invariant under the nonlinear T-duality on the background fields, e.g., equation (1.4). For concreteness, let us consider the sphere-level 4-point function of gravitons in type IIB string theory [22, 23]. The background flat metric in the Mandelstam variables is the string frame metric, hence, the Mandelstam variables are not invariant under the S-duality. In this case, one has to α' -expand the amplitude to be able to discuss the S-duality of the background fields. At the leading order in α' , the S-matrix element is invariant under the S-duality. At the next leading order in α' , the amplitude in the Einstein frame are proportional to $e^{-3\phi_0/2}$ where ϕ_0 is the dilaton background. This factor is not invariant under the nonlinear S-duality. It has been shown in [21] that the four-graviton couplings at order α'^4 , which produce R^4 couplings in the spacetime [22–25], become invariant under the S-duality when one includes the one-loop and the D-instanton effects. The amplitude at the higher orders of α' , which include the other dilaton factors, can be extended to the S-duality invariant form by adding the higher loops and the nonperturbative effects [26–30].

In some cases, the consistency of the quantum states of a given tree-level S-matrix element with the linear S-duality can be used to find a family of tree-level S-matrix elements. We call such S-matrix elements the S-dual S-matrix multiplet. For instance, consider the

sphere-level 4-point function of B-fields, which may be found by requiring the consistency of the four-graviton amplitude [22, 23] with the linear T-duality. This amplitude can not be extended to the S-dual form by adding only loops and the nonperturbative effects. In this case one should include the sphere-level 4-point function of the RR two-form as well. In this sense, one may find new tree-level S-matrix elements by using the compatibility of a given S-matrix element with the S-duality.

In this paper, we would like to examine the compatibility of disk-level S-matrix elements with the S-duality. In the case that a given S-matrix element represents the scattering from D₃-brane, its consistency with the S-duality may be used to find its S-dual S-matrix multiplet. All the elements are the disk-level amplitudes on the world volume of D₃-brane. In the case that a given disk-level S-matrix element represents the scattering from D₁-string or D₅-branes, its consistency with the S-duality can be used to fix the form of the S-matrix elements of F₁-string or NS₅-brane at strong coupling.

The outline of the paper is as follows: We begin in section 2 by studying the S-duality transformations of the disk-level 2-point functions representing the scattering from D₃-brane. We show how compatibility of the 2-point function of the B-field with S-duality predicts the form of some other disk-level amplitudes. In section 3, we show that the compatibility of the disk-level S-matrix elements of D₁-brane or D₅-brane with the S-duality generates the S-matrix elements of F₁-string or NS₅-brane at strong coupling. The new S-matrix elements imply that both F₁-string and NS₅-brane have open D₁-string excitation, as expected. In section 4, we argue that the contact terms of the new S-matrix elements at order $O(\alpha'^0)$ produce a Born-Infeld and Chern-Simons type effective action for both F₁-string and NS₅-brane which are consistent with the SL(2, ℝ)-covariant action proposed in [43].

2 S-duality of D₃-brane amplitudes

It is known that the supergravity effective action of type IIB is invariant under the S-duality [5]. The RR four-form is invariant under this duality. On the other hand, the D₃-brane couples linearly to the RR four-form, so the D₃-brane is also invariant under S-duality. We expect then the S-matrix elements on the world volume of D₃-brane to be invariant under the S-duality. Let us begin with the following disk-level one-point function in the string frame:

$$A(D_p; C_1^{(p+1)}) \sim T_p \epsilon^{a_0 \dots a_p} C_{1a_0 \dots a_p}^{(p+1)} \delta^{p+1}(p_1^a) \tag{2.1}$$

where T_p is the tension of D_p-brane in the string frame (1.2). We have normalized the amplitude by T_p in order to make it invariant under the T-duality. It is easy to verify that T-duality along the brane transforms it to $T_{p-1} \epsilon^{a_0 \dots a_{p-1}} C_{1a_0 \dots a_{p-1}}^{(p)} \delta^p(p_1^a)$. With this normalization, it is obvious that the amplitude (2.1) for D₃-brane case is invariant under the S-duality.

Next, consider the massless NSNS one-point function which is

$$A(D_p; \varepsilon_1) \sim T_p (\varepsilon_{1a}{}^a - \varepsilon_{1i}{}^i) \delta^{p+1}(p_1^a) \tag{2.2}$$

The standard S-matrix calculation for NSNS states gives the amplitude in the Einstein frame, so the above amplitude is in the Einstein frame. We normalized the amplitude by T_p to make it invariant under T-duality. To verify that the amplitude is invariant under the T-duality, one must consider the combination of the graviton and the dilaton 1-point functions because T-duality maps dilaton to graviton. The dilaton amplitude can be read from (2.2) by using the polarization tensor $\varepsilon_{\mu\nu} = (\eta_{\mu\nu} - p_\mu \ell_\nu - p_\nu \ell_\mu)/2$ where the auxiliary vector satisfies $\ell \cdot p = 1$. The sum of the graviton and the dilaton amplitudes in the Einstein frame becomes

$$A(D_p; 1) \sim T_p(2h_{1a}{}^a + (p-3)\phi_1)\delta^{p+1}(p_1^a) \quad (2.3)$$

where we have also used the traceless of the graviton polarization tensor. The expression inside the parentheses becomes $2(h_{1a}{}^a - 2\phi_1)$ in the string frame which is invariant under the linear T-duality [19]. For the case of D₃-brane, the last term in (2.3) is zero and the first term is invariant under the S-duality, since the Einstein frame metric is invariant.

Now consider the disk-level two-point function of one $C^{(4)}$ and one NSNS state. This amplitude in the string frame is given by (1.1) in which the appropriate kinematic factor is given in (1.6). In the Einstein frame $g_{\mu\nu}$, which is related to the string frame metric $G_{\mu\nu}$ as $G_{\mu\nu} = e^{(\phi_0+\phi)/2}g_{\mu\nu}$ where ϕ_0 is the constant dilaton background and ϕ is its quantum fluctuation, the amplitude becomes

$$A(D_3; \varepsilon_1, \varepsilon_2) \sim T_{D_3}\alpha'^2 K(D_3; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-te^{-\phi_0/2}/4)\Gamma(-se^{-\phi_0/2})}{\Gamma(1-te^{-\phi_0/2}/4-se^{-\phi_0/2})} \delta^4(p_1^a + p_2^a) \quad (2.4)$$

where the kinematic factor is

$$K(D_3; C_1^{(4)}, h_2) = \epsilon^{a_0 \dots a_3} e^{-\phi_0} \left[\frac{1}{2!3!} F_{1ia_1 \dots a_3j,a}^{(5)} R_2^a{}_{a_0}{}^{ij} - \frac{1}{4!} F_{1a_0 \dots a_3j,i}^{(5)} \hat{R}_2^{ij} \right] \quad (2.5)$$

The quantum states $C_1^{(4)}$ and h_2 are invariant under the S-duality, however, the S-matrix element is not invariant under the nonlinear S-duality on the dilaton background. To study the S-duality of the background, we have to α' -expand the Gamma functions. This expansion is

$$\frac{\Gamma(-te^{-\phi_0/2}/4)\Gamma(-se^{-\phi_0/2})}{\Gamma(1-te^{-\phi_0/2}/4-se^{-\phi_0/2})} = \frac{4e^{\phi_0}}{st} - \frac{\pi^2}{24} + O(\alpha'^2 e^{-\phi_0}) \quad (2.6)$$

One can easily observe that the leading term of the amplitude which is α'^0 order is invariant under the S-duality. The α'^2 order terms has the dilaton factor $e^{-\phi_0}$ which is not invariant under the nonlinear S-duality. The higher order of α' has other dilaton factors. None of them are invariant under the S-duality.

The dilaton and axion transform similarly under the S-duality, hence, one expects each of the dilaton factors in the above amplitude to be extended to a function of both dilaton and axion to be invariant under the S-duality. In this way, one can find the exact dependence of the amplitude on the background dilaton and axion. Note that the 2-point function (2.4) has no axion background. It has been shown in [18, 31] that by adding the

one-loop and the D-instanton effects to the α'^2 -order terms, which can be done by replacing $e^{-\phi_0}$ with the regularized non-holomorphic Eisenstein series $E_1(\phi_0, C_0)$, one extends the α'^2 -order terms to the S-dual invariant form [18, 31]. The higher order of α' terms require other Eisenstein series to make them S-duality invariant [32].

Under the S-duality, $C^{(2)} \rightarrow B$ and $B \rightarrow -C^{(2)}$ [5]. The invariance of the S-matrix elements under the S-duality then indicates that the one-point function of the RR two form and the one-point function of B-field must be zero, as they are. So we consider the two-point function of these states which is given, in the string frame, by (1.1) and by the appropriate kinematic factor in (1.6). In the Einstein frame, the amplitude is the same as (2.4) in which the kinematic factor is:

$$K(D_3; C_1^{(2)}, B_2) = \epsilon^{a_0 \dots a_3} e^{-\phi_0} \left[F_1^{(3)}{}_{a_2 a_3 i, a} H_{2 a_0 a_1}{}^{a, i} - F_1^{(3)}{}_{a_2 a_3 a, i} H_{2 a_0 a_1}{}^{i, a} \right] \quad (2.7)$$

The expression inside the bracket can be written in the following $SL(2, \mathbb{R})$ invariant form [31]:

$$K(D_3; C_1^{(2)}, B_2) = \epsilon^{a_0 \dots a_3} e^{-\phi_0} \mathcal{F}^T{}_{a_0 a_1 a, i} \mathcal{N} \mathcal{F}_{a_2 a_3}{}^{i, a} \quad (2.8)$$

where the $SL(2, \mathbb{R})$ matrix \mathcal{N} is

$$\mathcal{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.9)$$

and $\mathcal{F} = dB$ where \mathcal{B} is

$$\mathcal{B} = \begin{pmatrix} B_2 \\ C_1^{(2)} \end{pmatrix} \quad (2.10)$$

Hence, the S-matrix element is invariant under the S-duality on the quantum states, whereas it is not invariant under the nonlinear S-duality on the background. Using the expansion (2.6), one finds again that the amplitude at order α'^0 is invariant under the S-duality. The dilaton factor in the α'^2 -order terms should again be replaced by $E_1(\phi_0, C_0)$ to make it invariant. The dilaton factors in all higher order can be extended to S-dual forms by adding appropriate loops and D-instanton effects.

Consider the following standard coupling on the world volume of D₃-brane at order $O(\alpha'^0)$:

$$T_{D3} \int C^{(2)} \wedge B \quad (2.11)$$

Using the S-duality transformation $C^{(2)} \rightarrow B$ and $B \rightarrow -C^{(2)}$, one finds that it is not invariant under the S-duality.² However, there are other terms in the corresponding S-matrix element at this order. They are the massless open and closed string poles resulting

²One may use the $SL(2, \mathbb{R})$ -doublets $\tilde{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and rotate them at the same time that rotate the doublet $\begin{pmatrix} B \\ C^{(2)} \end{pmatrix}$, to write this coupling in an $SL(2, \mathbb{R})$ -covariant family of couplings [43].

from the supergravity couplings and the D₃-brane couplings $B.f$ and $C^{(2)} \wedge f$ where f is the world volume gauge field. The combination of all these terms which are the α^0 -order terms of the disk-level 2-point function (2.7), are invariant under the S-duality.

So far we have discussed only the cases where the S-duality requires one to include the loops and the D-instanton effects to make them invariant under the S-duality transformation on the background. They make the tree-level S-matrix elements to be S-duality invariant. We now discuss the cases where the S-matrix elements are not invariant under the linear S-duality transformation on the quantum states. Hence, the compatibility of disk-level S-matrix elements with the S-duality requires one to include some other S-matrix elements at the tree-level. Let us consider, for example, the disk-level 2-point function of B-field on the world volume of D₃-brane. This amplitude in the Einstein frame is given by (2.4) in which the kinematic factor is the transformation of (1.5) to the Einstein frame,

$$K(D_3; B_1, B_2) = e^{-\phi_0} \left[e^{-\phi_0} \left(\frac{1}{6} H_{1ijk,a} H_2^{ijk,a} + \frac{1}{3} H_{1abc,i} H_2^{abc,i} - \frac{1}{2} H_{1bci,a} H_2^{bci,a} \right) \right] \quad (2.12)$$

Since $B \rightarrow -C^{(2)}$ under the S-duality, it is obvious that this S-matrix element is neither invariant under the linear S-duality on the quantum states B_1, B_2 , nor under nonlinear S-duality on the background. To make it invariant under the linear S-duality of the quantum states, one needs similar 2-point function of the RR two-form.

The disk-level 2-point function of the RR two-form in the string frame is given by (1.1). The kinematic factor for the case of D₃-brane is [31]

$$K(D_3; C_1^{(2)}, C_2^{(2)}) = e^{2\phi_0} \left[\left(\frac{1}{6} F_{1ijk,a}^{(3)} F_2^{(3)ijk,a} + \frac{1}{3} F_{1abc,i}^{(3)} F_2^{(3)abc,i} - \frac{1}{2} F_{1bci,a}^{(3)} F_2^{(3)bci,a} \right) \right] \quad (2.13)$$

We have normalized the amplitude by $T_3 e^{2\phi_0}$ to make it consistent with the T-duality. To clarify this point, consider implementing T-duality on the world volume of the brane. Under T-duality, the first term, for example, transforms to $F_{1ijk_y,a}^{(4)} F_2^{(4)ijk_y,a}$ where y is the Killing index. We need the flat metric η^{yy} to contract the y indices. This arises from the nonlinear T-duality on the background dilaton factor which transforms as $e^{2\phi_0} \rightarrow e^{2\phi_0} \eta^{yy}$.

In the Einstein frame, the amplitude is given by (2.4) and the following kinematic factor

$$K(D_3; C_1^{(2)}, C_2^{(2)}) = e^{-\phi_0} \left[e^{\phi_0} \left(\frac{1}{6} F_{1ijk,a}^{(3)} F_2^{(3)ijk,a} + \frac{1}{3} F_{1abc,i}^{(3)} F_2^{(3)abc,i} - \frac{1}{2} F_{1bci,a}^{(3)} F_2^{(3)bci,a} \right) \right] \quad (2.14)$$

This kinematic factor is similar to the kinematic factor of two B-fields (2.12), as predicted by the S-duality.

The S-duality predicts even the disk-level S-matrix elements in the presence of constant axion background. Since the RR two-form and the B-field appear as doublet under the S-duality transformation, the following combination is invariant under the S-duality [5]:

$$(B_1, C_1^{(2)}) \mathcal{M} \begin{pmatrix} B_2 \\ C_2^{(2)} \end{pmatrix} = e^{-\phi_0} B_1 B_2 + e^{\phi_0} C_1^{(2)} C_2^{(2)} - e^{\phi_0} C_0 (B_1 C_2^{(2)} + B_2 C_1^{(2)}) + e^{\phi_0} C_0^2 B_1 B_2$$

where $B_1, C_1^{(2)}$ and $B_2, C_2^{(2)}$ are the polarizations of the external states and the matrix \mathcal{M} is the following function of the background dilaton and axion:

$$\mathcal{M} = e^{\phi_0} \begin{pmatrix} e^{-2\phi_0} + C_0^2 & -C_0 \\ -C_0 & 1 \end{pmatrix} \quad (2.15)$$

The consistency of the disk-level 2-point function of B-fields in the zero axion background, i.e., equations (2.4) and (2.12), with the S-duality then predicts the disk-level 2-point function of the RR two-form in the zero axion background, i.e., $e^{\phi_0} C^{(2)} C^{(2)}$, which is given by (2.4) and (2.14), and the disk-level 2-point functions $e^{\phi_0} C_0 B_1 C_2^{(2)}$ and $e^{\phi_0} C_0^2 B_1 B_2$ in the presence of non-zero axion background. They all combine into a 2-point function given by (2.4) in which the kinematic factor is

$$K(D_3; \varepsilon_1, \varepsilon_2) = e^{-\phi_0} \left[\frac{1}{6} \mathcal{F}_1^T{}_{ijk,a} \mathcal{M} \mathcal{F}_2^{ijk,a} + \frac{1}{3} \mathcal{F}_1^T{}_{abc,i} \mathcal{M} \mathcal{F}_2^{abc,i} - \frac{1}{2} \mathcal{F}_1^T{}_{bci,a} \mathcal{M} \mathcal{F}_2^{bci,a} \right] \quad (2.16)$$

where $\mathcal{F}_1 = d\mathcal{B}_1$ and \mathcal{B}_1 is

$$\mathcal{B}_1 = \begin{pmatrix} B_1 \\ C_1^{(2)} \end{pmatrix}$$

Similarly for \mathcal{F}_2 . Including the appropriate loops and nonperturbative effects to the above disk-level S-matrix multiplet, one can make it invariant under the nonlinear S-duality on the background.

The disk-level 2-point functions $e^{\phi_0} C_0 B_1^{(2)} C_2^{(2)}$ and $e^{\phi_0} C_0^2 B_1^{(2)} B_2^{(2)}$ in the presence of non-zero axion background can be calculated with the zero axion 3-point function $e^{\phi_0} C_3 B_1^{(2)} C_2^{(2)}$ and the 4-point function $e^{\phi_0} C_4 C_3 B_1^{(2)} B_2^{(2)}$, respectively, in which the axion field in the RR scalar vertex operator is a constant. Note that, in general, the disk-level 3-point function and the 4-point function are much more complicated than the 2-point function. However, when RR scalar is constant they should be reduced to (2.4) with the kinematic factor (2.16). It would be interesting to perform these calculations.

The above discussions can be applied for any other disk-level S-matrix element of D₃-brane to find its S-dual S-matrix multiplet. One may also extend the above discussions to the sphere-level S-matrix elements because the vacuum corresponding to the sphere-level is invariant under the S-duality. In the next section we turn to the cases in which the vacuum is not invariant under the S-duality.

3 S-matrix for F₁-string and NS₅-brane

We have seen in the previous section how the S-duality invariance of the zero-axion S-matrix elements on the world volume of D₃-brane may fix the appearance of the axion background in the S-matrix elements. The invariance of the S-matrix elements is related to the fact that the D₃-brane is invariant under the S-duality. The D₁-brane and D₅-brane are not invariant under the S-duality, hence, one does not expect the S-matrix elements on the world volume of these branes to be invariant under the S-duality. The S-duality

transformation that maps $C^{(2)} \rightarrow B$, transforms D₁-brane to F-string and D₅-brane to NS₅-brane. Hence, in these cases we expect the S-matrix elements to be $SL(2, \mathbb{Z})$ -covariant. Using this proposal, in general, one may find the S-matrix elements on the world volume of (p, q) -strings and (p, q) -5-branes by applying S-duality transformation on the S-matrix element of D₁-brane and D₅-brane, respectively. This is possible only if one knows the form of the latter S-matrix elements in the presence of axion background. To clarify this, consider the transformation of dilaton-axion field, i.e., $\tau = C + ie^{-\phi}$, under the S-duality

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d} ; ad - bc = 1 \tag{3.1}$$

If the axion is zero on the left-hand side, then it simplifies to

$$ie^{-\phi} \longrightarrow \tau' = \frac{ie^{-\phi} + (bd + ace^{-2\phi})}{c^2 e^{-2\phi} + d^2} \tag{3.2}$$

which indicates that axion is not zero after duality transformation. So if one begins with the zero-axion S-matrix elements of D₁-brane or D₅-brane, and applies the above $SL(2, \mathbb{Z})$ transformation, then the axion in the transformed S-matrix elements is not zero. On the other hand, since we do not include the axion resulting from the $SL(2, \mathbb{Z})$ transformation of axion, the axion in dual S-matrix does not appear correctly.

To avoid this difficulty, we use the particular $SL(2, \mathbb{Z})$ transformation which maps D₁-brane to F-string and D₅-brane to NS₅-brane. Under this transformation,

$$\tau \xrightarrow{s} \tau' = -\frac{1}{\tau} = \frac{ie^{-\phi} - C}{C^2 + e^{-2\phi}} \tag{3.3}$$

which indicates if the axion is zero on the left-hand side, it remains zero after duality transformation. So in this section we show how the zero-axion S-matrix elements of F-string and NS₅-brane can be read from the zero-axion S-matrix elements of D₁-brane and D₅-brane, respectively.

Let us begin with 1-point functions. Since D₁-string and F₁-string couples linearly to the RR two-form and the B-field, respectively, one finds the following transformation on the 1-point function:

$$T_{D1} \epsilon^{a_0 a_1} C_{a_0 a_1}^{(2)} \delta^2(p_1^a) \xrightarrow{s} T_{F1} \epsilon^{a_0 a_1} B_{a_0 a_1} \delta^2(p_1^a) \tag{3.4}$$

where the Einstein frame tensions are $T_{D1} = 1/(2\pi\alpha'\sqrt{g_s})$ and $T_{F1} = \sqrt{g_s}/(2\pi\alpha')$. While the first coupling can be confirmed by the disk-level 1-point function in which the RR vertex operator is in $(-1/2, -3/2)$ -picture, the second coupling which is a standard coupling, has no such description. There is also the following transformation on the Einstein frame 1-point function of graviton and dilaton (2.3):

$$T_{D1} (h_{1a}{}^a - \phi_1) \delta^2(p_1^a) \xrightarrow{s} T_{F1} (h_{1a}{}^a + \phi_1) \delta^2(p_1^a) \tag{3.5}$$

The graviton couplings on both sides are the standard couplings in the Nambu-Goto action.

We now consider the disk-level two-point function of one $C^{(2)}$ and one NSNS state on the world volume of D₁-brane. This amplitude in the string frame is given by (1.1)

in which the appropriate kinematic factor is given in (1.6). In the Einstein frame the amplitude becomes

$$A(D_1; \varepsilon_1, \varepsilon_2) \sim T_{D_1} \alpha'^2 K(D_1; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-te^{-\phi_0/2}/4)\Gamma(-se^{-\phi_0/2})}{\Gamma(1-te^{-\phi_0/2}/4-se^{-\phi_0/2})} \delta^2(p_1^a + p_2^a) \quad (3.6)$$

where the kinematic factor is

$$K(D_1; C_1^{(2)}, h_2) = \epsilon^{a_0 a_1} e^{-\phi_0} \left[F_{1i a_1 j, a}^{(3)} R_2^a{}_{a_0}{}^{ij} - F_{1a_0 a_1 j, i}^{(3)} \hat{R}_2^{ij} \right] \quad (3.7)$$

The amplitude (3.6) is mapped under the S-duality to the following 2-point function on the world-volume of the F_1 -string:

$$A(F_1; \varepsilon_1, \varepsilon_2) \sim T_{F_1} \alpha'^2 K(F_1; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-te^{\phi_0/2}/4)\Gamma(-se^{\phi_0/2})}{\Gamma(1-te^{\phi_0/2}/4-se^{\phi_0/2})} \delta^2(p_1^a + p_2^a) \quad (3.8)$$

where the kinematic factor is

$$K(F_1; B_1, h_2) = \epsilon^{a_0 a_1} e^{\phi_0} \left[H_{1i a_1 j, a} R_2^a{}_{a_0}{}^{ij} - H_{1a_0 a_1 j, i} \hat{R}_2^{ij} \right] \quad (3.9)$$

The above transformation is the extension of the transformations (3.4) and (3.5) to 2-point function.

In general, we expect all 2-point function on the world volume of F_1 -string at strong coupling to be given by (3.8) in which the kinematic factor is related to the kinematic factor of D_1 -brane by the S-duality transformation, i.e.,

$$K(D_1; \varepsilon_1, \varepsilon_2) \xrightarrow{s} K(F_1; \varepsilon_1, \varepsilon_2) \quad (3.10)$$

Similarly, one can find all n-point functions on the world-volume of F_1 -string.

As in the case of T-duality, the new S-matrix elements can be found by using nonlinear S-duality on the background fields and the linear S-duality on the quantum fluctuations. In fact, the axion and the dilaton are the only fields which transform nonlinearly under the S-duality. The background axion is zero in our discussion. The linear S-duality transformation of the quantum state of the axion can be found from (3.3) which is

$$C \xrightarrow{s} -e^{2\phi_0} C \quad (3.11)$$

where $e^{2\phi_0}$ is the background dilaton factor. Therefore, the axion state in the disk-level n-point function of D_1 -string is mapped to $-e^{2\phi_0} C$ in the tree-level n-point function of F_1 -string.

In the string frame, the amplitude (3.8) becomes

$$A(F_1; \varepsilon_1, \varepsilon_2) \sim T_{F_1} \alpha'^2 K(F_1; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-te^{\phi_0}/4)\Gamma(-se^{\phi_0})}{\Gamma(1-te^{\phi_0}/4-se^{\phi_0})} \delta^2(p_1^a + p_2^a) \quad (3.12)$$

where the string tension is $T_{F_1} = 1/(2\pi\alpha')$. The gamma functions represent the s - and t -channels. The poles in the t -channel are at $\frac{g_s t}{2} = 0, 2, 4, \dots$ and the poles in the s -channel are at $\frac{g_s s}{2} = 0, \frac{1}{2}, 1, \dots$. These two channels are similar to the s - and t -channels

of the disk-level scattering from D-branes [16]. In that case the poles in the t -channel are at $\frac{t}{2} = 0, 2, 4, \dots$, and the poles in the s -channel are at $\frac{s}{2} = 0, \frac{1}{2}, 1, \dots$. In terms of the tension of the intermediate string, the t -channel poles in the D-brane amplitude are at $-(p_1 + p_2)^2/(2\pi T_{F1}) = 0, 2, 4, \dots$, whereas the t -channel poles in the F_1 -string amplitude are at $-(p_1 + p_2)^2/(2\pi T_{D1}) = 0, 2, 4, \dots$. Hence, the extra factor of g_s in the F_1 -string amplitude (3.12) dictates that the intermediate string is D_1 -string, as expected.

The magnetic dual of the transformation (3.4) is

$$T_{D5}\epsilon^{a_0\dots a_5}C_{a_0\dots a_5}^{(6)}\delta^6(p_1^a) \xrightarrow{s} T_{NS5}\epsilon^{a_0\dots a_5}B_{a_0\dots a_5}\delta^6(p_1^a) \quad (3.13)$$

The Einstein frame tensions are $T_{D5} = \sqrt{g_s}/[4\pi^2(2\pi\alpha')^3]$ and $T_{NS5} = 1/[4\pi^2(2\pi\alpha')^3\sqrt{g_s}]$. Repeating the same steps as we have done for (3.4), one finds the following 2-point function for the NS_5 -brane at strong coupling in the Einstein frame:

$$A(NS_5; \varepsilon_1, \varepsilon_2) \sim T_{NS5}\alpha'^2 K(NS_5; \varepsilon_1, \varepsilon_2) \frac{\Gamma(-te^{\phi_0/2}/4)\Gamma(-se^{\phi_0/2})}{\Gamma(1-te^{\phi_0/2}/4-se^{\phi_0/2})} \delta^6(p_1^a + p_2^a) \quad (3.14)$$

where the kinematic factor $K(NS_5; \varepsilon_1, \varepsilon_2)$ is related to the kinematic factor of D_5 -brane by the S-duality transformation, i.e.,

$$K(D_5; \varepsilon_1, \varepsilon_2) \xrightarrow{s} K(NS_5; \varepsilon_1, \varepsilon_2) \quad (3.15)$$

where $K(D_5; \varepsilon_1, \varepsilon_2)$ is the kinematic factor of the D_5 -brane in the Einstein frame. We expect in a similar way all other S-matrix elements can be found.

We have seen that the S-matrix elements are invariant/covariant under the global S- and T-dualities. String theory is also invariant under the global supersymmetry. Hence, one expects the S-matrix elements to be invariant under the supersymmetry as well. In this case we call the S-matrix elements which are interconnected by the supersymmetry, a supersymmetric S-matrix multiplet. When the supersymmetry transformations [33] are used to transform all the bosonic and the fermionic components of the multiplet, the supersymmetry parameter ϵ must be canceled. In other words, the supersymmetric S-matrix multiplet should satisfy the Ward identity associated with the global supersymmetry transformations. It has been shown in [16] the disk-level 2-point functions satisfy the Ward identities corresponding to all the gauge symmetries. It would be interesting to show that they satisfy the Ward identity corresponding to the global supersymmetry as well.

The F_1 -string/ NS_5 -brane S-matrix elements that we have found are the S-dual of the disk-level D_1 -string/ D_5 -brane S-matrix elements, hence, they are valid at strong coupling. Since the S-matrix elements are invariant under the supersymmetry, one expects the above F_1 -string/ NS_5 -brane S-matrix elements to be valid for any coupling. However, the loops and the non-perturbative effects in these S-matrix elements which are the S-dual of the corresponding effects in D_1 -string/ D_5 -brane S-matrix elements, have non-zero contributions at arbitrary coupling. We have seen in the previous section that these effects have no contribution in $O(\alpha'^0)$ -order terms. Therefore, the $O(\alpha'^0)$ -order terms of the above S-matrix elements are valid at any arbitrary coupling.

3.1 Massless poles

The above amplitudes for F1-string and NS₅-brane indicate that there are massless poles in both open and closed D₁-string channels. Let us examine these poles. The scattering amplitude of one RR scalar (the axion quantum fluctuation) and one RR two-form on the world-volume of F₁-string is given by (3.8). At the leading order in α' it is

$$A(F_1; C_1, C_2^{(2)}) \sim T_{F1} \alpha'^2 e^{3\phi_0} \epsilon^{a_0 a_1} \left[F_{1i,a}^{(1)} F_{2a_0 a_1}^{(3) a,i} - F_{1a,i}^{(1)} F_{2a_0 a_1}^{(3) i,a} \right] \left(-\frac{1}{e^{\phi_0} s t} + \dots \right) \quad (3.16)$$

where the metric is the Einstein metric. The corresponding amplitude for D₁-string is

$$A(D_1; C_1, B_2) \sim T_{D1} \alpha'^2 e^{-\phi_0} \epsilon^{a_0 a_1} \left[F_{1i,a}^{(1)} H_{2a_0 a_1}^{a,i} - F_{1a,i}^{(1)} H_{2a_0 a_1}^{i,a} \right] \left(-\frac{1}{e^{-\phi_0} s t} + \dots \right)$$

One can easily observe that the F₁-string amplitude is the transformation of the D₁-string amplitude under the S-duality.

Now we are going to reproduce the above amplitudes in effective field theory. Using the following standard coupling in the type IIB supergravity in the Einstein frame [5]:

$$\int d^{10}x \sqrt{-g} \mathcal{H}_{\mu\nu\rho}^T \mathcal{M} \mathcal{H}^{\mu\nu\rho} \quad (3.17)$$

where the matrix \mathcal{M} is given in (2.15) and

$$\mathcal{H}_{\mu\nu\rho} = \begin{pmatrix} H_{\mu\nu\rho} \\ F_{\mu\nu\rho}^{(3)} \end{pmatrix}, \quad (3.18)$$

and the standard linear coupling of the B-field to F₁-string (3.4), one can calculate the massless closed D-string pole in the scattering

$$C + \text{F}_1\text{-string} \longrightarrow C^{(2)} + \text{F}_1\text{-string} \quad (3.19)$$

The Feynman amplitude becomes

$$\mathcal{A}_t(F_1) \sim T_{F1} \frac{F_1^{(1)\mu} F_{2\mu ab}^{(3)} \epsilon^{ab}}{t} e^{2\phi_0} \quad (3.20)$$

On the other hand, the supergravity coupling (3.17) and the linear coupling of the RR two-form to D₁-string (3.4) can be used to calculate the massless closed string pole in the following scattering:

$$C + \text{D}_1\text{-string} \longrightarrow B^{(2)} + \text{D}_1\text{-string} \quad (3.21)$$

The Feynman amplitude in this case becomes

$$\mathcal{A}_t(D_1) \sim T_{D1} \frac{F_1^{(1)\mu} H_{2\mu ab} \epsilon^{ab}}{t} \quad (3.22)$$

Comparing this amplitude with (3.20) and using the linear transformation of the axion (3.11), one finds that the massless closed string poles are related to each other by the S-duality, which is consistent with our proposal for the string amplitude (3.16).

The massless open string pole in the scattering (3.21) can be calculated by using the standard brane couplings in the Einstein frame: $T_{D1}B_{ab}f^{ab}e^{\phi_0/2}$, $T_{D1}\epsilon^{ab}f_{ab}C$ and $T_{D1}f_{ab}f^{ab}e^{\phi_0/2}$. The Feynman amplitude becomes

$$\mathcal{A}_s(D_1) \sim T_{D1} \frac{\epsilon^{ab}F_{1a}^{(1)}B_{2bc}{}^c}{s} \tag{3.23}$$

On the other hand, the massless open D-string pole in (3.19) can be found by assuming the brane couplings $T_{F1}C_{ab}\hat{f}^{ab}e^{-\phi_0/2}$, $T_{F1}\epsilon^{ab}\hat{f}_{ab}Ce^{2\phi_0}$ and $T_{F1}\hat{f}_{ab}\hat{f}^{ab}e^{-\phi_0/2}$ in the Einstein frame where \hat{f}_{ab} is the S-dual of the gauge field f_{ab} . The Feynman amplitude becomes

$$\mathcal{A}_s(F_1) \sim T_{F1} \frac{\epsilon^{ab}F_{1a}^{(1)}C_{2bc}{}^c}{s} e^{2\phi_0} \tag{3.24}$$

Comparing (3.23) with (3.24), one again finds that the massless open string poles are related to each other by S-duality, which is consistent with the proposal for the string amplitude (3.16).

One can find the contact terms at order $O(\alpha'^0)$ by subtracting the above massless poles from the α' -expansion of the tree-level 2-point function. In the case of D₁-string, one finds

$$A(D_1; C_1, B_2) - \mathcal{A}_t(D_1) - \mathcal{A}_s(D_1) \sim T_{D1}C_1B_{2ab}\epsilon^{ab} + O(\alpha'^2) \tag{3.25}$$

which is a standard term in the Chern-Simons part of the D₁-string action. Similar calculation for F₁-string gives

$$A(F_1; C_1, C_2^{(2)}) - \mathcal{A}_t(F_1) - \mathcal{A}_s(F_1) \sim T_{F1}C_1C_{2ab}\epsilon^{ab}e^{2\phi_0} + O(\alpha'^2) \tag{3.26}$$

This is a coupling in the world volume of F₁-string. One may extend the above calculations to the other scattering amplitudes and find other couplings in the effective action of F₁-string/NS₅-brane. In the next section we discuss these couplings.

4 F₁-string and NS₅-brane effective actions

The dynamics of the D-branes of type II superstring theories is well-approximated by the effective world-volume field theory which consists of the Dirac-Born-Infeld (DBI) and the Chern-Simons (CS) actions. The DBI action describes the dynamics of the brane in the presence of NS-NS background fields, which can be found by requiring its consistency with the nonlinear T-duality [34, 35]. On the other hand, the CS part describes the coupling of D-branes to the R-R potential [36, 37]. These actions in the string frame for D₁-brane and D₅-brane are³

$$\begin{aligned} S_{D1} &= -T_{D1} \int d^2x e^{-\phi} \sqrt{-\det(g_{ab} + B_{ab})} + T_{D1} \int [C^{(2)} + CB] \\ S_{D5} &= -T_{D5} \int d^6x e^{-\phi} \sqrt{-\det(g_{ab} + B_{ab})} \\ &\quad + T_{D5} \int [C^{(6)} + C^{(4)} \wedge B + \frac{1}{2}C^{(2)} \wedge B \wedge B + \frac{1}{3!}CB \wedge B \wedge B] \end{aligned} \tag{4.1}$$

³We are using the convention in which the asymptotic value of the dilaton is zero. In this convention the D-brane tension and the D-brane charge are identical [40].

All the closed string fields in the actions are pull-back of the bulk fields onto the world-volume of branes. The abelian gauge field can be added to the actions as $B \rightarrow B + 2\pi\alpha' f$. This makes the action to be invariant under the B-field gauge transformation. These actions can be naturally extended to the nonabelian case by using the symmetric trace prescription [38, 39], and by including the Myers terms [40]. The above actions can be confirmed by the contact terms of the D-brane S-matrix elements at order $O(\alpha'^0)$.

We have seen in section 3, how to find the S-matrix elements of F₁-string and NS₅-brane at zero axion background by applying the particular S-duality transformation (3.3) on the disk-level S-matrix element of D₁-string and D₅-brane, respectively. If one knew the disk-level S-matrix elements in non-zero axion background, then the S-duality would produce the S-matrix elements of F₁-string and NS₅-brane at non-zero axion background. In that case also the contact terms of D₁-string/D₅-brane at order $O(\alpha'^0)$ would be mapped to the contact terms of F₁-string/NS₅-brane at order $O(\alpha'^0)$. On the other hand, the contact terms of D₁-string/D₅-brane produce the effective action (4.1). Hence, the contact terms of F₁-string/NS₅-brane at order $O(\alpha'^0)$ should produce effective actions which are related to (4.1) by the S-duality.

To apply the S-duality (3.3) on (4.1), it is better to first write them in the Einstein frame. There is no metric in the Chern-Sioms parts, so they remain unchanged. The DBI parts in the Einstein frame become

$$\begin{aligned}
 S_{D1}(DBI) &= -T_{D1} \int d^2x e^{-\phi/2} \sqrt{-\det(g_{ab} + e^{-\phi/2} B_{ab})} \\
 S_{D5}(DBI) &= -T_{D5} \int d^6x e^{\phi/2} \sqrt{-\det(g_{ab} + e^{-\phi/2} B_{ab})}
 \end{aligned}
 \tag{4.2}$$

The tensions in the Einstein frame are

$$T_{D1} = 1/(2\pi\alpha' \sqrt{g_s}) ; T_{D5} = \sqrt{g_s}/[4\pi^2(2\pi\alpha')^3]
 \tag{4.3}$$

Note that if we had used the conversion in which the asymptotic value of the dilaton in (4.1) were non-zero, then the string coupling g_s would not appear in the above tensions.

The particular S-duality transformation that maps D₁-string to F₁-string and D₅-brane to NS₅-brane is

$$\begin{aligned}
 C^{(2)} &\xrightarrow{s} B ; B \xrightarrow{s} -C^{(2)} \\
 C^{(6)} &\xrightarrow{s} B^{(6)} ; B^{(6)} \xrightarrow{s} -C^{(6)} \\
 e^{-\phi} &\xrightarrow{s} \frac{1}{C^2 e^\phi + e^{-\phi}} ; C \xrightarrow{s} -\frac{C}{C^2 + e^{-2\phi}} \\
 g_{\mu\nu} &\xrightarrow{s} g_{\mu\nu} ; C^{(4)} \xrightarrow{s} C^{(4)}
 \end{aligned}
 \tag{4.4}$$

Using these transformation, one finds that the actions (4.2) are mapped to the following actions:

$$\begin{aligned}
 S_{\text{F1}} &= T_{\text{F1}} \int \left[-d^2x \left(C^2 e^\phi + e^{-\phi} \right)^{-1/2} \sqrt{-\det \left(g_{ab} - \frac{C_{ab}}{(C^2 e^\phi + e^{-\phi})^{1/2}} \right)} + B + \frac{CC^{(2)}}{C^2 + e^{-2\phi}} \right] \\
 S_{\text{NS5}} &= T_{\text{NS5}} \int \left[-d^6x \left(C^2 e^\phi + e^{-\phi} \right)^{1/2} \sqrt{-\det \left(g_{ab} - \frac{C_{ab}}{(C^2 e^\phi + e^{-\phi})^{1/2}} \right)} \right. \\
 &\quad \left. + B^{(6)} - C^{(4)} \wedge C^{(2)} + \frac{1}{2} B \wedge C^{(2)} \wedge C^{(2)} + \frac{1}{3!} \frac{CC^{(2)} \wedge C^{(2)} \wedge C^{(2)}}{C^2 + e^{-2\phi}} \right]
 \end{aligned} \tag{4.5}$$

where the Einstein frame tensions are

$$T_{\text{F1}} = \frac{1}{2\pi\alpha'} \left(C_0^2 e^{\phi_0} + e^{-\phi_0} \right)^{-1/2}; \quad T_{\text{NS5}} = \frac{1}{4\pi^2(2\pi\alpha')^3} \left(C_0^2 e^{\phi_0} + e^{-\phi_0} \right)^{1/2} \tag{4.6}$$

The closed string fields in (4.5) are pull-back of the bulk fields onto the world-volume of branes. The abelian gauge field are added to the actions by the replacement $C^{(2)} \rightarrow C^{(2)} + 2\pi\alpha' \hat{f}$. Then the above actions are invariant under the RR two-form gauge transformation. The quadratic couplings at order $O(\alpha'^0)$ that considered in the previous section are consistent with the above actions. As we argued before in section 3, the $O(\alpha'^0)$ -order terms of the S-matrix elements receive no loops or nonperturbative corrections, hence, the above actions are expected to be valid for any string coupling.

The gauge field \hat{f}_{ab} and the transverse scalar fields in the definition of the pull-back operation in (4.5) which are the transformation of the corresponding fields in (4.1) under the S-duality, are the massless open D-string excitation of F_1 -string/NS₅-brane. The transformation $f \xrightarrow{S} -\hat{f}$ under the S-duality has been considered in [41] in proposing an S-dual action for superstring. The above actions can be extended to the nonabelian case by using the symmetric trace prescription, and by including the Myers terms in which $C^{(2)}, C^{(6)}$ are replaced by $B^{(2)}, B^{(6)}$.

Finally, let us compare our results with the results in [43]. An $\text{SL}(2, \mathbb{R})$ -covariant action for all D_p -branes of type IIB string theory which is based on the assumption that the Chern-Simons part is gauge invariant, has been proposed in [42, 43]. The gauge field in the action of D_1 -brane has been integrated out in [43], and hence the action for F_1 -string in [43] has no RR two-form. Doing the same thing here, one would find the same result as in [43]. The action (4.5) for NS₅-brane is consistent with the $\text{SL}(2, \mathbb{R})$ -covariant action for (p, q) -5-brane proposed in [43] (see equation (3.5) in [43] for the special charge of $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$). Note that in the convention [43], the asymptotic value of the dilaton is non-zero, hence the tension of D_5 -brane and NS₅-brane are constant that depends only on α' which have been dropped in [43].

We have seen in section 2 that the S-matrix elements on the world volume of D_3 -brane can be combined into the $\text{SL}(2, \mathbb{Z})$ -invariant multiplets. In particular, the $O(\alpha'^0)$ -order terms of the disk-level S-matrix elements can be combined into the $\text{SL}(2, \mathbb{Z})$ -invariant

multiplets without adding the loops and the non-perturbative effects. These $O(\alpha'^0)$ -order terms include massless poles and contact terms. Hence, the contact terms which produce the DBI and Chern-Simons actions of the D_3 -brane, are not invariant under the S-duality. In this respect, it is similar to the discussions in [44–46] that show even though the D_3 -brane action is not invariant under the S-duality, however, the equations of motion resulting from this action and the supergravity are invariant under the S-duality.

Alternatively, one may use the $SL(2, \mathbb{R})$ -doublets $\tilde{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and rotate them at the same time that rotate the world volume and bulk polarization tensors, to write the S-matrix elements on the world volume of D_3 -brane in an $SL(2, \mathbb{R})$ -covariant family of S-matrix elements. In this case, the intermediate string propagating on the world volume of D_3 -brane would be the (p, q) -string. This would be unlike the $SL(2, \mathbb{Z})$ -invariant case that the intermediate string is always F_1 -string. The massless poles and the contact terms of the $SL(2, \mathbb{R})$ -covariant S-matrix elements at order $O(\alpha'^0)$ should be separately covariant. The contact terms should then be reproduced by the $SL(2, \mathbb{R})$ -covariant action proposed in [43].

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