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S duality of the D_3-brane S matrix

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There is a conjecture in the literature that indicates the tree-level S-matrix elements of graviton become symmetric under the $SL(2, Z)$ transformation after including the loops and the nonperturbative effects. Using the Ward identity corresponding to the global S-duality transformations, this conjecture can be extended to other S-matrix elements as well. While the $SL(2, Z)$ transformation on the background dilaton is nonlinear, the Ward identity dictates the S-duality transformations on external states should be the linearized $SL(2, R)$ transformations. We examine in detail various S-matrix elements involving massless closed string and/or open string vertex operators on the world volume of the D_3-brane in favor of this conjecture.

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I. INTRODUCTION

It is known that type II superstring theory is invariant under T duality [1–5] and S duality [5–11]. At the classical level, these dualities appear in the equations of motion and in their solutions [12–15]. At the quantum level, these dualities should appear in the S-matrix elements. The contact terms of the sphere-level S-matrix elements of graviton are speculated to be invariant under the S duality after including the loops and the nonperturbative effects [16–32]. This idea has been extended to the S-matrix elements on the world volume of D_3-branes as well [33–35]. The S duality may also be used to find the S-matrix elements on the world volume of the F_1-string/NS_5-brane from the corresponding S-matrix elements on the world volume of the D_1-string/D_5-brane [36].

The S duality holds order by order in α' and is nonperturbative in the string loop expansion [5]. In order to study the S duality of a S-matrix element, one has to first α' expand the amplitude in the Einstein frame and then study its S duality at each order of α'. Let us consider the disk-level S-matrix element of two gravitons on the world volume of D_3-branes whose S duality has been studied in [33]. The leading α'-order terms of this amplitude are invariant under the S duality because the graviton in the Einstein frame is invariant. On the other hand, the nonleading order terms which include the background dilaton factors as well are speculated to become symmetric after including the loops and the nonperturbative effects [16]. Using the fact that the S-matrix elements should satisfy the Ward identity corresponding to the global S-duality transformations, the above proposal has been extended to all S-matrix elements in [36]. The S-dual Ward identity indicates that an n-point function must transform to an n-point function under the S duality. Hence, in order to study the S-dual Ward identity of a S-matrix element one has to consider the linearized $SL(2, R)$ transformations on the external states [36]. More specifically, the S-matrix element of $n-$ closed $+\, m-$ open strings must transform to the S-matrix element of $n-$ closed $+\, m-$ open strings; hence, the S-duality transformations of the external closed strings and the external open strings must be separately linear. The transformation of the background fields, however, should be the nonlinear S-duality transformation. In general, imposing the invariance of the S-matrix element under this later transformation requires one to include the loops and the nonperturbative contributions to the tree-level S-matrix element.

All S-matrix elements of two massless Ramond-Ramond (RR) and/or Neveu Schwarz–Neveu Schwarz states [37,38] on the world volume of the D_3-brane have been analyzed in [35,36] in favor of this proposal. In this paper we would like to test this proposal by examining in detail the S-matrix elements which involve the open string states as well. In particular, using the linear S-duality transformation of the gauge field [39], we will show that the leading α'-order terms of the S-matrix elements combine into linear S-duality invariant multiplets. The background dilaton factors in the nonleading order terms which are not invariant under the S duality may then be extending to the $SL(2, Z)$ invariant functions after including the loops and the nonperturbative effects, as in [33].

The leading α'-order terms of the disk-level S-matrix elements, in general, have both contact terms as well as massless poles. The contact terms along which produce the low energy effective actions may not satisfy the S-dual Ward identity. In fact, the low energy effective action of a single D_3-brane which is given by the combination of the Abelian Born-Infeld action [40,41] and the Chern-Simons action [42,43] is not invariant under the S duality. However, it is known that their equations of motion are invariant under the S duality [39,44,45].

In general, an effective action can be separated into two parts, i.e.,

$$S = S_1 + S_2, \tag{1}$$

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where S_1 includes the couplings which are invariant under the linear S duality, and S_2 includes the couplings which must be combined, in the momentum space, with some massless poles to become invariant under the linear S duality. We will show that the Born-Infeld action has such a structure. That is, its F^4 terms are invariant under the linear S duality; however, its F^6 terms and higher may be combined with some massless poles to become invariant under the linear S duality. We will discuss how this property may be used iteratively to find all couplings of the effective action of D_3-branes.

The outline of this paper is as follows: In Sec. II, using the standard S-duality transformations at the linear order, we show that the S-matrix elements of one massless open string and one massless closed string vertex operators on the world volume of a D_3-brane can be written in a linear S-duality invariant form. In Sec. III we show that the leading α'-order terms of the disk-level S-matrix element of two open string and one closed string states can be combined into linear S-duality invariant combinations. Following [33,34], the background dilaton factors in the nonleading order terms may be extended to the nonlinear $SL(2,Z)$ invariant functions. In Sec. IV, we show the same procedure can be applied for the S-matrix element of four Abelian gauge field vertex operators. We discuss our results in Sec. V.

II. ONE GAUGE FIELD AMPLITUDES

In this section we consider the S-matrix elements of one gauge field and one closed string vertex operator. These amplitudes in the string frame have been calculated in [46]. In the Einstein frame, they are

$$\begin{aligned}
\mathcal{A}(D_3;\zeta_1,B_2) &\sim -T_3 e^{-\phi_0}F_{1ab}B_2^{ab}\delta^4(k_1^a+p_2^a),\\
\mathcal{A}(D_3;\zeta_1,C_2^{(2)}) &\sim \frac{T_3}{2}F_{1a_0a_1}C_{2a_2a_3}\epsilon^{a_0\cdots a_3}\delta^4(k_1^a+p_2^a),
\end{aligned}\tag{2}$$

where ϕ_0 is the constant dilaton background, ζ_1 is the polarization of the gauge field, and $F_{1ab}=i(k_{1a}\zeta_{1b}-k_{1b}\zeta_{1a})$ is its field strength.[1] B_2 and $C_2^{(2)}$ are the polarization of the B field and the RR two-form, respectively. We have normalized the amplitudes such that they become consistent with T duality. We have not, however, fixed the numerical factor of the amplitudes in this paper. To study the T duality one should first change the Einstein frame metric $g^E_{\mu\nu}$ to the string frame metric $g^S_{\mu\nu}$ as $g^S_{\mu\nu}=e^{\phi_0/2}g^E_{\mu\nu}$, and then apply the linear T-duality transformation as in [47,48]. In particular, the T duality along a world-volume direction maps the string frame tension as $T_3\delta^4(k_1^a+p_2^a)\to T_2\delta^3(k_1^a+p_2^a)$ and the string frame coupling $F_{1ab}B_2^{ab}\to F_{1ab}B_2^{ab}+\cdots$, where dots refer to some couplings involving the transverse scalar fields in which we are not interested in this section. The D_3-brane tension in the string frame is $T_3=e^{-\phi_0}(2\pi)^{-3}(\alpha')^{-2}$, where α' is in the string frame. In the Einstein frame $\alpha'\to e^{-\phi_0/2}\alpha'$ and hence the tension becomes $T_3=(2\pi)^{-3}(\alpha')^{-2}$ which is invariant under the S duality.

Now we have to show that the above amplitudes can be combined into a linear $SL(2,R)$ invariant form. The nonlinear transformation of the gauge field and the axion dilaton, $\tau=C+ie^{-\phi}$, are given by [39]

$$\begin{aligned}
F_{ab}&\to sF_{ab}+r*G_{ab},\\
G_{ab}&\to pG_{ab}-q*F_{ab},\\
\tau&\to\frac{p\tau+q}{r\tau+s},
\end{aligned}\tag{3}$$

where the antisymmetric tensor G_{ab} is given by

$$G_{ab}=-\frac{2}{T_3}\frac{\partial L}{\partial F^{ab}},\tag{4}$$

where L is the Lagrangian. Using $**=-1$, one can write the transformation of gauge field as

$$\begin{pmatrix}*F\\ G\end{pmatrix}\to(\Lambda^{-1})^T\begin{pmatrix}*F\\ G\end{pmatrix},\qquad \Lambda=\begin{pmatrix}p&q\\ r&s\end{pmatrix}.\tag{5}$$

The B field and the RR two-form also appear as a doublet under the $SL(2,R)$ transformation [45]

$$\mathcal{B}\equiv\begin{pmatrix}B\\ C^{(2)}\end{pmatrix}\to(\Lambda^{-1})^T\begin{pmatrix}B\\ C^{(2)}\end{pmatrix}.\tag{6}$$

Unlike the transformations (3), the above transformation is linear.

The S-matrix elements (2) involve both the background fields as well as the external open and closed string fluctuations. The invariance of the S-matrix elements under the linear S duality is such that the $SL(2,R)$ transformation on the background fields is nonlinear and on the external states is linear [36]. For the closed string amplitudes considered in [36], this transforms an n-point function to another n-point function. In the cases that the S-matrix elements involve both open and closed strings, the transformation on the external states must be in such a way that an S-matrix element of $n-$ closed $+\ m-$ open string vertex operators transforms to another S-matrix element of $n-$ closed $+\ m-$ open vertex operators. So the transformation of the open and closed strings should be separately linear, i.e., the open string should transform to open string, and the closed string should transform to closed string. Since the gauge field in the amplitudes (2) is an open string quantum fluctuation, we have to consider vector to vector transformation on this field. To have a linear vector field in G_{ab}, we have to consider the quadratic vector terms in the Lagrangian. Therefore, we have to consider the following D_3-brane action in the Einstein frame [39]:

[1]Our index convention is that the Greek letters $(\mu,\nu,\ldots)$ are the indices of the space-time coordinates, the letters $(a,b,c,\ldots)$ are the world-volume indices, and the letters $(i,j,k,\ldots)$ are the normal bundle indices.

$$L = T_3(-\tfrac{1}{4}e^{-\phi_0}F_{ab}F^{ab} + \tfrac{1}{4}C_0F_{ab}(*F)^{ab}), \tag{7}$$

where $(*F)^{ab} = \epsilon^{abcd}F_{cd}/2$ and ϕ_0, C_0 are the background fields. Note that there is no higher derivative corrections to the quadratic terms in (7). The antisymmetric tensor G_{ab} becomes

$$G_{ab} = e^{-\phi_0}F_{ab} - C_0(*F)_{ab}, \tag{8}$$

which is linear in the vector field and has no higher derivative corrections at this order.

Now, considering the transformation (5) for the closed string fields and the following linearized transformation:

$$\mathcal{F}_{ab} \equiv \begin{pmatrix} (*F)_{ab} \\ e^{-\phi_0}F_{ab} - C_0(*F)_{ab} \end{pmatrix} \rightarrow (\Lambda^{-1})^T\begin{pmatrix} (*F)_{ab} \\ e^{-\phi_0}F_{ab} - C_0(*F)_{ab} \end{pmatrix} \tag{9}$$

for the open string field, the amplitudes (2) can be extended to

$$\mathcal{A}(D_3;\zeta_1,\mathcal{B}_2) \sim T_3(\mathcal{F}_1^T)_{ab}\mathcal{N}(\mathcal{B}_2)^{ab}\delta^4(k_1^a + p_2^a), \tag{10}$$

where $\mathcal{F}_1$ and $\mathcal{B}_2$ are the polarizations of the $\mathcal{F}$ field and the $\mathcal{B}$ field, respectively. The $SL(2,R)$ matrix $\mathcal{N}$ is

$$\mathcal{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{11}$$

which has the property

$$\mathcal{N} = \Lambda\mathcal{N}\Lambda^T. \tag{12}$$

The amplitude (10) is manifestly invariant under the linear $SL(2,R)$ transformation. There is no background dilaton factor left over in (10); hence, there would be no need to include loops and nonperturbative effects to make the amplitude fully invariant under the *S* duality. The tree-level *S*-matrix element of 1 − closed + 1 − open string vertex operators is the only *S*-matrix element which is fully invariant under the linear *S* duality. In all other cases, one needs to add the loops and the nonperturbative effects to make the tree-level amplitudes fully invariant under the *S* duality. The observation that the amplitude (10) is invariant under the *S* duality is consistent with the fact that the loop effects in the 1 − closed + 1 − open amplitude are zero.

The *S*-dual amplitude (10) includes the *S*-matrix elements (2) as well as the following 2-point function in the presence of a constant axion:

$$\mathcal{A}(D_3,C_0;\zeta_1,B_2) \sim \frac{T_3}{2}C_0F_{1a_0a_1}B_{2a_2a_3}\epsilon^{a_0\cdots a_3}\delta^4(k_1^a + p_2^a), \tag{13}$$

which is a standard coupling in the Chern-Simons part of the D_3-brane action. This *S*-matrix element can be calculated with a disk-level 3-point function of one gauge field, one *B* field, and one RR scalar vertex operator in which the RR scalar is a constant. For the nonconstant RR field, the amplitude has the complicated structure of the *S*-matrix element of two closed and one open string vertex operators [49]; however, for a constant field it should be reduced to (13). The disk-level 3-point function of one gauge field, one *B* field, and one RR vertex operator has been recently calculated in [50]. It is easy to verify that for a constant RR scalar, it reduces to (13).

III. TWO GAUGE FIELDS AMPLITUDES

The *S*-matrix element of two gauge fields and one closed string is nonzero when the closed string is dilaton, RR scalar, or graviton [46,51]. Since graviton is invariant under the *S* duality, one expects the *S*-matrix element of one graviton and two gauge fields to be invariant under the linear *S* duality. On the other hand, the dilaton and the RR scalar transform as (3). So we expect the *S*-matrix elements of the dilaton and the axion to combine into a linear *S*-dual multiplet. Let us first consider the graviton amplitude.

The *S*-matrix element of two gauge fields and one graviton is given in [46,51]. In the Einstein frame, it is

$$\mathcal{A}(D_3;\zeta_1,\zeta_2,h_3) \sim T_3e^{-\phi_0}\left(F_{1a}^{c}F_{2bc}h_3^{ab} - \frac{1}{4}F_{1ab}F_2^{ab}h_{3c}^{c}\right)\frac{\Gamma(1-2te^{-\phi_0/2})}{[\Gamma(1-te^{-\phi_0/2})]^2}, \tag{14}$$

where h_3 is the polarization of the graviton and the Mandelstam variable t is $t = -\alpha' k_1 \cdot k_2$. There is also a conservation of momentum factor $\delta^4(k_1^a + k_2^a + p_3^a)$. Here again we have normalized the amplitude such that it becomes consistent with linear *T* duality. The α' in the Mandelstam variable t, which is in the Einstein frame, is invariant under the *S* duality; however, the dilaton factor in the gamma functions in (14) is not invariant under the *S* duality. Hence, we have to expand the gamma functions in order to study the *S* duality of this amplitude. The α' expansion of the gamma functions is

$$\frac{\Gamma(1-2te^{-\phi_0/2})}{[\Gamma(1-te^{-\phi_0/2})]^2} = 1 + t^2\zeta(2)e^{-\phi_0} + 2t^3\zeta(3)e^{-3\phi_0/2} + \frac{19}{4}t^4\zeta(4)e^{-2\phi_0} + \cdots. \tag{15}$$

The first term is invariant; hence, to show that the leading α'-order term of (14) is invariant under the linear *S* duality one should be able to write the kinematic factor in (14) in linear $SL(2,R)$ invariant form. To this end, consider the matrix $\mathcal{M}$

$$\mathcal{M} = e^{\phi}\begin{pmatrix} |\tau|^2 & C \\ C & 1 \end{pmatrix}, \tag{16}$$

which transforms under the $SL(2, R)$ transformation as[2]

$$\mathcal{M} \to \Lambda \mathcal{M} \Lambda^T. \tag{17}$$

Using this matrix, one finds

$$(\mathcal{F}_1^T)_a^c \mathcal{M}_0 \mathcal{F}_{2bc} = e^{-\phi_0}[(*F_1)_a^c (*F_2)_{bc} + F_{1a}^c F_{2bc}], \tag{18}$$

where $\mathcal{M}_0$ is the matrix $\mathcal{M}$ for constant background fields ϕ_0 and C_0. Using the identity

$$\epsilon_a^{cde}\epsilon_{bc}^{fg} = -\eta_{ab}(\eta^{df}\eta^{eg} - \eta^{dg}\eta^{ef}) + \delta_a^f(\delta_b^d\eta^{eg} - \delta_b^e\eta^{dg}) - \delta_a^g(\delta_b^d\eta^{ef} - \delta_b^e\eta^{df}), \tag{19}$$

one finds

$$(\mathcal{F}_1^T)_a^c \mathcal{M}_0 \mathcal{F}_{2bc} = e^{-\phi_0}[-\tfrac{1}{2}F_{1cd}F_2^{cd}\eta_{ab} + F_{1a}^c F_{2bc} + F_{1b}^c F_{2ac}]. \tag{20}$$

Using the above relation, one observes that the kinematic factor in (14) is invariant under the linear S duality.[3]

Therefore, the leading term of the amplitude (14) which is of α'^0 order, is invariant under the linear S duality. All other terms are the higher derivative terms. Since the linear S-duality transformation (9) has no higher derivative corrections, all the higher derivative terms of (14) are the higher derivatives of the kinematic factor. Hence, they all are invariant under the linear S duality. The α'^2-order term, however, has the constant dilaton factor $e^{-\phi_0}$ which is not invariant under the nonlinear S duality. The terms with the higher order of α' have other dilaton factors. None of them are invariant under the S duality. Since the background dilaton and axion transform nonlinearly as (3) under the S duality, one should extend each of the dilaton factors to a function of both dilaton and axion to make them invariant under the S duality. In this way, one can find the exact dependence of the amplitude on the background dilaton and axion. By adding the one-loop and the D-instanton effects to the α'^2-order term, which may be done by replacing $e^{-\phi_0}$ with the regularized nonholomorphic Eisenstein series $E_1(\phi_0, C_0)$, one may extend the α'^2-order term to the S-dual invariant form [33–35]. The dilaton factor $\zeta(3)e^{-3\phi_0/2}$ in the α'^3-order term may be extended to the nonholomorphic Eisenstein series $E_{3/2}(\phi_0, C_0)$ [34]. In general, the dilaton factor $\zeta(n)e^{-n\phi_0/2}$ may be extended to the nonholomorphic Eisenstein series $E_{n/2}(\phi_0, C_0)$ after including the loops and the nonperturbative effects.

Therefore, the amplitude (14) may be extended to

$$\mathcal{A}(D_3; \zeta_1, \zeta_2, h_3) \sim \frac{T_3}{2}(\mathcal{F}_1^T)_a^c \mathcal{M}_0 \mathcal{F}_{2bc} h_3^{ab}(1 + \alpha_2 t^2 E_1(\phi_0, C_0) + \alpha_3 t^3 E_{3/2}(\phi_0, C_0) + \cdots), \tag{22}$$

where α_n's are some number, i.e., $\alpha_2 = 1$, $\alpha_3 = 2$, and so on. This amplitude is invariant under the linear $SL(2, R)$ transformation on the external states and is invariant under the nonlinear $SL(2, Z)$ transformation on the background fields. One may expect the replacement of the dilaton factors in the tree-level amplitude with the appropriate nonholomorphic Eisenstein series includes all the loops and the noperturbative corrections to the tree-level amplitude; however, this does not mean that an exact S-matrix element can be found in this way. In general, there are many new terms in the loop amplitudes which have structure different than those in the tree level. We expect the dilaton factors in the new terms at each loop order to become invariant under the S duality after including the higher loop effects.

Now consider the S-matrix elements of the dilaton and axion which are given in [46,51]. In the Einstein frame they are

$$\mathcal{A}(D_3; \zeta_1, \zeta_2, \phi_3) \sim T_3 e^{-\phi_0} F_{1ab}F_2^{ab}\phi_3 \frac{\Gamma(1 - 2te^{-\phi_0/2})}{[\Gamma(1 - te^{-\phi_0/2})]^2},$$
$$\mathcal{A}(D_3; \zeta_1, \zeta_2, C_3) \sim T_3 F_{1ab}(*F_2)^{ab} C_3 \frac{\Gamma(1 - 2te^{-\phi_0/2})}{[\Gamma(1 - te^{-\phi_0/2})]^2}, \tag{23}$$

where ϕ_3 is the polarization of the dilaton, and C_3 is the polarization of the axion. These polarizations are one; however, for clarity we keep them in the amplitudes. There is also a conservation of momentum factor $\delta^4(k_1^a + k_2^a + p_3^a)$ in each amplitude.

To write the kinematic factors in (23) in a linear S-dual form, consider the variation of the matrix $\mathcal{M}$ in (16). It is given by

[2]Note that the matrix $\mathcal{M}$ here is the inverse of the matrix $\mathcal{M}$ in [39].

[3]The kinematic factor of two Abelian gauge fields and two transverse scalars can be read from the expansion of Dirac-Born-Infeld (DBI) action. In the Einstein frame it is

$$K(\zeta_1, \zeta_2, \Phi_3, \Phi_4) = e^{-\phi_0}(F_{1a}^c F_{2bc}\Phi_3^{i,a}\Phi_4^{j,b}\eta_{ij} - \tfrac{1}{4}F_{1ab}F_2^{ab}\Phi_3^{i,c}\Phi_4^{j,d}\eta_{ij}\eta_{cd}) + (1 \leftrightarrow 2),$$

where commas denote partial differentiation in the momentum space. This is similar to the kinematic factor in (14). The transverse scalar fields are invariant under the S duality. Using (20), one can write this kinematic factor as

$$K(\zeta_1, \zeta_2, \Phi_3, \Phi_4) = (\mathcal{F}_1^T)_a^c \mathcal{M}_0 \mathcal{F}_{2bc}\Phi_3^{i,a}\Phi_4^{j,b}\eta_{ij}, \tag{21}$$

which is manifestly invariant under the linear S duality.

$$\delta\mathcal{M}=\begin{pmatrix}-(e^{-\phi}-C^2e^{\phi})\delta\phi+2Ce^{\phi}\delta C & Ce^{\phi}\delta\phi+e^{\phi}\delta C\\ Ce^{\phi}\delta\phi+e^{\phi}\delta C & e^{\phi}\delta\phi\end{pmatrix}. \tag{24}$$

This matrix transforms under the $SL(2,R)$ transformation as

$$\delta\mathcal{M}\to\Lambda\delta\mathcal{M}\Lambda^T. \tag{25}$$

Consider the case in which the variations are the external states, i.e., $\delta\phi=\phi_3$ and $\delta C=C_3$, and the dilaton and the axion are the constant background fields ϕ_0 and C_0, respectively. We call this matrix $\delta\mathcal{M}_3$. Using this matrix, one finds the following relation:

$$(\mathcal{F}_1^T)_{ab}\delta\mathcal{M}_3\mathcal{F}_2^{ab}=2e^{-\phi_0}F_{1ab}F_2^{ab}\phi_3+2F_{1ab}(*F_2)^{ab}C_3, \tag{26}$$

which is invariant under the linear S duality. Using this relation, one can write the kinematic factor in $\mathcal{A}(D_3;\zeta_1,\zeta_2,\phi_3)+\mathcal{A}(D_3;\zeta_1,\zeta_2,C_3)$ in linear $SL(2,R)$ invariant form. Adding the loops and the nonperturbative effects as in the previous case, one may extend the dilaton factors in the α' expansion of the gamma functions to the $SL(2,Z)$ invariant nonholomorphic Eisenstein series. Therefore, one may write the amplitudes (23) as

$$\begin{aligned}&\mathcal{A}(D_3;\zeta_1,\zeta_2,\phi_3+C_3)\\&\sim\frac{T_3}{2}(\mathcal{F}_1^T)_{ab}\delta\mathcal{M}_3\mathcal{F}_2^{ab}(1+\alpha_2t^2E_1(\phi_0,C_0)\\&\quad+\alpha_3t^3E_{3/2}(\phi_0,C_0)+\cdots),\end{aligned} \tag{27}$$

which is manifestly invariant under the linear $SL(2,R)$ transformation on the quantum fluctuations and is invariant under the nonlinear $SL(2,Z)$ transformations on the background fields ϕ_0, C_0.

IV. FOUR GAUGE FIELDS AMPLITUDE

The disk-level scattering amplitude of four gauge bosons on the world volume of a single D$_3$-brane and for 1234 ordering of the vertex operators is calculated in [52]. In the Einstein frame, it is

$$\begin{aligned}&\mathcal{A}_{1234}\sim T_3\alpha'^2K(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\\&\times\frac{\Gamma(-se^{-\phi_0/2})\Gamma(-te^{-\phi_0/2})}{\Gamma(1-se^{-\phi_0/2}-te^{-\phi_0/2})}\delta^4(k_1^a+k_2^a+k_3^a+k_4^a),\end{aligned} \tag{28}$$

where $s=-2\alpha'k_1\cdot k_2$, $t=-2\alpha'k_1\cdot k_4$, and the kinematic factor is [52]

$$\begin{aligned}K=&-e^{-2\phi_0}k_1\cdot k_2(\zeta_1\cdot k_4\zeta_3\cdot k_2\zeta_2\cdot\zeta_4\\&+\zeta_2\cdot k_3\zeta_4\cdot k_1\zeta_1\cdot\zeta_3+\zeta_1\cdot k_3\zeta_4\cdot k_2\zeta_2\cdot\zeta_3\\&+\zeta_2\cdot k_4\zeta_3\cdot k_1\zeta_1\cdot\zeta_4)-e^{-2\phi_0}k_2\cdot k_3k_2\cdot k_4\zeta_1\\&\cdot\zeta_2\zeta_3\cdot\zeta_4+\{1,2,3,4\to1,3,2,4\}\\&+\{1,2,3,4\to1,4,3,2\}.\end{aligned} \tag{29}$$

This kinematic factor is symmetric under any permutation of the external states and satisfies the Ward identity associated with the gauge transformation. The α' expansion of the gamma functions is

$$\begin{aligned}&\frac{\Gamma(-se^{-\phi_0/2})\Gamma(-te^{-\phi_0/2})}{\Gamma(1-se^{-\phi_0/2}-te^{-\phi_0/2})}\\&=\frac{e^{\phi_0}}{st}-\frac{\pi^2}{6}-\zeta(3)(s+t)e^{-\phi_0/2}\\&\quad-\frac{\pi^4}{360}(4s^2+st+4t^2)e^{-\phi_0}+\cdots.\end{aligned}$$

The total amplitude includes all noncyclic permutation of the external states, i.e.,

$$\begin{aligned}\mathcal{A}=&\mathcal{A}_{1234}+\mathcal{A}_{1243}+\mathcal{A}_{1324}+\mathcal{A}_{1342}\\&+\mathcal{A}_{1423}+\mathcal{A}_{1432}.\end{aligned} \tag{30}$$

The α' expansion of the amplitude $\mathcal{A}$ can be written as

$$\begin{aligned}\mathcal{A}\sim&T_3\alpha'^2K(\zeta_1,\zeta_2,\zeta_3,\zeta_4)\delta^4(k_1^a+k_2^a+k_3^a+k_4^a)\\&\times\sum_{n=-2}^{\infty}a^{(n)},\end{aligned} \tag{31}$$

where $a^{(n)}$'s are functions of the Mandelstam variables. For the Abelian case in which we are interested, these functions are [53]

$$\begin{aligned}&a^{(-2)}=a^{(-1)}=0,\\&a^{(0)}=-\pi^2,\\&a^{(1)}=0,\\&a^{(2)}=-\frac{\pi^2}{4}(t^2+s^2+u^2)\zeta(2)e^{-\phi_0},\\&a^{(3)}=-\pi^2stu\zeta(3)e^{-3\phi_0/2},\\&a^{(4)}=-\frac{9\pi^2}{48}(s^2+t^2+u^2)^2\zeta(4)e^{-2\phi_0},\cdots,\end{aligned} \tag{32}$$

where $s+t+u=0$.

To show that the amplitude satisfies the Ward identity corresponding to the global S-duality transformations, we first use the observation in [54] that indicates the leading contact terms of $\mathcal{A}$ are reproduced by the quartic terms of the BI action, i.e., $\mathrm{Tr}(F^4)/8-(\mathrm{Tr}(F^2))^2/32$, where the traces are over the world volume of the matrix F_{ab}. Using this observation, one can rewrite the kinematic factor in (31) as

$$K = \mathcal{K}_{1234} + \mathcal{K}_{1243} + \mathcal{K}_{1324} + \mathcal{K}_{1342} + \mathcal{K}_{1423} + \mathcal{K}_{1432}, \tag{33}$$

where

$$\mathcal{K}_{1234} = e^{-2\phi_0}[\tfrac{1}{2}\mathrm{Tr}(F_1F_2F_3F_4) - \tfrac{1}{16}\mathrm{Tr}(F_1F_2)\mathrm{Tr}(F_3F_4) - \tfrac{1}{16}\mathrm{Tr}(F_4F_1)\mathrm{Tr}(F_2F_3)]. \tag{34}$$

Then using the relation (20), one can write K in the following simple form:

$$K = \tfrac{1}{4}[\mathrm{Tr}(\mathcal{F}_1^T\mathcal{M}_0\mathcal{F}_2\mathcal{F}_3^T\mathcal{M}_0\mathcal{F}_4) + \mathrm{Tr}(\mathcal{F}_1^T\mathcal{M}_0\mathcal{F}_3\mathcal{F}_2^T\mathcal{M}_0\mathcal{F}_4) + \mathrm{Tr}(\mathcal{F}_1^T\mathcal{M}_0\mathcal{F}_4\mathcal{F}_2^T\mathcal{M}_0\mathcal{F}_3)], \tag{35}$$

which is invariant under the linear $SL(2,R)$ transformation. Therefore, the amplitude (30) at order α'^2 is invariant under the linear $SL(2,R)$ transformation. Apart from the dilaton factors, all higher order terms are the derivatives of the leading order terms which are then invariant under the linear S duality. The dilaton factor $\zeta(2)e^{-\phi_0}$ in the α'^4-order terms may be extended to the regularized nonholomorphic Eisenstein series $E_1(\phi_0, C_0)$ after including the loop and the nonperturbative effects [16,33]. The dilaton factor $\zeta(3)e^{-3\phi_0/2}$ in the α'^5-order terms may be extended to the nonholomorphic Eisenstein series $E_{3/2}(\phi_0, C_0)$ [34], and so on.

The α' expansion of the S-matrix element of two gauge fields and two scalars is the same as (31), i.e.,

$$\mathcal{A} \sim T_3\alpha'^2 K(\zeta_1, \zeta_2, \Phi_3, \Phi_4)\delta^4(k_1^a + k_2^a + k_3^a + k_4^a) \times \sum_{n=-2}^{\infty} a^{(n)}. \tag{36}$$

The kinematic factor is the same as the four gauge boson case (29) in which the condition $\Phi \cdot k = 0$ is imposed and an extra factor of e^{ϕ_0} is added because the indices of the scalars is uppercase, i.e., Φ^i, whereas the indices of the gauge fields are lowercase, i.e., A_a. It is shown in (21) that the kinematic factor can be written in linear $SL(2,R)$ invariant form. So the amplitude can be extended to a S-dual form by including the loops and nonperturbative effects as in the case of four gauge bosons.

Finally, the α' expansion of the S-matrix element of four scalars is the same as (31), i.e.,

$$\mathcal{A} \sim T_3\alpha'^2 K(\Phi_1, \Phi_2, \Phi_3, \Phi_4)\delta^4(k_1^a + k_2^a + k_3^a + k_4^a) \times \sum_{n=-2}^{\infty} a^{(n)}. \tag{37}$$

The kinematic factor is the same as the four gauge boson case (29) in which the condition $\Phi \cdot k = 0$ is imposed and an extra factor of $e^{2\phi_0}$ is added. So there is no dilaton in the kinematic factor. The scalars are invariant under the S duality,[4] hence, the kinematic factor is invariant under the S duality. The amplitude can be extended to the $SL(2,Z)$ invariant form as in the case of four gauge fields. This ends our illustration of consistency of the S-matrix elements on the world volume of the Abelian D_3-brane with the Ward identity corresponding to the global S-duality transformations.

V. DISCUSSION

In this paper, by working on some examples, we have shown that the S-matrix elements of n − closed + m − open string vertex operators on the world volume of a single D_3-brane at the leading order of α' can be combined into multiplets which are invariant under the linear $SL(2,R)$ transformations. The extra dilaton factors in the nonleading order terms may become invariant under S duality after including the loops and the nonperturbative effects [16,33].

The S-duality transformations on the quantum fluctuations (classical fields) must be linear (nonlinear) [36]. Moreover, since the S-matrix element of n open strings is totally different from the S-matrix element of n closed strings, the linear S duality should transform open string to open string and closed string to closed string. In all examples we have worked out, the leading α'-order terms of the multiplets are invariant under the above linear $SL(2,R)$ transformations.

The D_3-brane is invariant under the S duality, so the fundamental string excitation of the D_3-brane at weak coupling should transform to the (p,q)-string excitation of the D_3-brane under the $SL(2,Z)$ transformation. Moreover, the external string should transform to the (p,q) string as well. The massless spectrum of these strings is invariant under the linear S duality, e.g., the $C^{(2)}$ of the fundamental closed string at weak coupling maps to $B^{(2)}$ of the closed D string at strong coupling, or the electric components of the gauge field strength F of the fundamental open string at weak coupling maps to the magnetic components of the gauge field strength of the open D string at strong coupling.

In general, one expects an S-matrix element at weak coupling transforms under the linear S duality to another S-matrix element at strong coupling which is different from the original one. Consider, for example, the weak

[4]The invariance of the transverse scalar fields under the S duality for the Abelian case that we are interested in for this paper is as expected because there is no way to construct the combination of the scalars to have either two antisymmetric world-volume indices to contract with the world-volume form under the S duality, or three antisymmetric traverse indices to contract with the transverse volume form under the S duality. For the non-Abelian case, however, the second possibility can occur which has been considered in [55].

coupling S-matrix element (14) which is at zero axion background and has two external massless open and one massless closed fundamental string. This amplitude transforms under the linear S duality to the S-matrix element of two open and one closed D string at strong coupling. At zero axion background, it is given by

$$\begin{aligned}\mathcal{A}(D_3;\zeta_1,\zeta_2,h_3)\\ \sim T_3 e^{-\phi_0}\bigg(F_{1a}^{c}F_{2bc}h_3^{ab}-\frac{1}{4}F_{1ab}F_2^{ab}h_{3c}^{c}\bigg)\\ \times\frac{\Gamma(1-2te^{\phi_0/2})}{[\Gamma(1-te^{\phi_0/2})]^2},\end{aligned} \tag{38}$$

where now the gauge fields and the graviton are the massless modes of the external D strings. The poles of the gamma functions show the open D-string excitation of the D_3-brane [36]. The above amplitude cannot be calculated in the perturbative string theory. The kinematic factors in the two amplitudes are the same; however, the gamma functions are different as expected. The difference between the above amplitude and the amplitude (14) stems from the fact that the original amplitude (14) does not include the loops and the nonperturbative effects. Obviously, if one includes these effects which may be given by (22), then the transformed amplitude would be the same as the original one.

The linear S-duality invariance of the leading α'-order terms of the S-matrix elements indicates that the low energy effective field theory on the world volume of the D_3-brane may not be invariant under the linear S duality. To see this consider the following discussion: The leading α'-order terms of the S-matrix element (31) are reproduced by the F^4 terms of the BI action [54]. Hence, the F^4 terms of BI action are invariant under the linear S-duality transformations. On the other hand, up to a total derivative term the F^2 term of this action which is not invariant under the S duality is zero when the gauge field is on shell. So the on-shell BI action is invariant under the linear S duality up to F^4 terms. What happens for the F^6 and the higher order terms? Are they invariant under the linear S duality as well? Consider the expansion of the BI action in the Einstein frame, i.e.,

$$\begin{aligned}&\sqrt{-\det(\eta_{ab}+e^{-\phi_0/2}F_{ab})}-1\\ &=-\frac{e^{-\phi_0}}{4}\mathrm{Tr}(F^2)-\frac{e^{-2\phi_0}}{8}\bigg[\mathrm{Tr}(F^4)-\frac{1}{4}(\mathrm{Tr}(F^2))^2\bigg]\\ &\quad-\frac{e^{-3\phi_0}}{12}\bigg[\mathrm{Tr}(F^6)-\frac{3}{8}\mathrm{Tr}(F^2)\mathrm{Tr}(F^4)\\ &\quad+\frac{1}{32}(\mathrm{Tr}(F^2))^3\bigg]+\cdots.\end{aligned} \tag{39}$$

One may use the relations specific to four dimensions to write the above terms in alternative ways. For example, one can show the following relation for F^6 terms:

$$\mathrm{Tr}\,(F^6)-\tfrac{3}{4}\mathrm{Tr}(F^2)\mathrm{Tr}(F^4)+\tfrac{1}{8}(\mathrm{Tr}(F^2))^3=0. \tag{40}$$

Using this relation one may rewrite the F^6 terms in (39) in a different form.

Now, using (20) one finds the following expressions are invariant under the linear $SL(2,R)$ transformation:

$$\begin{aligned}&0=\mathrm{Tr}(\mathcal{F}^T\mathcal{M}_0\mathcal{F}),\\ &4e^{-2\phi_0}\bigg[\mathrm{Tr}(F^4)-\frac{1}{4}(\mathrm{Tr}(F^2))^2\bigg]\\ &\quad=\mathrm{Tr}(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}),\\ &8e^{-3\phi_0}\bigg[\mathrm{Tr}(F^6)-\frac{3}{4}\mathrm{Tr}(F^2)\mathrm{Tr}(F^4)+\frac{1}{8}(\mathrm{Tr}(F^2))^3\bigg]\\ &\quad=\mathrm{Tr}(\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}\mathcal{F}^T\mathcal{M}_0\mathcal{F}).\end{aligned} \tag{41}$$

The first two terms are the same as the first two terms in the on-shell BI action (39). However, the F^6 terms in (41) are not the same as the corresponding terms in (39). In fact, they add up to zero. More generally, one can show that the couplings which are invariant under the linear S duality are

$$\begin{aligned}&\mathrm{Tr}(\overbrace{\mathcal{F}^T\mathcal{M}_0\mathcal{F}\cdots\mathcal{F}^T\mathcal{M}_0\mathcal{F}}^{(2n-1)})=0,\\ &\mathrm{Tr}(\overbrace{\mathcal{F}^T\mathcal{M}_0\mathcal{F}\cdots\mathcal{F}^T\mathcal{M}_0\mathcal{F}}^{(2n)})=4e^{-2n\phi_0}[\mathrm{Tr}(F^4)\\ &\qquad-\tfrac{1}{4}(\mathrm{Tr}(F^2))^2]^n,\end{aligned} \tag{42}$$

where in the first case the number of $\mathcal{F}^T\mathcal{M}_0\mathcal{F}$ is odd and in the second case it is even. There are no such simple relations for the corresponding terms in (39).

The reason that the F^6 terms of the BI action are not invariant under the linear $SL(2,R)$ transformation is that the S-matrix element of the six gauge field vertex operators at leading order has both contact terms and massless poles. The contact terms are reproduced by the F^6 of the BI action (39), and the massless poles are reproduced by the F^4 terms of (39). The massless pole in the $(k_1+k_2+k_3)^2$ channel, for example, is given by the following Feynman amplitude:

$$\mathcal{A}_{123}=V^a(\zeta_1,\zeta_2,\zeta_3,A)G_{ab}(A)V^b(A,\zeta_4,\zeta_5,\zeta_6), \tag{43}$$

where the propagator can be read from the F^2 term of (39) and the vertices can be read from the F^4 terms of (39). Neither the F^6-massless poles nor the F^6-contact terms are invariant under the linear S duality. According to the S-duality proposal, the combination of these two terms, i.e. (F^6 − massless poles + F^6 − contact terms), must be invariant under the linear S duality.[5] A similar observation has been made in [48] in trying to extend the anomalous

[5]It has been shown in [56,57] that the tree-level scattering amplitudes generated by BI action conserve helicity which might be a result of the S duality.

Chern-Simons couplings at order α'^2 to be consistent with the linear T duality. In that case also one observes that only the combination of the contact terms and massless poles is fully invariant under the T duality.

One may find the F^{2n} terms of the BI action by imposing the linear S duality of ($F^{2n}-$massless poles $+ F^{2n} -$ contact terms). Starting from F^4 terms which are invariant under the linear S duality, one can construct F^6-massless poles. Then imposing the linear S duality, one may find F^6-contact terms. Using the F^4- and the F^6-contact terms, one can calculate the F^8-massless poles. Then using the linear S duality, one may find the F^8-contact terms, and so on. This calculation should confirm the Abelian BI action as the effective action of a single D-brane.

When there is more than one coincident D-brane, the Abelian BI action should be extended to a non-Abelian gauge theory. A non-Abelian extension of the BI action has been proposed in [58,59] which includes the symmetric trace prescription. This proposal works for F^2 and F^4 terms; however, there are various discussions that indicate the $F^6, F^8, \ldots$ terms of the D-branes are not correctly captured by the non-Abelian BI action [60,61]. One may extend the above linear S duality of ($F^{2n}-$massless poles $+ F^{2n} -$ contact terms) to the non-Abelian case to find the F^n terms of the non-Abelian D_3-branes action. It would be interesting then to study the Ward identity corresponding to the global S-duality transformations for the non-Abelian cases.

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