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On the structure of groups whose exterior or tensor square is a p-group

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ABSTRACT

It is well known that if G is a nilpotent (infinite) p-group of bounded exponent, then $G \otimes G$ (resp. $G \wedge G$) is also an (infinite) p-group. We study the converse under some restrictions. We also prove that if G is a finitely generated group with G^{ab} capable, then the finiteness of $G \wedge G$ implies that of G.

1. Introduction and motivation

The tensor square $G \otimes G$ of G is the special case of the non-abelian tensor product of two groups G and H when $G = H$, and G acts on itself by conjugation. More precisely, it is generated by the symbols $g \otimes h$ subject to the relations

$$gg' \otimes h = \left({}^{g}g' \otimes {}^{g}h\right)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)\left({}^{h}g \otimes h'\right)$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. Some classical results on this topic can be found in [2,3].

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The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1_\otimes$ on $G \otimes G$. The image of $g \otimes h$ in $G \wedge G$ is denoted by $g \wedge h$ for all $g, h \in G$. From the defining relation of $G \otimes G$, it follows that the commutator map $\kappa : G \otimes G \to [G, G]$ given by $\kappa(g \otimes h) = [g, h]$ is a homomorphism. Clearly, κ has all elements $g \otimes g$ $(g \in G)$ in its kernel, hence it induces a homomorphism $k' : G \wedge G \to G'$. The kernel is $\mathcal{M}(G)$, the Schur multiplier of G (for more information, see [2,3]).

The tensor square and exterior square of G inherit many properties from G; for example, if G is finite, a p-group, nilpotent, solvable, polycyclic, or locally finite, then so are $G \otimes G$ and $G \wedge G$ [5,9–11,13].

The main part of this paper answers the following question: when $G \otimes G$ (resp. $G \wedge G$) is a p-group what are the possible structures for G?

We also prove that for a group G with finitely generated abelianization, $G \otimes G$ is a p-group if and only if G is a p-group. We show that the condition that G^{ab} be finitely generated is essential and cannot be removed.

The structure of a group whose exterior square is a p-group, is a semidirect product of a p-group by a cyclic one, but the conditions under which we can deduce this fact are more general than that for the tensor square.

Recall that a group G is called capable if $G \cong E/Z(E)$ for some group E. It was proved in [6, Theorem 4] that G is capable if and only if the exterior center subgroup $Z^\wedge(G)$ is trivial, where

$$Z^\wedge(G) = \{g \in Z(G)\colon\ g \wedge x = 1 \text{ for all } x \in G\}.$$

By the result of [6], the exterior center coincides with the epicenter $Z^*(G)$ which is defined as

$$\bigcap \{\phi Z(E)\colon\ (E, \phi) \text{ is a central extension}\}.$$

Beyl et al. defined the epicenter of a group G, and they showed in [1, Corollary 2.2] that $Z^*(G)$ is intersection of all normal subgroups N of G such that G/N is capable.

As we mentioned, the finiteness of G implies that of $G \wedge G$. But the converse is not true in general. Here we prove for a finitely generated group G, the finiteness of $G \wedge G$ implies the finiteness of G provided that G^{ab} is a capable group. Note that the group $\mathbb{Z} \oplus \mathbb{Z}_2$ shows the capability of G^{ab} is essential.

The finiteness of $G \otimes G$ implies that of a finitely generated group G without any conditions on the abelianization, but through an example we will show the condition finitely generated is essential and cannot be removed.

To obtain the main results, we need some known results which will appear without proof. The following lemma follows from [2, Proposition 10].

Lemma 1.1. *Let G_1 and G_2 be arbitrary groups. Then*

$$(G_1 \times G_2) \wedge (G_1 \times G_2) \cong (G_1 \wedge G_1) \times (G_2 \wedge G_2) \times \left(G_1^{ab} \otimes G_2^{ab}\right).$$

An immediate consequence of Lemma 1.1, is the following lemma.

Lemma 1.2. *Let G_1 and G_2 be two groups with coprime exponents. Then*

$$Z^\wedge(G_1 \times G_2) = Z^\wedge(G_1) \times Z^\wedge(G_2).$$

2. The tensor and exterior square and p-groups

In the present section, we concentrate on the p-group property and prove a couple of results involving the tensor and exterior square. The following lemma is useful in the next investigation.

Lemma 2.1. *Let G be a finite group whose commutator subgroup is a p-group for some prime p. Then G is a semidirect product of P, the unique Sylow p-subgroup of G, by H in which $|H|$ and p are coprime.*

Proof. If G is a finite group and P is a normal Sylow subgroup, then G splits over P (see for example [8, Theorem 3.8]). Since G' is a p-group, the p-Sylow subgroup of G is normal and the result follows. $\square$

We recall a result of Tahara [12, Corollary 2.2.6]:

Theorem 2.2 *(Tahara 1972). Let G be the semidirect product of N by H, where $|N|$ and $|H|$ are coprime. Then*

$$\mathcal{M}(G) \cong \mathcal{M}(N)^H \oplus \mathcal{M}(H),$$

where $\mathcal{M}(N)^H$ is the H-stable subgroup of $\mathcal{M}(N)$.

The next theorem is about the exterior square of finite groups.

Theorem 2.3. *Let G be a finite group. Then $G \wedge G$ is a p-group if and only if G is a semidirect product of a p-group by a cyclic group of order coprime to p.*

Proof. First assume that $G \wedge G$ is a p-group. Since G' and $\mathcal{M}(G)$ are isomorphic to factor group and a subgroup of $G \wedge G$, respectively, they are p-groups. By invoking Lemma 2.1, G is a semidirect product of P by H in which $|H|$ and p are coprime. By Theorem 2.2, we have $\mathcal{M}(G) \cong \mathcal{M}(N)^H \oplus \mathcal{M}(H)$, and hence $\mathcal{M}(H) = 0$. Since H is finite abelian, H is cyclic via [12, Proposition 2.1.1(ii) and Corollary 2.2.12].

Conversely, let G be the semidirect product of a p-group P by a cyclic group C of order coprime to p; then Theorem 2.2 implies that $\mathcal{M}(G) \cong \mathcal{M}(P)^C$ is a p-group. On the other hand, $G/P \cong C$, which implies G' is a p-group. Since finite p-groups are closed with respect to forming extensions, $G \wedge G$ also is a p-group. $\square$

The last theorem shows that in finite case, if $G \wedge G$ is a p-group, then G need not be a p-group. In this case even in general, $G/Z^\wedge(G)$ is not a p-group.

The following theorem states conditions for a finite group G to conclude $G/Z^\wedge(G)$ being a p-group.

Theorem 2.4. *Let G be a finite group such that $G \wedge G$ is a p-group. Then $G/Z^\wedge(G)$ is p-group if and only if G is nilpotent.*

Proof. For a given group G, if $G/Z^\wedge(G)$ is a p-group, then so is its epimorphic image $G/Z(G)$. Therefore G is nilpotent.

Conversely, when G is nilpotent, it can be written as $P \times H$ where P is the Sylow p-subgroup of G and H is the product of all other Sylow subgroups. By applying Lemma 1.1,

$$G \wedge G \cong P \wedge P \times H \wedge H.$$

Since $G \wedge G$ is a p-group, we have $H \wedge H = 0$. So H is abelian and $\mathcal{M}(H) = H \wedge H$. This implies H is cyclic by [12, Proposition 2.1.1(ii) and Corollary 2.2.12]. Hence we have $G \cong P \times \mathbb{Z}_n$ for some n coprime to p. Invoking Lemma 1.2, we have

$$Z^\wedge(G) = Z^\wedge(P) \times Z^\wedge(\mathbb{Z}_n) = Z^\wedge(P) \times \mathbb{Z}_n.$$

Therefore $G/Z^\wedge(G) = P/Z^\wedge(P)$ is a p-group, as claimed. $\square$

The infinite case is not as straightforward as the finite case. In the following lemma and theorem we assume that $G \wedge G$ is a p-group and try to describe the structure of G. Of course we need to impose some restrictions on G, however we will show this restriction is essential. First for the abelian case we have:

Lemma 2.5. *Let G be a finitely generated abelian group such that $G \wedge G$ is a p-group, then G is the direct sum of a finite p-group and a cyclic group either of infinite order or of finite order coprime to p.*

Proof. Suppose that $G = T \oplus F$ in which T is a finite abelian group and F is a free abelian group of finite rank. Since $G \wedge G = T \wedge T \times F \wedge F \times T \otimes F$ is a p-group, it is easy to see that either $\operatorname{rank} F = 1$ and T is a p-group, or $\operatorname{rank} F = 0$. In the latter case decompose T into direct sum of its Sylow subgroups and write $T = T_p \times C$ where p and the order of C are coprime. Again we have $G \wedge G = T_p \wedge T_p \times C \wedge C \times T_p \otimes C$ which is a p-group, so we must have C to be cyclic. □

Theorem 2.6. *Let G be a group such that G^{ab} is finitely generated. If $G \wedge G$ is a p-group, then G is the semidirect product of a finite p-group by a cyclic group either of infinite order or of finite order coprime to p.*

Proof. Using Lemma 2.5, we have $G^{ab} = P^* \oplus C$ in which P^* is a finite p-group and C is a cyclic group either infinite or of order coprime to p. Let P be the preimage of P^* in G. Since G' is a p-group, P is a p-group. Now it is easy to see that $G = C \ltimes P$, as desired. □

The following example shows that the condition on G^{ab} to be finitely generated is essential.

Example 2.7. Let $G = \mathbb{Z}(p^\infty) \times \mathbb{Z}_q \times \mathbb{Z}_q$ in which p and q are distinct primes and $\mathbb{Z}(p^\infty)$ is a quasicyclic p-group. Since $\mathbb{Z}(p^\infty)$ is a divisible torsion group, we have $\mathbb{Z}(p^\infty) \otimes \mathbb{Z}_q = \mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p^\infty) = 0$ so $G \wedge G \cong \mathbb{Z}_q$ is a q-group, but G is not as the form introduced in Theorem 2.6.

To prove a converse of Theorem 2.6 we need to put some extra restrictions on groups. Recall from [7] the relative Schur multiplier of a pair of groups (G, N) is denoted by $\mathcal{M}(G, N)$. In the next contribution we need the following lemma whose proof can be found in [7].

Lemma 2.8. *Let G be the semidirect product of N by Q. Then*

(i) $\mathcal{M}(G) \cong \mathcal{M}(G, N) \oplus \mathcal{M}(Q)$,
(ii) $\mathcal{M}(G, N) \cong \ker(\mu : \mathcal{M}(G) \twoheadrightarrow \mathcal{M}(Q))$.

We also need the following theorem which comes from [4, Proposition 2.8].

Theorem 2.9. *Let G be a group and N a locally finite normal subgroup of G. If exponent of N is n, then the exponent of $\mathcal{M}(G, N)$ is n-bounded.*

We are in a position to decide whether for the groups in Theorem 2.6, the exterior square is a p-group. But the conditions differ as follows.

Theorem 2.10. *Let G be the semidirect product of a p-group P by an abelian group C. If $\mathcal{M}(C) = 0$, P/G' is of finite exponent, and G' is locally finite of finite exponent, then $G \wedge G$ is a p-group.*

Proof. Since C is abelian we have $G' \subseteq P$. P is locally finite of finite exponent because G' is locally finite and P/G' is abelian of finite exponent. Now Lemma 2.8 shows that $\mathcal{M}(G) \cong \mathcal{M}(G, P)$. By Theorem 2.9, the exponent of $\mathcal{M}(G, P)$ is p^α-bounded where p^α is the exponent of P. This implies $\mathcal{M}(G, P)$ to be a p-group. Therefore $\mathcal{M}(G)$ is a p-group and since G' is a p-group, $G \wedge G$ is a p-group. □

We can combine Theorems 2.6 and 2.10 to obtain the following.

Theorem 2.11. *Let G be a group with the following properties*:

(i) G' *is of finite exponent and locally finite*;
(ii) G/G' *is finite.*

Then $G \wedge G$ is a p-group if and only if $G = C \ltimes P$ where P is a finite p-group and C is a cyclic group of infinite order or of order coprime to p.

The analogous theorem to Theorem 2.11 for $G \otimes G$ gives a more restrictive structure for G. Again we start with abelian groups.

Lemma 2.12. *If G is a finitely generated abelian group and $G \otimes G$ is a p-group, then G is a finite p-group.*

Proof. By fundamental theorem of finitely generated abelian groups $G = T \oplus F$ in which T is a finite abelian group and F is a free abelian group of finite rank. Since $G \otimes G = T \otimes T \times F \otimes F \times T \otimes F \times F \otimes T$ is a p-group, we have $\operatorname{rank} F = 0$. Decompose T into direct sum of its Sylow subgroups and write $T = T_p \oplus C$ where p and the order of C are coprime. We have $T \otimes T = T_p \otimes T_p \times C \otimes C \times T_p \otimes C \times C \otimes T_p$. But $G \otimes G$ is a p-group, so we must have $C = 0$, hence G is a p-group, as required. □

We know that $G \otimes G$ (resp. $G \wedge G$) is an (infinite) p-group, when G is a nilpotent (infinite) p-group of bounded exponent. The following theorem shows the converse is true in some cases.

Theorem 2.13. *Let G be a group. If G^{ab} is finitely generated and $G \otimes G$ is a p-group, then so is G.*

Proof. Since $G^{ab} \otimes G^{ab}$ is a p-group, by Lemma 2.12 G^{ab} is a p-group. The commutator map κ shows that G', the derived subgroup of G, is also a p-group. Hence G itself is a p-group, as required. □

The following example shows the condition on G^{ab} to be finitely generated is essential and cannot be removed.

Example 2.14. Let $G = \mathbb{Z}(p^\infty) \times \mathbb{Z}_q$ in which p and q are distinct primes and $\mathbb{Z}(p^\infty)$ is a quasicyclic p-group. Since $\mathbb{Z}(p^\infty)$ is a divisible torsion group, we have $\mathbb{Z}(p^\infty) \otimes \mathbb{Z}_q = \mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p^\infty) = 0$, so $G \otimes G \cong \mathbb{Z}_q$ is a q-group but G is not a q-group.

3. Finiteness conditions for exterior and tensor square

In this section, we intend to consider the finiteness of $G \otimes G$ and $G \wedge G$. More precisely we want to know whether the finiteness of $G \otimes G$ or $G \wedge G$ implies the finiteness of G. The answer for $G \otimes G$ is always yes, provided that G^{ab} being finitely generated; that is, we will prove for a group G with finitely generated abelianization, the finiteness of $G \otimes G$ implies the finiteness of G. We also show that this condition is essential. But in the case of exterior square we need some further conditions; in general case the finiteness of $G \wedge G$ does not imply that of G ($\mathbb{Z} \oplus \mathbb{Z}_2$ shows it). Instead we will prove the finiteness of $G \wedge G$ implies $G/Z^\wedge(G)$ to be finite, and then we state conditions which lead to the finiteness of G. Among the results some inequalities between the order of $G \wedge G$ and the minimal number of the generators of $G/Z^\wedge(G)$ will be presented. The first result states the finiteness of $G \otimes G$ implies the finiteness of G in the finitely generated case.

Theorem 3.1. *Let G be a finitely generated group such that $G \otimes G$ is a finite group, then so is G.*

Proof. First assume that G is an abelian group. Write G as a direct sum of cyclic groups; since the tensor product distributes over the direct sum and $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{gcd(m,n)}$, it follows that if $G \otimes G$ is finite then so is G. Now in general case in which G need not be abelian, the epimorphism $G \to G^{ab}$ induces an epimorphism $G \otimes G \to G^{ab} \otimes G^{ab}$ which forces $G^{ab} \otimes G^{ab}$ to be finite. By the first part G^{ab} is a finite group and the commutator map κ shows the finiteness of G' and hence G is finite, as desired. □

Example 3.2. Similar to Example 2.14, we have $\mathbb{Z}(p^\infty) \otimes \mathbb{Z}(p^\infty) = 0$ and so it is a finite group but $\mathbb{Z}(p^\infty)$ is not a finite group.

In what follows, we use the exterior centralizer of g, which is defined by

$$C_G^\wedge(g) = \{h \in G \mid g \wedge h = 1_{G\wedge G}\}.$$

To prove theorems about the exterior square some preliminaries are needed.

Lemma 3.3. *Let $G/Z^\wedge(G)$ be finitely generated. If $\{g_1, \ldots, g_k\}$ generates G modulo $Z^\wedge(G)$, then*

$$Z^\wedge(G) = \bigcap_{i=1}^{k} C_G^\wedge(g_i).$$

Proof. Straightforward. □

We use the next theorem only in the finite case, but we state it in general case.

Lemma 3.4. *Let G be any group, then for all $g \in G$*

$$\left[G : C_G^\wedge(g)\right] \leqslant |G \wedge G|,$$

in which the inequality is between cardinals.

Proof. Let X be a right transversal to $C_G^\wedge(g)$ in G. Define $\phi : X \to G \wedge G$ by $\phi(x) = x \wedge g$. By proving that ϕ is one-to-one the result will follow. If $\phi(x_1) = \phi(x_2)$ then using the defining relation of $G \wedge G$ we have $x_1 x_2^{-1} \wedge g = ((x_1 \wedge g)(x_2 \wedge g)^{-1})^{x_2^{-1}} = 1$, so $x_1 x_2^{-1} \in C_G^\wedge(g)$ and hence $x_1 = x_2$. □

Using Lemmas 3.3 and 3.4 we have

Theorem 3.5. *Let G be a group with $G/Z^\wedge(G)$ finitely generated. If $G \wedge G$ is finite, then so is $G/Z^\wedge(G)$. Moreover $|G/Z^\wedge(G)| \leqslant |G \wedge G|^{d(G/Z^\wedge(G))}$ where $d(G/Z^\wedge(G))$ is the number of minimal generators of $G/Z^\wedge(G)$.*

It must be noted that the upper bound for the order of $G/Z^\wedge(G)$ is sharp. See the following two examples.

Example 3.6. It is known that for $G = Q_8$, $G \wedge G \cong \mathbb{Z}_2$ and $G/Z^\wedge(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $|G/Z^\wedge(G)| = |G \wedge G|^{d(G/Z^\wedge(G))}$ in this case.

Example 3.7. If $G = \mathbb{Z}_9 \times \mathbb{Z}_3$, then $G \wedge G = \mathbb{Z}_3$ and $G/Z^\wedge(G) = \mathbb{Z}_3 \times \mathbb{Z}_3$, so again in this case we have $|G/Z^\wedge(G)| = |G \wedge G|^{d(G/Z^\wedge(G))}$.

Remark 3.8. In case of nilpotent groups, we can say a little more. For p-groups in the above theorem one may replace "$\leqslant$" with "divides". Now as finite nilpotent groups are the direct product of their Sylow p-subgroups, by virtue of Lemmas 1.1 and 1.2 we can replace "$\leqslant$" with "divides" if G is nilpotent.

Considering the above remark we proved

Theorem 3.9. *Let G be a group with $G/Z^\wedge(G)$ being finitely generated; if $G \wedge G$ is finite then so is $G/Z^\wedge(G)$. Furthermore if G is nilpotent then $G/Z^\wedge(G)$ divides $|G \wedge G|^{d(G/Z^\wedge(G))}$.*

Finally, we present some results on the finiteness of G rather than that of $G/Z^\wedge(G)$.

Theorem 3.10. *Let G be a finitely generated group and G^{ab} is capable. If $G \wedge G$ is finite, then so is G.*

Proof. Since $G \wedge G$ is finite G' is finite. $Z^\wedge(G)$ is the intersection of all normal subgroup of G with capable factor, so the capability of G^{ab} implies $Z^\wedge(G) \subseteq G'$, thus G^{ab} is finite, and the result holds. □

Taking a look at the proof of Theorem 3.10, one may easily see that the capability of G^{ab} is used to show the finiteness of some quotient of G by a finite normal subgroup N (which is G' in the proof), so the following theorems can be proved in the same way.

Theorem 3.11. *If G is a finitely generated group and $G/\gamma_{c+1}(G)$ is capable. Then if $G \wedge G$ is finite then G is finite.*

Theorem 3.12. *If G is a finitely generated group and $G/\delta^n(G)$ is capable. Then if $G \wedge G$ is finite then G is finite.*

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