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Short Paper

Solving a Class of Separated Continuous Programming Problems Using Linearization and Discretization

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Abstract: In this paper we present a new approach for solving a class of separated continuous programming (SCP) problems using convex combination property of intervals. By convex combination property of intervals, we transform the SCP problems to the corresponding linear problems. Moreover, discretization method is applied to convert the new problem to a discrete-time problem which we solve it by linear programming methods. Finally, some numerical examples are given to show the efficiency of the proposed approach.

Keywords: SCP problem; Convex combination; Discretization; linear programming.

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1. Introduction

We consider the following class of separated continuous programming (SCP) problem:

$$\begin{aligned} &minimize && \int_0^T c(t)y(t)dt \qquad (1)\\ &subject\ to && B(t)y(t)+\int_0^t F(t,s,x(s))ds = p(t)\\ &&& G(t)y(t)\le q(t), \quad t\in[0,T]\\ &&& 0\le x(t)\le M, \quad y(t)\ge 0, \quad t\in[0,T] \end{aligned}$$

where $B(\cdot)$ is a given $k\times n$ matrix, $G(\cdot)$ is a given $s\times n$ matrix, and $F(\cdot,\cdot,\cdot)$ are given k vectors, $c(\cdot)$ is given n vector, $q(.)$ is a given s vector, M is given m vector, $y(\cdot)$ is a n vector, and $x(\cdot)$ is a m vector. The elements of vectors $F(\cdot,\cdot,\cdot)$, $c(\cdot)$, $p(\cdot)$, $d(\cdot)$ and Matrices $B(\cdot)$ and $G(\cdot)$ are continuous functions on $[0,T]$. In SCP problem (1) the vectors functions $x(\cdot)$ and $y(\cdot)$ are unknown. Firstly, Anderson [1,2] introduced separated continuous linear programs (SCLP). The 1987 book of Anderson and Nash [3] summarizes the theory developed by Anderson. Some special cases of SCLP were solved by Anderson and Philpott [4,5]. The series of papers on SCLP by Pullan [6–11] deals with solution structure, duality theory, and numerical algorithms. In addition, Erfanian [12] introduced the methods for solving SCLP and SCP by measure theory. Here we illustrate a new Technique for obtaining optimal solutions of SCP problem (1) that arise in communications, manufacturing and urban traffic control, etc. In this paper is used the convex combination property of interval to transform problem (1) to the corresponding linear problem. The vector function $x(\cdot)$ is bounded on interval $[0, T]$ and the bounds of component of this function are applied for convex combinations.

The structure of the paper is as follows. In Section 2, by convex combination property of intervals we firstly convert the SCP problem (1) to the corresponding linear problem. In Section 3, the new problem is transformed to the discrete-time problem using discretization methods. Finally, numerical examples are given in Section 4.

2. Linearization

In this section, we initially state and prove the following theorem:

Theorem 1 *Let $\Psi : [a_1, b_1] \times [a_2, b_2] \times A \to R$ be a continuous function where A is a compact and connected set in R^n, then for any arbitrary but fixed $(t, s) \in [a_1, b_1] \times [a_2, b_2]$. The set $\{\Psi(t, s, x) : x \in A\}$ is a closed interval in R.*

Proof. Let $(t, s) \in [a_1, b_1] \times [a_2, b_2]$ be given and $\Phi(x) = \Psi(t, s, x)$. Then $\Phi(\cdot)$ is a continuous function on set A. But, image of compact and connected set A by continuous function $\Phi(\cdot)$ is compact and connected set in R . So $\{\Phi(x) : x \in A\}$ is compact and connected set in R. Hence, $\{\Psi(t, s, x) : x \in A\}$ is a closed interval in R.

Now suppose $F = (F_1, \ldots, F_k)$ and $M = (M_1, \ldots, M_m)$. In addition, for any $t, s \in [0, T] \times [0, T]$ we may suppose the lower and upper bounds of interval $\{F_i(t, s, x) : x \in \prod_{j=1}^{m}[0, M_j]\}$ are $g_i(t, s)$ and $w_i(t, s)$ for $i = 1, 2, \ldots, k$, respectively. Thus for $(t, s) \in [0, T] \times [0, T]$ and $i = 1, 2, \ldots, k$

$$g_i(t, s) \leq F_i(t, s, x) \leq w_i(t, s) \tag{2}$$

So, we have for all $(t, s) \in [0, T] \times [0, T]$:

$$w_i(t, s) = \max\{F_i(t, s, x) : x \in \prod_{j=1}^{m}[0, M_j]\} \tag{3}$$
$$g_i(t, s) = \min\{F_i(t, s, x) : x \in \prod_{j=1}^{m}[0, M_j]\}$$

Theorem 2 *Let functions and for all be defined by relations (3). Then they are uniformly continuous on* $[0, T] \times [0, T]$.

Proof. We will show that $g_i(., .)$ for all $i = 1, 2, \ldots, k$ is a uniformly continuous function. It is sufficient that we show for any $\varepsilon > 0$, there exists $delta > 0$ such that if $(t_1, s_1) \in N_\delta(t_2, s_2)$ then $|g_i(t_1, s_1) - g_i(t_2, s_2)| < \varepsilon$ where $N_\delta(z)$ is a neighborhood of z by radius δ. We know each continuous function on compact set is a uniformly continuous function. Thus function $F_i(., ., .)$ on compact set $[0, T] \times [0, T] \times \prod_P j = 1^m[0, M_j]$ is a uniformly continuous function. It means that for any $\varepsilon > 0$, there is

$\delta > 0$, such that if $(t_1, s_1, y) \in N_\delta(t_2, s_2, x)$ then $|F_i(t_1, s_1, x) - F_i(t_2, s_2, x)| < \varepsilon$. Thus $F_i(t_1, s_1, x) < F_i(t_2, s_2, x) + \varepsilon$. Moreover, $g_i(t_1, s_1) \leq F_i(t_1, s_1, x)$, So we have $g_i(t_1, s_1) < F_i(t_2, s_2, x) + \varepsilon$ and $g_i(t_1, s_1) \leq g_i(t_2, s_2) + \varepsilon$ or $g_i(t_1, s_1) - g_i(t_2, s_2) \leq \varepsilon$. Also by a similar procedure, we have $g_i(t_2, s_2) - g_i(t_1, s_1) \leq \varepsilon$. Thus $|g_i(t_1, s_1) - g_i(t_2, s_2)| \leq \varepsilon$. The proof of uniformly continuity of the function $w_i(\cdot, \cdot)$, $i = 1, 2, \ldots, k$ is similar.

Now, by relation (2) and Theorem (1) we have for all $i = 1, 2, \ldots, k$:

$$\alpha_i(t) \leq \int_0^t F_i(t, s, x(s))ds \leq \beta_i(t), \quad t \in [0, T] \tag{4}$$

where $\alpha_i(t) = \int_0^t g_i(t, s)ds$ and $\beta_i(t) = \int_0^t w_i(t, s)ds$. By convex combination property of intervals and relation (4) we also have for any $t \in [0, T]$ and $i = 1, 2, \ldots, k$:

$$\int_0^t F_i(t, s, x(s))ds = (\beta_i()t) - \alpha_i(t)\lambda_i(t) + \alpha_i(t), \quad \lambda_i(t) \in [0, 1] \tag{5}$$

Thus we may convert the SCP problem (1) by relation (5) as follows:

$$\begin{aligned} &minimize && \int_0^T c(t)y(t)dt \\ &subject\ to && B(t)y(t) + \langle \beta(t) - \alpha(t), \lambda(t)\rangle = p(t) - \alpha(t) \\ &&& G(t)y(t) \leq q(t) \\ &&& 0 \leq \lambda(t) \leq 1, \quad y(t) \geq 0, \quad t \in [0, T] \end{aligned} \tag{6}$$

where

$$\lambda(\cdot) = (\lambda_1(\cdot), \ldots, \lambda_k(\cdot))^t, \alpha(\cdot) = (\alpha_1(\cdot), \ldots, \alpha_k(\cdot))^t, \beta(\cdot) = (\beta_1(\cdot), \ldots, \beta_k(\cdot))^t$$

and

$$\langle \beta(t) - \alpha(t), \lambda(t)\rangle = ((\beta_1(t) - \alpha_1(t))\lambda_1(t), \ldots, (\beta_k(t) - \alpha_k(t))\lambda_k(t))^t$$

Note that decision variables in the continuous-time problem (6) are $y(\cdot)$ and $\lambda(\cdot)$. in the next section, problem (6) is converted to a corresponding discrete-time problem.

3. Discrete-time problem

In this section, discretization method is used to transform continuous problem (6) to the corresponding discrete form. We write problem (6) as follows:

$$\begin{aligned} &minimize && \int_0^T \sum_{j=1}^n c_j(t)y_j(t)dt \\ &subject\ to && \sum_{j=1}^n B_{ij}(t)y_j(t) + (\beta_i(t) - \alpha_i(t))\lambda_i(t) = p_i(t) - \alpha_i(t) \\ &&& \sum_{j=1}^n G_{rj}(t)y_j(t) \leq q_r(t) \\ &&& 0 \leq \lambda_i(t) \leq 1, \quad y_j(t) \geq 0, \quad t \in [0, T] \\ &&& r = 1, 2, \ldots, s, \quad i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, n \end{aligned} \tag{7}$$

where $B_{ij}(\cdot)$ and $G_{rj}(\cdot)$ are elements of matrixes $B(\cdot)$ and $G(\cdot)$, respectively. For transformation problem (7) to the discrete form, we choose the large number w and consider the equidistance points

$t_0 = 0 < t_1 < t_2 < \ldots < t_m = T$ of interval $[0, T]$ which $t_v = \frac{T}{w}v$ for all $v = 0, 1, \ldots, w$. Now, by trapezoidal approximation, problem (7) is converted to the following problem:

$$\begin{aligned} &minimize && \sum_{j=1}^{n}\left(\frac{T}{2m}\,c_j(t_0)y_{j,0} + \frac{T}{m}\sum_{v=1}^{n-1} c_j(t_v)y_{j,v} + \frac{T}{2m}\;c_j(t_m)y_{j,m}\right) \qquad (8)\\ &subject\ to && \sum_{j=1}^{n} B_{ij}(t_v)y_{j,v} + (\beta_i(t_v) - \alpha_i(t_v)), \lambda_{iv}\rangle = p_i(t_v) - \alpha_i(t_v)\\ &&& \sum_{j=1}^{n} G_{rj}(t_v)y_{j,v} \le q_r(t_v)\\ &&& 0 \le \lambda_{i,v} \le 1, \quad y_{j,v}(t) \ge 0, \quad t \in [0, T]\\ &&& r = 1, 2, \ldots, s, \quad i = 1, 2, \ldots, k, \quad v = 1, 2, \ldots, w, \quad j = 1, 2, \ldots, n \end{aligned}$$

where $\lambda_{i,v} = \lambda_i(t_v)$ and $y_{j,v} = y_j(t_v)$ for all $v = 0, 1, \ldots, m$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$ are decision variables of this problem. By solving problem (8), we can obtain the optimal solutions $\lambda^*_{i,v}$ and $y^*_{j,v}$ for all $v = 0, 1, \ldots, m$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, k$.

4. Numerical examples

Here we obtain the numerical solutions of several SCP problems by proposed approach in above. For solving linear programming problem (8) we use the simplex method (see [8]) in Matlab software.

Example 1 *Consider the following SCP problem:*

$$\begin{aligned} &minimize && \int_0^1 (2t-1)y(t)dt \qquad (9)\\ &subject\ to && y(t) + \int_0^t (s + tx(s))ds = t^2\\ &&& 0 \le x(t) \le 2, \quad y(t) \ge 0, \quad t \in [0, 1] \end{aligned}$$

Assume $F(t, s, x) = s + tx$ *for* $(t, s, x) \in [0, 1] \times [0, 1] \times [0, 2]$. *So by (3) for* $(t, s) \in [0, 1] \times [0, 1]$*:*

$$\begin{aligned} g(t, s) &= \min\{s + tx : x \in [0, 2]\} = s\\ w(t, s) &= \max\{s + tx : x \in [0, 2]\} = s + 2t \end{aligned}$$

In addition, we have:

$$\begin{aligned} \alpha(t) &= \int_0^t g(t, s)ds = \int_0^t sds = \frac{t^2}{2}, \quad t \in [0, 1]\\ \beta(t) &= \int_0^t w(t, s)ds = \int_0^t (s + 2t)ds = \frac{5t^2}{2}, \quad t \in [0, 1] \end{aligned}$$

We assume that $w = 100$ *and* $t_v = \frac{v}{100}$ *for all* $v = 1, 2, \ldots, 100$ *and write problem (8) as follows:*

$$\begin{aligned} &minimize && \frac{1}{200}\,(2t_v - 1)y_0 + \frac{1}{100}\sum_{v=1}^{99}(2t_v - 1)y_v + \frac{1}{200}\,(2t_v - 1)y_{100} \qquad (10)\\ &subject\ to && y_v + 2t_v^2\lambda_v = \frac{t_v^2}{2}\\ &&& 0 \le \lambda_v \le 1, \quad y_v \ge 0, \quad v = 0, 1, 2, \ldots, 100 \end{aligned}$$

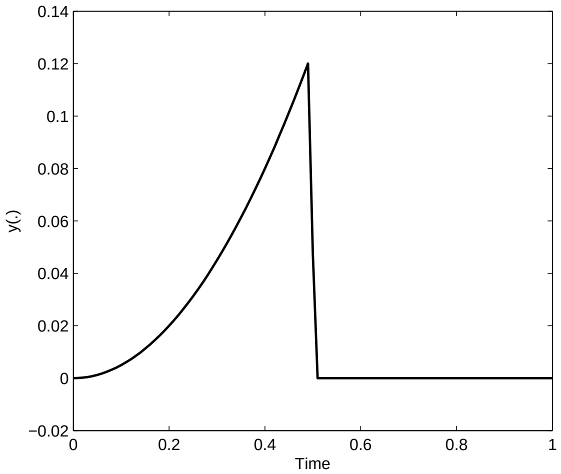

Figure 1. The optimal solution $y^*(\cdot)$ of Example 1

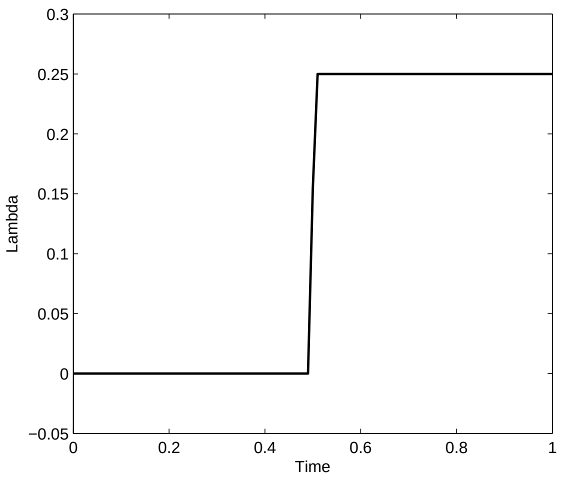

Figure 2. The optimal solution $\lambda^*(\cdot)$ of Example 1

We give the optimal solution $y^*(\cdot)$ of problem (9) by solving problem (10) which is illustrated in Fig. 1. Also the optimal solution $\lambda^*(\cdot)$ of problem (10) is showed in Fig. 2. The obtained optimal solution of objective function is -0.0052.

Example 2 *Consider the following SCP problem:*

$$\begin{aligned} &minimize && \int_0^1 \sin(3\pi t)y(t)dt \\ &subject\ to && y(t) - \int_0^t \tan(\frac{\pi}{8}x^3(s) + s)ds = 0.2e^{-t} \\ &&& y(t) \leq 1 \\ &&& 0 \leq x(t) \leq 1, \quad y(t) \geq 0, \quad t \in [0,1] \end{aligned} \tag{11}$$

We have $p(t) = 0.2e^{-t}$, $F(t,s,x) = -\tan(\frac{\pi}{8}x^3(s) + s)$ and $c(t) = \sin(3\pi t)$ for all $(t,s,x) \in [0,1] \times [0,1] \times [0,2]$. By relations (3):

$$\begin{aligned} g(t,s) &= \min\left\{-\tan(\frac{\pi}{8}x^3 + s) : x \in [0,1]\right\} = -\tan(\frac{\pi}{8} + s), \quad (t,s) \in [0,1] \times [0,1] \\ w(t,s) &= \max\left\{-\tan(\frac{\pi}{8}x^3 + s) : x \in [0,1]\right\} = -\tan(s), \quad (t,s) \in [0,1] \times [0,1] \end{aligned}$$

Hence

$$\begin{aligned} \alpha(t) &= -\int_0^t \tan(\frac{\pi}{8} + s)ds = \ln(\cos(\frac{\pi}{8}) + t) - \ln(\cos(\frac{\pi}{8})), \quad t \in [0,1] \\ \beta(t) &= -\int_0^t \tan(s)ds = \ln(\cos(t)), \quad t \in [0,1] \end{aligned}$$

We assume $w = 100$ and $t_v = \frac{v}{100}$ for all $v = 0, 1, \ldots, 100$. We obtain the optimal solution $y^*(\cdot)$ of problem (11) by solving the corresponding problem (8) which is illustrated in Fig. 1. In addition the corresponding optimal solution $\lambda^*(\cdot)$ of problem (11) is showed in Fig. 2. Moreover the obtained optimal solution of objective function is 0.0340.

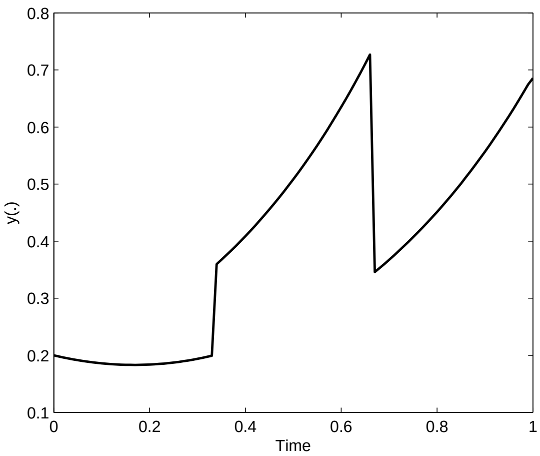

Figure 3. The optimal solution $y^*(\cdot)$ of Example 2

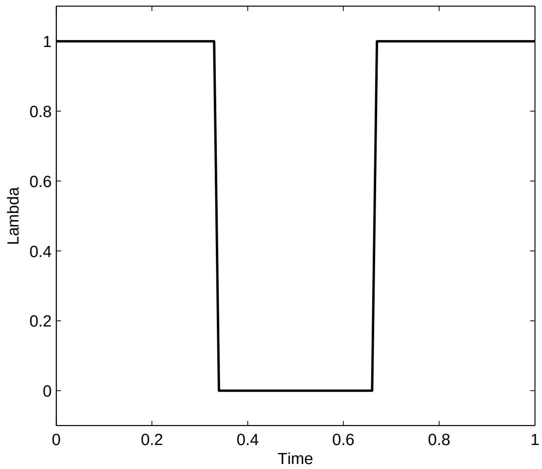

Figure 4. The optimal solution $\lambda^*(\cdot)$ of Example 2

5. Conclusion

In this paper, the convex combination property of intervals was applied to linearization the nonlinear terms in constraints of SCP problems. By this approach, we can convert the nonlinear problems to the corresponding linear problems. We also used the discretization method and obtained a discrete-time problem which is a linear programming problem. The linear programming methods, such as simplex method, can be used for solving the obtained linear programming problem. Thus by this approach, we can solve a wide class of SCP problems.

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