# The Capacity Region of Fading Multiple-Access Relay Channels with Common Information

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***Abstract*— In this paper, we derive a capacity region for Gaussian fading multiple access degraded relay channel with partial channel state information (CSI) at the transmitters (CSIT) and perfect CSI at the receiver (CSIR). The achievable rate region obtained by Kramer-Wijngaarden is a special case of our region.**

***Keywords-Gaussian fading channel; channel state information (CSI); discrete multiple access relay channel.***

## I. INTRODUCTION

Relaying has been proposed as a means to increase coverage area of wireless networks. Relay nodes in cooperation with the users, act as a distributed multi antenna system. Nowadays, there has been much research on a multi-user extension of the relay channel, e.g. multiple-access relay channel (MARC). In [1], MARC is introduced, where some sources communicate with one single destination with the help of a relay node. An example of such a channel model is the cooperative uplink of some mobile stations to the base station with the help of the relay in a cellular based mobile communication system. Fig. 1 shows an N-source MARC. Many recent results concerning coding strategies on the MARC can be found in [2]-[4]. An achievable rate region of the MAC with feedback was established in [5]. In [6], rate regions for multiple access relay channels with relay-source feedback is obtained. All above works are studied in non-fading environments and with sources without common information.

The rest of the paper is organized as follows: In section II, preliminaries and channel model are introduced. The main results are presented in section III, where in subsection III-A, the capacity problem of fading MARC with partial CSIT and perfect CSIR with common information is studied and in subsection III-B the capacity region of Gaussian fading MARC with common information is investigated. We prove the main results in section IV. Finally, the paper is concluded in Section V.

## II. PRELIMINARIES AND CHANNEL MODEL

In this paper, we use the following notations: random variables (r.v.) are denoted by uppercase letters and lowercase letters are used to show their realizations. The



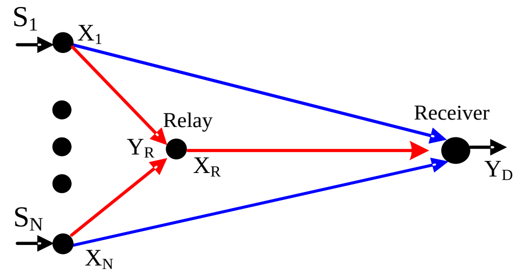


Figure 1. N-source multiple-access relay channel

probability distribution function (p.d.f) of a r.v. $X$ with alphabet set $\mathcal{X}$ is denoted by $P_X(x)$ where $x \in \mathcal{X}$ ; $P_{X|Y}(x|y)$ denote the conditional p.d.f of $X$ given $Y$. A sequence of r.v.'s $(X_{j,q}, \dots, X_{j,n})$ with the same alphabet set $\mathcal{X}$ is denoted by $X_{j,q}^n$ and its realization is denoted by $(x_{j,q}, \dots, x_{j,n}), x_{j,t} \in \mathcal{X}, t = q, \dots, n$ and $j$ is index for $j^{th}$ sender. For the case of $X_{j,1}^n$, the second subscript is dropped, occasionally. The set of all $\epsilon$-typical $n$-sequences $X^n$ with respect to the p.d.f $P_X(x)$, is denoted by $A_\epsilon^n(X)$. The notation $E(.)$ indicates the expectation operator, where sometimes to be more precise we use $E_X(.)$ to denote expectation with respect to the distribution of the r.v. $X$. we also use $h(.)$ to represent the differential entropy. The set of the real number and also the set of all nonnegative real number is denoted by $R$ and $R_+$, respectively. Finally, $C(x) \triangleq \frac{1}{2} log\ (1 + x)$.

$\mathbf{H} \in R_+^{2\times(N+1)}$ is a random matrix representing the state process of the channel (the matrix of fading coefficients) and the elements of $\mathbf{H}$ are r.v.'s which belong to $R_+$ as follows,

$$\boldsymbol{H}_t = \begin{bmatrix} h_{D1,t} & \dots\ h_{DN,t} & h_{DR,t} \\ h_{R1,t} & \dots\ h_{RN,t} & 0 \end{bmatrix}, \quad 1 \le t \le n \qquad (1)$$

and $F_i \in \theta_i (\ i = 1, \dots, N, R)$ is a r.v. representing the CSIT available at the $i^{th}$ transmitter which is a deterministic function of $H$: $F_i = \gamma_i(H)$. In addition, the function $\varphi_i: \theta_i \to R_+$ satisfying the following constraint:

$$E\{\varphi_i(F_i)\} \le P_i\,, \quad i = 1, \dots, N, R \qquad (2)$$

denotes the power allocation policy for the $i^{th}$ transmitter and relay. $\alpha_i: \theta_i \to [0,1]$ and $\beta_i: \theta_i \to [0,1]$, $i = 1, \dots, N$, are two arbitrary (limited) deterministic functions that take their values from the interval [0,1]. At the $t^{th}$ transmission

$X_{k,t}$, $k \in \{1,\dots,N\}$, and $X_{R,t}$ are sent and the channel outputs are:

$$Y_{D,t} = \sum_{k=1}^{N} h_{Dk,t} X_{k,t} + h_{DR,t} X_{R,t} + Z_{2t}, \quad 1 \le t \le n \tag{3}$$

$$Y_{R,t} = \sum_{k=1}^{N} h_{Rk,t} X_{k,t} + Z_{1t}, \quad 1 \le t \le n \tag{4}$$

where $N$ is the number of senders, $\boldsymbol{Z_i} = (Z_{i1},\dots,Z_{it},\dots,Z_{in})$, $i = 1,2$, is a sequence of independent identically distributed (i.i.d) normal random variables with zero mean and variance $N_i$, $i = 1,2$,.

## III. Main Results

***Theorem 1***: Let $[U, V_1, \dots, V_N]$ be a sequence of auxiliary r.v.'s with alphabets $U \in \mathcal{U}$ and $V_i \in \mathcal{V}_i, i = 1,\dots,N$, then the capacity region of fading multiple access relay channels with partial CSIT, perfect CSIR and with common information denoted by $\mathcal{C}_{common}^{F-MARC}$ is given by

$$\mathcal{C}_{common}^{F-MARC} = \bigcup \left\{ \begin{matrix} (R_0, R_1, \dots, R_N) \in R_+^{N+1} \\ \forall A \subseteq \{1,\dots,N\}: \\ \sum_{k\in A} R_k \le min\langle I_1', I_2' \rangle \\ \sum_{k=0}^{N} R_k \le min\langle I_3', I_4' \rangle \end{matrix} \right\} \tag{5}$$

$$I_1' = I(\{V_k\}_{k\in A}; Y_R | H, U, \{V_k\}_{k\notin A}, X_R)$$

$$I_2' = I(\{V_k\}_{k\in A} X_R; Y_D | H, U, \{V_k\}_{k\notin A})$$

$$I_3' = I(\{V_k\}_{k=1}^{N}, U; Y_R | H, X_R)$$

$$I_4' = I(\{V_k\}_{k=1}^{N}, X_R, U; Y_D | H)$$

where the joint p.d.f of r.v.'s $(H, F_1, \dots, F_N, F_R, U, V_1, \dots, V_N, X_R)$ is given by:

$$P_{H,F_1,\dots,F_N,F_R,U,V_1,\dots,V_N,X_R}(h, f_1, \dots, f_N, f_R, u, v_1, \dots, v_N, x_R) \tag{6}$$

$$= P_{H,F_1,\dots,F_N,F_R}(h, f_1, \dots, f_N, f_R) P_U(u) P_{X_R}(x_R) \prod_{i=1}^{N} P_{V_i|UX_R}(v_i | u x_R)$$

Furthermore, $\{g_i(.): \mathcal{V}_i \times \mathcal{U} \times \theta_i \to R\}$ is a set of deterministic functions such that $X_i = g_i(V_i, U, F_i)$, $i = 1,\dots,N$, and a set of relay functions $\{g_{R,t}\}_{t=1}^{n}$ such that $X_{R,t} = g_{R,t}(Y_R^{t-1}, F_R), 1 \le t \le n$, satisfies the power constraint policy of the transmitters and relay: $E\{X_i^2\} \le P_i, i = 1,\dots,N$ and $E\{X_R^2\} \le P_R$.

***Theorem 2:*** Consider the Gaussian fading MARC with a common message, with partial CSIT and perfect CSIR. The capacity region denoted by $\mathcal{C}_{common}^{GF-MARC}$ is given by:

$$\mathcal{C}_{common}^{GF-MARC} = \bigcup \left\{ \begin{matrix} (R_0, R_1, \dots, R_N) \in R_+^{N+1} \\ \forall A \subseteq \{1,\dots,N\}: \\ \sum_{k\in A} R_k \le min\{I_1, I_2\} \\ \sum_{k=0}^{N} R_k \le min\{I_3, I_4\} \end{matrix} \right\} \tag{7}$$

$$I_1 = E_H\left[C\left(\frac{\sum_{k\in A} h_{Rk}^2 \varphi_k(F_k)\left(1-\alpha_k^2(F_k)\right)(1-\beta_k)}{N_1}\right)\right]$$

$$I_2 = E_H\left[C\left(\frac{h_{DR}^2 \varphi_R(F_R) + \sum_{k\in A}\left(\begin{matrix} h_{Dk}^2 \varphi_k(F_k)\left(1-\alpha_k^2(F_k)\right) \\ +2h_{DR}h_{Dk}\sqrt{\beta_k \varphi_R(F_R)\varphi_k(F_k)}\left(1-\alpha_k^2(F_k)\right)\end{matrix}\right)}{N_2}\right)\right]$$

$$I_3 = E_H\left[C\left(\frac{\begin{matrix}\sum_{k=1}^{N} h_{Rk}^2 \varphi_k(F_k)\left(2-\beta_k-\alpha_k^2(F_k)+\alpha_k^2(F_k)\beta_k\right) \\ +2\sum_{1\le k<i<N} h_{Rk}h_{Ri}\sqrt{\varphi_i(F_i)\varphi_k(F_k)}\alpha_i(F_i)\alpha_k(F_k)\end{matrix}}{N_1}\right)\right]$$

$$I_4 = E_H\left[C\left(\frac{\begin{matrix} h_{DR}^2\varphi_R(F_R) \\ +\sum_{k=1}^{N} h_{Dk}^2\varphi_k(F_k) + 2h_{DR}h_{Dk}\sqrt{\beta_k\varphi_R(F_R)\varphi_k(F_k)\left(1-\alpha_k^2(F_k)\right)} \\ +2\sum_{1\le k<i<N} h_{Dk}h_{Di}\sqrt{\varphi_i(F_i)\varphi_k(F_k)}\alpha_i(F_i)\alpha_k(F_k)\end{matrix}}{N_2}\right)\right]$$

## IV. Proof of The Main Results

### A. *Proof of Theorem 1:*

***Achievability part:***

We consider $B$ blocks, each of $n$ symbols. We use superposition block Markov coding. In each block, $b = 1,2,\dots,B$, we use the same set of codebooks:

$$\mathcal{C} = \{u^n(j), x_R^n(m), v_1(j,m,l_1), \dots, v_N(j,m,l_N)\}$$

$$j \in [1:2^{nR_0}], \qquad m \in \left[1:2^{nR_1}\right] \times \dots \times \left[1:2^{nR_N}\right],$$

$$l_k \in [1:2^{nR_k}], \quad k = 1,\dots,N, \tag{8}$$

*Random codebook generation*: First fix a choice of

$$P_{U,V_1,\dots,V_N,X_R}(u, v_1, \dots, v_N, x_R)$$

$$= P_U(u) P_{X_R}(x_R) \prod_{i=1}^{N} P_{V_i|UX_R}(v_i|u x_R) \tag{9}$$

and the set of deterministic functions $\{g_i(.): \mathcal{V}_i \times \mathcal{U} \times \theta_i \to R\}$ and $X_{R,t} = g_{R,t}(Y_R^{t-1}, F_R)$.

1) Generate $2^{nR_0}$ independent identically distributed $n$-sequences $u^n$, each drawn according to $P(u^n) = \prod_{t=1}^{n} P_u(u_t)$. Index them as $u^n(j), j \in [1:2^{nR_0}]$.
2) Generate $2^{n\sum_{k=1}^{N} R_k}$ independent identically distributed $n$-sequence $x_R^n$, each drawn according to $P(x_R^n) = \prod_{t=1}^{n} P(x_{R,t})$ and index $x_R^n$ as $x_R^n(m)$, $m = (m_1, \dots, m_N) \in [1:2^{nR_1}] \times \dots \times [1:2^{nR_N}]$.

3) For each $\{x_R^n(m), u^n(j)\}$, generate $2^{nR_k}, k = 1, \dots, N$, independent identically $n$-sequence $v_k^n$, each drawn according to $P\left(v_k^n \middle| x_R^n(m), u^n(j)\right) = \prod_{t=1}^n P(v_{k,t}|x_{R,t}(m), u_t(j))$. Index them as $v_k^n(j, m, l_k)$, $l_k \in [1: 2^{nR_k}]$.

*Encoding:* Encoding is performed in *B+1* blocks. The encoding strategy is shown in Table I.

*Decoding and error Analysis*: It can be shown that the relay, after determining $x_R^n$ from $y_R^n$, uses jointly decoding and can decode reliably if

$$\sum_{k \in A} R_k \leq I(\{V_k\}_{k \in A}; Y_R | H, U, \{V_k\}_{k \notin A}, X_R) \tag{10}$$

and

$$\sum_{k=0}^N R_k \leq I\left(\{V_k\}_{k=1}^N U; Y_R | H, X_R\right) \tag{11}$$

and the receiver can decode with arbitrarily small probability of error if

$$\sum_{k \in A} R_k \leq I(\{V_k\}_{k \in A} X_R; Y_D | H, U, \{V_k\}_{k \notin A}) \tag{12}$$

and

$$\sum_{k=0}^N R_k \leq I\left(\{V_k\}_{k=1}^N X_R U; Y_D | H\right) \tag{13}$$

TABLE I. ENCODING STRATEGY

| Block 1 | Block 2 | $\cdots$ | Block B + 1 |
|---|---|---|---|
| $u_1^n(w_{0,1})$ | $u_2^n(w_{0,2})$ | $\cdots$ | $u_{B+1}^n(w_{0,B+1})$ |
| $x_{R,1}^n(1, \dots, 1)$ | $x_{R,2}^n(w_{1,1}, \dots, w_{N,1})$ | $\cdots$ | $x_{R,B+1}^n(w_{1,B}, \dots, w_{N,B})$ |
| $v_{1,1}^n(w_{0,1}, w_{1,1}, 1)$ | $v_{1,2}^n(w_{0,2}, w_{1,2}, w_{1,1})$ | $\dots$ | $v_{1,B+1}^n(1,1, w_{1,B})$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ |
| $v_{N,1}^n(w_{0,1}, w_{N,1}, 1)$ | $v_{N,2}^n(w_{0,2}, w_{N,2}, w_{N,1})$ | $\dots$ | $v_{N,B+1}^n(1,1, w_{N,B})$ |

***Converse part:***

To prove the converse part, we first derive an outer bound on $C_{common}^{F-MARC}$ in the following lemma.

*Lemma:* $C_{common}^{F-MARC}$ is outer bounded by:

$$C_{common}^{F-MARC} \subseteq \bigcup \left\{ \begin{array}{c} (R_0, R_1, \dots, R_N) \in R_+^{N+1} \\ \forall A \subseteq \{1, \dots, N\}: \\ \sum_{k \in A} R_k \leq min\langle I_1'', I_2'' \rangle \\ \sum_{k=0}^N R_k \leq min\langle I_3'', I_4'' \rangle \end{array} \right\} \tag{14}$$

$$I_1'' = I(\{X_k\}_{k \in A}; Y_R | H, U, \{X_k\}_{k \notin A}, X_R)$$

$$I_2'' = I(\{X_k\}_{k \in A} X_R; Y_D | H, U, \{X_k\}_{k \notin A})$$

$$I_3'' = I\left(\{X_k\}_{k=1}^N, U; Y_R | H, X_R\right)$$

$$I_4'' = I\left(\{X_k\}_{k=1}^N, X_R, U; Y_D | H\right)$$

where the joint p.d.f of r.v.'s $(H, F_1, \dots, F_N, F_R, U, X_1, \dots, X_N, X_R)$

$$P_{H,F_1,\dots,F_N,F_R,U,X_1,\dots,X_N,X_R}(h, f_1, \dots, f_N, f_R, u, x_1, \dots, x_N, x_R) \tag{15}$$

$$= P_{H,F_1,\dots,F_N,F_R}(h, f_1, \dots, f_N, f_R) P_U(u) P_{X_R}(x_R) \prod_{i=1}^N P_{X_i|UX_R}(x_i|ux_R)$$

that satisfy the power constraint: $E\{X_i^2\} \leq P_i, i = 1, \dots, N$, $E\{X_R^2\} \leq P_R$.

*Proof:* The proof is done using Fano's inequality and the input deterministic functions. We have used the degraded property of relay channel in our proof. The details are omitted for the brevity.

The achievable rate region that was obtained is also an outer bound for the above outer bound i.e. the outer bound is a subset of the inner bound; consequently, it is proved that the achievable rate region is a capacity region. It is proved by substituting $(U, X_1, \dots, X_N)$ as defined in following in the rate region described by (14); one can see that the rate region in (5) is obtained.

$$X_k = \sqrt{\varphi_k(F_k)} \left( \alpha_k(F_k) U + \sqrt{\left(1 - \alpha_k^2(F_k)\right)} V_k \right) \tag{16}$$

$$X_R = \sqrt{\varphi_R(F_R)} \left(\sum_{k=1}^N \sqrt{\beta_k(F_k)} V_k\right), \quad \sum_{k=1}^N \beta_k(F_k) = 1 \tag{17}$$

■

*B. Proof of Theorem 2:*

To prove of theorem, let $[U, V_1, \dots, V_N]$ be a sequence of Gaussian distributed r.v.'s with zero mean and unit variance, independent of each other and also independent of the fading matrix $\mathbf{H}$. In addition, let $\varphi_i: \theta_i \to R_+, i = 1, \dots, N, R$, be a set of power allocation policy functions satisfying the power constraints in $E\{X_i^2\} \leq P_i, i = 1, \dots, N$, $E\{X_R^2\} \leq P_R$ and $\alpha_i: \theta_i \to [0,1]$ and $\beta_i: \theta_i \to [0,1], i = 1, \dots, N,$ are two arbitrary (limited) deterministic functions that take their values from the interval [0,1]. We define the r.v.'s $[X_R, X_1, \dots, X_N]$ as (16) and (17). It can be clearly seen that the r.v.'s $[X_R, X_1, \dots, X_N]$ satisfy:

$$E_H[\varphi_i(F_i)] \leq P_i, i = 1, \dots, N \tag{18}$$

Proof of achievability is the same as that of theorem1 with regard to the above suppositions. For converse part, by substituting $[U, V_1, \dots, V_N]$ and $[X_R, X_1, \dots, X_N]$ in the capacity region (5), the theorem 2 is proved. The proof is omitted for the brevity.

■

## V. CONCLUSION

We investigated the fading multiple-access degraded relay channel with common message, partial CSIT and perfect CSIR, obtained a capacity region and extended the result to the Gaussian case. The achievable rate in [1] for the channels without fading or side information and without common messages is a special case of our work.

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