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# Solving infinite horizon nonlinear optimal control problems using an extended modal series method

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**Abstract:** This paper presents a new approach for solving a class of infinite horizon nonlinear optimal control problems (OCPs). In this approach, a nonlinear two-point boundary value problem (TPBVP), derived from Pontryagin's maximum principle, is transformed into a sequence of linear time-invariant TPBVPs. Solving the latter problems in a recursive manner provides the optimal control law and the optimal trajectory in the form of uniformly convergent series. Hence, to obtain the optimal solution, only the techniques for solving linear ordinary differential equations are employed. An efficient algorithm is also presented, which has low computational complexity and a fast convergence rate. Just a few iterations are required to find an accurate enough suboptimal trajectory-control pair for the nonlinear OCP. The results not only demonstrate the efficiency, simplicity, and high accuracy of the suggested approach, but also indicate its effectiveness in practical use.

**Key words:** Infinite horizon nonlinear optimal control problem, Pontryagin's maximum principle, Two-point boundary value problem, Extended modal series method
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## 1 Introduction

Optimal control is one of the most active research topics in control theory and has a wide range of applications in different fields such as physics, economy, aerospace, chemical engineering, and robotics (Garrard and Jordan, 1977; Manousiouthakis and Chmielewski, 2002; Notsu *et al.*, 2008; Tang *et al.*, 2009). For linear time-invariant systems, the theory and application of optimal control are highly developed (Bryson, 2002; Yousefi *et al.*, 2010). However, optimal control of nonlinear systems is much more challenging and has been studied extensively for decades.

To solve nonlinear optimal control problems (OCPs), many computational techniques have been developed. One familiar scheme is the state-dependent Riccati equation (SDRE) method (Cimen, 2008). Although this technique has been widely used in various applications, its major limitation is that it requires solving a sequence of matrix Riccati algebraic equations. This property may use up a lot of computing time and memory space. Another scheme is called the approximating sequence of Riccati equations (ASRE) (Banks and Dinesh, 2000). From a practical point of view, the ASRE is attractive; however, it suffers from computational complexity since it requires solving a sequence of time-varying matrix Riccati differential equations.

For solving nonlinear OCPs, a numerical approach has been suggested by Huang and Lin (1995) and Abu-Khalaf *et al.* (2006). This approach finds the Taylor series solution of the Hamilton-Jacobi-Isaacs (HJI) equation associated with the nonlinear $H_\infty$ control problem. The coefficients of the Taylor series are generated by solving one Riccati algebraic equation and a sequence of linear algebraic equations. However, deriving each linear equation in the sequence requires a number of matrix computations, which may introduce computational complexity.

To determine the optimal control law, there is another approach using dynamic programming (Bellman, 1952). This approach leads to the Hamilton-Jacobi-Bellman (HJB) equation which in most cases is hard to solve. An excellent literature review on the methods for solving the HJB equation was provided by Beard *et al*. (1997) who also considered a successive Galerkin approximation (SGA) approach. In the SGA, a sequence of generalized HJB equations is solved iteratively to obtain a sequence of approximations leading eventually to the solution of the HJB equation. However, the proposed sequence may converge very slowly or even diverge.

The optimal control law can also be derived using Pontryagin's maximum principle (Pontryagin, 1959). For nonlinear OCPs, this approach leads to a nonlinear two-point boundary value problem (TPBVP) that unfortunately, in general, cannot be solved analytically. Therefore, many researchers have tried to find an approximate solution for nonlinear TPBVPs (Ascher *et al.*, 1995). In recent years some better results have been obtained. For instance, a new successive approximation approach (SAA) was proposed by Tang (2005) where, instead of directly solving the nonlinear TPBVP derived from the maximum principle, a sequence of nonhomogeneous linear time-varying TPBVPs is solved iteratively. The sensitivity approach, proposed by Tang *et al.* (2002), is similar to the SAA. It requires solving iteratively only a sequence of nonhomogeneous linear time-varying TPBVPs to determine the infinite series presenting the optimal control law. Note that solving time-varying equations is much more difficult than solving time-invariant ones.

Recently, a novel numerical technique was proposed for solving nonlinear OCPs, which is based on differential transformation (DT) (Hwang *et al.*, 2009). Using DT, a nonlinear TPBVP, derived from the maximum principle, is transformed into a system of nonlinear algebraic equations. Then, through inverse DT, the optimal solution is obtained in the form of a finite-term Taylor series. A system of nonlinear algebraic equations can be solved numerically by various methods such as Newton's method or the nonlinear least square method. Hwang *et al*. (2009) used a trust region Newton's method for solving such a system of nonlinear algebraic equations. Unfortunately, Newton's method should start from a good initial guess, sufficiently close to the exact solution. Otherwise, the convergence of this method cannot be ensured.

In recent years, a new technique, called the modal series method, has been developed in the field of nonlinear system analysis (Pariz, 2001; Pariz *et al.*, 2003; Shanechi *et al.*, 2003; Wu *et al.*, 2007; Khatibi and Shanechi, 2011). This method, which was initially introduced by Pariz (2001), provides the solution of autonomous nonlinear systems in terms of fundamental and interacting modes. This solution, which is called modal series, yields a good deal of physical insight into the system behavior. In contrast to the perturbation method (Murdock, 1999), the modal series method does not depend on the small/large physical parameters in the system model. In addition, unlike the traditional non-perturbation techniques, such as Lyapunov's artificial small parameter method (Lyapunov, 1892) and Adomian's decomposition method (Adomian and Adomian, 1984), the modal series converges uniformly to the exact solution.

The aim of this paper is to extend the modal series method to solve a class of infinite horizon nonlinear OCPs. By this extension, a nonlinear TPBVP, derived from the maximum principle, is transformed into a sequence of linear time-invariant TPBVPs. Solving the latter problems in a recursive manner leads to the optimal control law and the optimal trajectory in the form of uniformly convergent series.

## 2 Statement of the problem

Consider an infinite horizon nonlinear OCP described by

$$\begin{aligned} &\min\ J=\frac{1}{2}\int_{t_0}^{\infty}(\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t)+\boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{u}(t))\,\mathrm{d}t \\ &\text{s.t.}\quad \begin{cases}\dot{\boldsymbol{x}}(t)=\boldsymbol{F}(\boldsymbol{x}(t))+\boldsymbol{B}\boldsymbol{u}(t),\ \ t>t_0,\\ \boldsymbol{x}(t_0)=\boldsymbol{x}_0,\end{cases} \end{aligned} \tag{1}$$

where $\boldsymbol{x}\in\mathbb{R}^n$ and $\boldsymbol{u}\in\mathbb{R}^m$ are the state and control vectors respectively, $\boldsymbol{Q}\in\mathbb{R}^{n\times n}$ and $\boldsymbol{R}\in\mathbb{R}^{m\times m}$ are positive semi-definite and positive definite matrices respectively, $\boldsymbol{F}$: $\mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear analytic vector field where $\boldsymbol{F}(\mathbf{0})=\mathbf{0}$ (and hence $\boldsymbol{x}=\mathbf{0}$ is an equilibrium point of the system), $\boldsymbol{B}$ is a constant matrix of appropriate dimension, and $\boldsymbol{x}_0 \in \mathbb{R}^n$ is the initial state vector. Also, it is assumed that the pair $(\boldsymbol{J}_{\boldsymbol{F}}(\mathbf{0}), \boldsymbol{B})$ is controllable and the pair $(\boldsymbol{J}_{\boldsymbol{F}}(\mathbf{0}), \mathrm{sqrt}(\boldsymbol{Q}))$ is observable, where $\boldsymbol{J}_{\boldsymbol{F}}$ is the Jacobian matrix of $\boldsymbol{F}$, and $\mathrm{sqrt}(\boldsymbol{Q})$ is the square root of matrix $\boldsymbol{Q}$. These assumptions guarantee the existence of a smooth optimal solution on a certain region of state space containing the equilibrium point (McCaffrey and Banks, 2005).

According to Pontryagin's maximum principle, the optimality conditions are obtained as the following nonlinear TPBVP:

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \boldsymbol{F}(\boldsymbol{x}(t)), \\ \dot{\boldsymbol{\lambda}}(t) = -\boldsymbol{Q}\boldsymbol{x}(t) - \boldsymbol{\Psi}(\boldsymbol{x}(t), \boldsymbol{\lambda}(t)), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0,\ \boldsymbol{\lambda}(\infty) = \mathbf{0}, \end{cases} \tag{2}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^n$ is the co-state vector, and $\boldsymbol{\Psi}(\boldsymbol{x}(t), \boldsymbol{\lambda}(t)) = \left(\dfrac{\partial \boldsymbol{F}(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\right)^{\mathrm{T}} \boldsymbol{\lambda}(t)$. The optimal control law is given by

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t), \quad t > t_0. \tag{3}$$

**Lemma 1** The solution of the nonlinear TPBVP (2) is analytic with respect to $\boldsymbol{x}_0$.

**Proof** Let the pair $(\boldsymbol{x}(\cdot), \boldsymbol{\lambda}(\cdot))$ be a solution of TPBVP (2). Define $(\boldsymbol{x}_0, \boldsymbol{\lambda}_0)=(\boldsymbol{x}(t_0), \boldsymbol{\lambda}(t_0))$. Then, $(\boldsymbol{x}(\cdot), \boldsymbol{\lambda}(\cdot))$ is the solution of the following initial value problem (IVP):

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \boldsymbol{F}(\boldsymbol{x}(t)), \\ \dot{\boldsymbol{\lambda}}(t) = -\boldsymbol{Q}\boldsymbol{x}(t) - \boldsymbol{\Psi}(\boldsymbol{x}(t), \boldsymbol{\lambda}(t)), \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0,\ \boldsymbol{\lambda}(t_0) = \boldsymbol{\lambda}_0. \end{cases} \tag{4}$$

Since the nonlinear terms in Eq. (4) are analytic, $(\boldsymbol{x}(\cdot), \boldsymbol{\lambda}(\cdot))$, as the solution of IVP (4), is analytic with respect to $\boldsymbol{x}_0$ (Arnold, 1992). Thus, $(\boldsymbol{x}(\cdot), \boldsymbol{\lambda}(\cdot))$ as the solution of TPBVP (2) is analytic with respect to $\boldsymbol{x}_0$, and the proof is complete.

Unfortunately, problem (2) is a nonlinear TPBVP that can be solved analytically only in a few simple cases. To overcome this difficulty, we will extend the modal series method in the next section.

## 3 Extending the modal series method and optimal control design strategy

The modal series method was originally proposed and further improved to solve a class of IVPs of the form $\dot{\boldsymbol{x}}(t) = \boldsymbol{F}(\boldsymbol{x}(t))$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ (Pariz, 2001; Pariz *et al.*, 2003; Shanechi *et al.*, 2003; Khatibi and Shanechi, 2011). In this section, we extend this method for solving a nonlinear TPBVP (2). To this end, first we need a Taylor series expansion of the nonlinear non-polynomial terms. Therefore, if the nonlinear terms in Eq. (2), i.e., $\boldsymbol{F}(\boldsymbol{x}(t))$ and $\boldsymbol{\Psi}(\boldsymbol{x}(t), \boldsymbol{\lambda}(t))$, are not polynomial, they should be expanded in the Taylor series around the equilibrium point $(\boldsymbol{x}, \boldsymbol{\lambda})=(\mathbf{0}, \mathbf{0})$, which yields

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) + \boldsymbol{A}_{10}\boldsymbol{x}(t) + \frac{1}{2!}\begin{bmatrix} \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{H}_{20}^{1}\boldsymbol{x}(t) \\ \vdots \\ \boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{H}_{20}^{n}\boldsymbol{x}(t) \end{bmatrix} + \cdots, \tag{5a}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\boldsymbol{Q}\boldsymbol{x}(t) - \bar{\boldsymbol{A}}_{01}\boldsymbol{\lambda}(t) - \begin{bmatrix} \boldsymbol{x}^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^{1}\boldsymbol{\lambda}(t) \\ \vdots \\ \boldsymbol{x}^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^{n}\boldsymbol{\lambda}(t) \end{bmatrix} - \cdots, \tag{5b}$$

where $\boldsymbol{A}_{10} = \left.\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\mathbf{0}}$, $\boldsymbol{H}_{20}^{i} = \left.\dfrac{\partial^2 \boldsymbol{F}_i}{\partial \boldsymbol{x}^2}\right|_{\boldsymbol{x}=\mathbf{0}}$, $\bar{\boldsymbol{A}}_{01} = \left.\dfrac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{\lambda}}\right|_{\substack{\boldsymbol{x}=\mathbf{0} \\ \boldsymbol{\lambda}=\mathbf{0}}}$, $\bar{\boldsymbol{H}}_{11}^{i} = \left.\dfrac{\partial^2 \boldsymbol{\Psi}_i}{\partial \boldsymbol{\lambda}\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\mathbf{0} \\ \boldsymbol{\lambda}=\mathbf{0}}}$, in which $\boldsymbol{F}_i$ and $\boldsymbol{\Psi}_i$ are the $i$th components of vector fields $\boldsymbol{F}$ and $\boldsymbol{\Psi}$, respectively. Moreover, in the Taylor series of $\boldsymbol{\Psi}$ only the mixed partial derivatives which are of the first order with respect to $\boldsymbol{\lambda}$ are non-zero at the origin.

The solution of the nonlinear TPBVP (2) for arbitrary initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ and for all $t>t_0$ can be expressed as

$$\begin{cases} \boldsymbol{x}(t) = \boldsymbol{\Lambda}(\boldsymbol{x}_0, t), \\ \boldsymbol{\lambda}(t) = \boldsymbol{\Gamma}(\boldsymbol{x}_0, t), \end{cases} \tag{6}$$

where $\boldsymbol{\Lambda}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\boldsymbol{\Gamma}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are analytic vector fields with respect to the initial condition $\boldsymbol{x}_0$ (Lemma 1). In addition, it is easy to show that $\boldsymbol{\Lambda}(\mathbf{0}, t) = \boldsymbol{\Gamma}(\mathbf{0}, t)=\mathbf{0}$, $\forall t>t_0$. Therefore, we can expand Eq. (6) as a Maclaurin series about $\boldsymbol{x}_0$ as follows:

$$\boldsymbol{x}(t)=\boldsymbol{\Lambda}(\boldsymbol{x}_0,t)=\underbrace{\left.\frac{\partial\boldsymbol{\Lambda}(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\boldsymbol{x}_0}_{\boldsymbol{g}_1(t)}+\underbrace{\frac{1}{2!}\begin{bmatrix}\boldsymbol{x}_0^{\mathrm{T}}\left(\left.\frac{\partial^2\Lambda_1(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0^2}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\right)\boldsymbol{x}_0\\ \vdots\\ \boldsymbol{x}_0^{\mathrm{T}}\left(\left.\frac{\partial^2\Lambda_n(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0^2}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\right)\boldsymbol{x}_0\end{bmatrix}}_{\boldsymbol{g}_2(t)}+\cdots=\sum_{i=1}^{\infty}\boldsymbol{g}_i(t), \tag{7a}$$

$$\boldsymbol{\lambda}(t)=\boldsymbol{\Gamma}(\boldsymbol{x}_0,t)=\underbrace{\left.\frac{\partial\boldsymbol{\Gamma}(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\boldsymbol{x}_0}_{\boldsymbol{h}_1(t)}+\underbrace{\frac{1}{2!}\begin{bmatrix}\boldsymbol{x}_0^{\mathrm{T}}\left(\left.\frac{\partial^2\Gamma_1(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0^2}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\right)\boldsymbol{x}_0\\ \vdots\\ \boldsymbol{x}_0^{\mathrm{T}}\left(\left.\frac{\partial^2\Gamma_n(\boldsymbol{x}_0,t)}{\partial\boldsymbol{x}_0^2}\right|_{\boldsymbol{x}_0=\boldsymbol{0}}\right)\boldsymbol{x}_0\end{bmatrix}}_{\boldsymbol{h}_2(t)}+\cdots=\sum_{i=1}^{\infty}\boldsymbol{h}_i(t), \tag{7b}$$

where $\Lambda_i$ and $\Gamma_i$ are the $i$th components of vector fields $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ respectively, and $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ are vector functions in which their components are linear combinations of all terms depending on the multiplication of $i$ elements of vector $\boldsymbol{x}_0$. For example, components of $\boldsymbol{g}_2(t)$ and $\boldsymbol{h}_2(t)$ contain linear combinations of all terms of the form $x_{0,k}x_{0,l}$ for $k, l\in\{1, 2, \ldots, n\}$, where $x_{0,j}$ is the $j$th element of vector $\boldsymbol{x}_0$. Moreover, since $\boldsymbol{\Lambda}$ and $\boldsymbol{\Gamma}$ are analytic with respect to $\boldsymbol{x}_0$, the existence and uniform convergence of the Maclaurin series in Eqs. (7a) and (7b) are guaranteed. Let $\Phi_1$ and $\Phi_2$ be the convergence domains of the Maclaurin series in Eqs. (7a) and (7b) for all $t>t_0$ where $\Phi_1\subseteq\mathbb{R}^n$ and $\Phi_2\subseteq\mathbb{R}^n$ are subsets of the initial state space. Assume that $\Phi_3=\Phi_1\cap\Phi_2$ is non-empty and let the initial condition be $\varepsilon\boldsymbol{x}_0$, i.e., $\boldsymbol{x}(t_0)=\varepsilon\boldsymbol{x}_0$, where $\varepsilon$ is an arbitrary scalar parameter such that $\varepsilon\boldsymbol{x}_0\in\Phi_3$. Since $\Phi_3$ is assumed to be non-empty, such a parameter exists. This parameter simplifies only the calculations and its value does not have any significance. Similar to Eqs. (7a) and (7b), we can write

$$\begin{cases}\boldsymbol{x}(t)=\boldsymbol{\Lambda}(\varepsilon\boldsymbol{x}_0,t)=\sum_{i=1}^{\infty}\varepsilon^i\boldsymbol{g}_i(t),\\ \boldsymbol{\lambda}(t)=\boldsymbol{\Gamma}(\varepsilon\boldsymbol{x}_0,t)=\sum_{i=1}^{\infty}\varepsilon^i\boldsymbol{h}_i(t).\end{cases} \tag{8}$$

Since $\varepsilon\boldsymbol{x}_0\in\Phi_3$, Eq. (8) must satisfy Eqs. (5a) and (5b). Satisfying Eq. (8) in Eqs. (5a) and (5b) and rearranging with respect to the order of $\varepsilon$ yield

$$\begin{cases}\varepsilon^1\dot{\boldsymbol{g}}_1(t)+\varepsilon^2\dot{\boldsymbol{g}}_2(t)+\cdots=\varepsilon^1(\boldsymbol{A}_{10}\boldsymbol{g}_1(t)-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_1(t))\\ \quad+\varepsilon^2\left(\boldsymbol{A}_{10}\boldsymbol{g}_2(t)-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_2(t)+\frac{1}{2!}\begin{bmatrix}\boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^1\boldsymbol{g}_1(t)\\ \vdots\\ \boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^n\boldsymbol{g}_1(t)\end{bmatrix}\right)+\cdots,\\ \varepsilon^1\dot{\boldsymbol{h}}_1(t)+\varepsilon^2\dot{\boldsymbol{h}}_2(t)+\cdots=\varepsilon^1(-\boldsymbol{Q}\boldsymbol{g}_1(t)-\bar{\boldsymbol{A}}_{01}\boldsymbol{h}_1(t))\\ \quad+\varepsilon^2\left(-\boldsymbol{Q}\boldsymbol{g}_2(t)-\bar{\boldsymbol{A}}_{01}\boldsymbol{h}_2(t)-\begin{bmatrix}\boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^1\boldsymbol{h}_1(t)\\ \vdots\\ \boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^n\boldsymbol{h}_1(t)\end{bmatrix}\right)+\cdots.\end{cases} \tag{9}$$

Since Eq. (9) must hold for any $\varepsilon$ as long as $\varepsilon\boldsymbol{x}_0\in\Phi_3$, terms with the same order of $\varepsilon$ on each side must be equal. This procedure yields

$$\varepsilon^1:\begin{cases}\dot{\boldsymbol{g}}_1(t)=\boldsymbol{A}_{10}\boldsymbol{g}_1(t)-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_1(t),\\ \dot{\boldsymbol{h}}_1(t)=-\boldsymbol{Q}\boldsymbol{g}_1(t)-\bar{\boldsymbol{A}}_{01}\boldsymbol{h}_1(t),\end{cases} \tag{10a}$$

$$\varepsilon^2:\begin{cases}\dot{\boldsymbol{g}}_2(t)=\boldsymbol{A}_{10}\boldsymbol{g}_2(t)-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_2(t)+\frac{1}{2!}\begin{bmatrix}\boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^1\boldsymbol{g}_1(t)\\ \vdots\\ \boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^n\boldsymbol{g}_1(t)\end{bmatrix},\\ \dot{\boldsymbol{h}}_2(t)=-\boldsymbol{Q}\boldsymbol{g}_2(t)-\bar{\boldsymbol{A}}_{01}\boldsymbol{h}_2(t)-\begin{bmatrix}\boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^1\boldsymbol{h}_1(t)\\ \vdots\\ \boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^n\boldsymbol{h}_1(t)\end{bmatrix},\end{cases} \tag{10b}$$

and so on. Note that Eq. (10a) is a system of homogeneous linear time-invariant ordinary differential equations (ODEs). From Eq. (10a), $\boldsymbol{g}_1(t)$ and $\boldsymbol{h}_1(t)$ can be easily obtained. Assume that $\boldsymbol{g}_1(t)$ and $\boldsymbol{h}_1(t)$ have been obtained from Eq. (10a) in the first step. Then, $\boldsymbol{g}_2(t)$ and $\boldsymbol{h}_2(t)$ can be easily obtained from Eq. (10b) in the second step since Eq. (10b) is a system of nonhomogeneous linear time-invariant ODEs. Note that the nonhomogeneous terms of Eq. (10b) are calculated using the solution of Eq. (10a). Continuing as above, $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ for $i\geq 2$ can be easily obtained in the $i$th step by solving only a system of nonhomogeneous linear time-invariant ODEs. At each step, nonhomogeneous terms are calculated using the information obtained from previous steps. Therefore, solving the presented sequence is a recursive process.

To obtain the boundary conditions for the sequence, set $t=t_0$ and $t=\infty$ in Eq. (8) as follows:

$$\begin{cases} \varepsilon \boldsymbol{x}_0 = \boldsymbol{x}(t_0) = \boldsymbol{\Lambda}(\varepsilon \boldsymbol{x}_0, t_0) = \sum_{i=1}^{\infty} \varepsilon^i \boldsymbol{g}_i(t_0), \\ \boldsymbol{0} = \boldsymbol{\lambda}(\infty) = \boldsymbol{\Gamma}(\varepsilon \boldsymbol{x}_0, \infty) = \sum_{i=1}^{\infty} \varepsilon^i \boldsymbol{h}_i(\infty). \end{cases} \tag{11}$$

Equating the coefficients of the same powers of $\varepsilon$ in Eq. (11), we obtain

$$\begin{cases} \boldsymbol{g}_1(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{h}_1(\infty) = \boldsymbol{0} \end{cases} \text{ and } \begin{cases} \boldsymbol{g}_i(t_0) = \boldsymbol{0} \\ \boldsymbol{h}_i(\infty) = \boldsymbol{0} \end{cases} \text{ for } i \geq 2. \tag{12}$$

Finally, in accordance with the previous discussions, the following theorem can be stated, which is the major result of this section:

**Theorem 1** Let $\Phi_3$ be the intersection of convergence domains of series in Eqs. (7a) and (7b). For any $\boldsymbol{x}_0\in\Phi_3$, the solution of the nonlinear TPBVP (2) can be expressed as

$$\boldsymbol{x}(t) = \sum_{i=1}^{\infty} \boldsymbol{g}_i(t), \quad \boldsymbol{\lambda}(t) = \sum_{i=1}^{\infty} \boldsymbol{h}_i(t), \tag{13}$$

where $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ for $i\geq 1$ are obtained by solving recursively only the following sequence of linear time-invariant TPBVPs:

$$\varepsilon^1: \begin{cases} \dot{\boldsymbol{g}}_1(t) = \boldsymbol{A}_{10}\boldsymbol{g}_1(t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_1(t), \\ \dot{\boldsymbol{h}}_1(t) = -\boldsymbol{Q}\boldsymbol{g}_1(t) - \bar{\boldsymbol{A}}_{01}\boldsymbol{h}_1(t), \\ \boldsymbol{g}_1(t_0) = \boldsymbol{x}_0, \ \boldsymbol{h}_1(\infty) = \boldsymbol{0}, \end{cases} \tag{14a}$$

$$\varepsilon^2: \begin{cases} \dot{\boldsymbol{g}}_2(t) = \boldsymbol{A}_{10}\boldsymbol{g}_2(t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_2(t) \\ \qquad + \frac{1}{2!}\begin{bmatrix} \boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^1\boldsymbol{g}_1(t) \\ \vdots \\ \boldsymbol{g}_1^{\mathrm{T}}(t)\boldsymbol{H}_{20}^n\boldsymbol{g}_1(t) \end{bmatrix}, \\ \dot{\boldsymbol{h}}_2(t) = -\boldsymbol{Q}\boldsymbol{g}_2(t) - \bar{\boldsymbol{A}}_{01}\boldsymbol{h}_2(t) - \begin{bmatrix} \boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^1\boldsymbol{h}_1(t) \\ \vdots \\ \boldsymbol{g}_1^{\mathrm{T}}(t)\bar{\boldsymbol{H}}_{11}^n\boldsymbol{h}_1(t) \end{bmatrix}, \\ \boldsymbol{g}_2(t_0) = \boldsymbol{0}, \ \boldsymbol{h}_2(\infty) = \boldsymbol{0}, \end{cases} \tag{14b}$$

$$\vdots$$

$$\varepsilon^i: \begin{cases} \dot{\boldsymbol{g}}_i(t) = \boldsymbol{A}_{10}\boldsymbol{g}_i(t) - \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_i(t) \\ \qquad + \boldsymbol{G}_i(\boldsymbol{g}_1(t),\dots,\boldsymbol{g}_{i-1}(t)), \\ \dot{\boldsymbol{h}}_i(t) = -\boldsymbol{Q}\boldsymbol{g}_i(t) - \bar{\boldsymbol{A}}_{01}\boldsymbol{h}_i(t) \\ \qquad - \boldsymbol{H}_i(\boldsymbol{g}_1(t),\dots,\boldsymbol{g}_{i-1}(t),\boldsymbol{h}_1(t),\dots,\boldsymbol{h}_{i-1}(t)), \\ \boldsymbol{g}_i(t_0) = \boldsymbol{0}, \ \boldsymbol{h}_i(\infty) = \boldsymbol{0}, \end{cases} \tag{14c}$$

and so on, where the nonhomogeneous terms $\boldsymbol{G}_i$ and $\boldsymbol{H}_i$ in Eq. (14c) are determined by equating the coefficients of $\varepsilon^i$ in Eq. (9).

**Corollary 1** Consider the infinite horizon nonlinear OCP (1) with $\boldsymbol{x}_0\in\Phi_3$. The optimal trajectory-control pair for all $t>t_0$ is

$$\boldsymbol{x}^*(t) = \sum_{i=1}^{\infty} \boldsymbol{g}_i(t), \quad \boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{\infty} \boldsymbol{h}_i(t). \tag{15}$$

The following theorem shows uniform convergence of the obtained series solution in Eq. (15) to the optimal solution:

**Theorem 2** Define sequences $\{\boldsymbol{x}^{(k)}(t)\}_{k=1}^{\infty}$, $\{\boldsymbol{\lambda}^{(k)}(t)\}_{k=1}^{\infty}$, and $\{\boldsymbol{u}^{(k)}(t)\}_{k=1}^{\infty}$ as follows:

$$\begin{aligned} &\boldsymbol{x}^{(k)}(t) \triangleq \sum_{i=1}^{k} \boldsymbol{g}_i(t), \quad \boldsymbol{\lambda}^{(k)}(t) \triangleq \sum_{i=1}^{k} \boldsymbol{h}_i(t), \\ &\boldsymbol{u}^{(k)}(t) \triangleq -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(k)}(t). \end{aligned} \tag{16}$$

Then, for the infinite horizon nonlinear OCP (1) with $\boldsymbol{x}_0\in\Phi_3$, the sequences $\{\boldsymbol{x}^{(k)}(t)\}_{k=1}^{\infty}$ and $\{\boldsymbol{u}^{(k)}(t)\}_{k=1}^{\infty}$ converge uniformly to $\boldsymbol{x}^*(t)$ and $\boldsymbol{u}^*(t)$, respectively.

**Proof** According to the previous discussions, the Maclaurin series $\sum_{i=1}^{\infty}\boldsymbol{g}_i(t)$ and $\sum_{i=1}^{\infty}\boldsymbol{h}_i(t)$ converge uniformly to the exact solution of the nonlinear TPBVP (2). That is,

$$\begin{cases} \boldsymbol{x}(t) = \lim_{k\to\infty}\sum_{i=1}^{k} \boldsymbol{g}_i(t) \Leftrightarrow \boldsymbol{x}^{(k)}(t) \xrightarrow{\text{Uniformly}} \boldsymbol{x}(t), \\ \boldsymbol{\lambda}(t) = \lim_{k\to\infty}\sum_{i=1}^{k} \boldsymbol{h}_i(t) \Leftrightarrow \boldsymbol{\lambda}^{(k)}(t) \xrightarrow{\text{Uniformly}} \boldsymbol{\lambda}(t). \end{cases} \tag{17}$$

The control sequence $\{\boldsymbol{u}^{(k)}(t)\}_{k=1}^{\infty}$ depends only on the co-state vector sequence $\{\boldsymbol{\lambda}^{(k)}(t)\}_{k=1}^{\infty}$ through a linear operator. Therefore, we have

$$\boldsymbol{u}^*(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{\infty}\boldsymbol{h}_i(t)$$
$$= -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\left(\lim_{k\to\infty}\sum_{i=1}^{k}\boldsymbol{h}_i(t)\right) = \lim_{k\to\infty}\left(-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{k}\boldsymbol{h}_i(t)\right)$$
$$= \lim_{k\to\infty}(-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(k)}(t)) = \lim_{k\to\infty}\boldsymbol{u}^{(k)}(t). \quad (18)$$

That is, the control sequence $\{\boldsymbol{u}^{(k)}(t)\}_{k=1}^{\infty}$ converges uniformly to the optimal control law $\boldsymbol{u}^*(t)$, and the proof is complete.

## 4 Approximate solution of a nonlinear TPBVP and its domain of validity

It is almost impossible to obtain the solution of the nonlinear TPBVP (2) as in Eq. (13), since Eq. (13) contains infinite series. Therefore, in practical applications, the $M$th order approximate solution is obtained by replacing ∞ with a finite positive integer $M$ in Eq. (13) as follows:

$$\boldsymbol{x}^{(M)}(t) = \sum_{i=1}^{M}\boldsymbol{g}_i(t), \quad \boldsymbol{\lambda}^{(M)}(t) = \sum_{i=1}^{M}\boldsymbol{h}_i(t). \quad (19)$$

For this approximate solution, we define the domain of validity through the following definition:

**Definition 1** Let $\delta>0$ be fixed. For a given $M$, the domain of validity for the $M$th order approximate solution (19), denoted by $\Phi(M)$, is defined as

$$\Phi(M) \triangleq \{\boldsymbol{x}_0 \in \mathbb{R}^n : E(\boldsymbol{x}_0) < \delta\}, \quad (20)$$

where

$$E(\boldsymbol{x}_0) = \int_{t_0}^{\infty}(\| E_1(\boldsymbol{x}_0,t)\|_2^2 + \| E_2(\boldsymbol{x}_0,t)\|_2^2)\mathrm{d}t, \quad (21)$$

in which

$$E_1(\boldsymbol{x}_0,t) = \dot{\boldsymbol{x}}^{(M)}(t) + \boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(M)}(t) - \boldsymbol{F}(\boldsymbol{x}^{(M)}(t)), \quad (22a)$$

$$E_2(\boldsymbol{x}_0,t) = \dot{\boldsymbol{\lambda}}^{(M)}(t) + \boldsymbol{Q}\boldsymbol{x}^{(M)}(t) + \boldsymbol{\Psi}(\boldsymbol{x}^{(M)}(t),\boldsymbol{\lambda}^{(M)}(t)), \quad (22b)$$

and $\boldsymbol{x}^{(M)}(t)$ and $\boldsymbol{\lambda}^{(M)}(t)$ are as in Eq. (19) wherein $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ depend on $\boldsymbol{x}_0$ (see Eqs. (7a) and (7b)).

$\Phi(M)$ is the range of initial conditions for which the $\boldsymbol{x}^{(M)}(t)$ and $\boldsymbol{\lambda}^{(M)}(t)$ in Eq. (19) satisfy the nonlinear TPBVP (2) with the global error $E(\boldsymbol{x}_0)$ less than a small enough constant $\delta>0$.

Here, we suggest a practical technique to obtain the domain of validity for the $M$th order approximate solution (19). We can obtain the domain of validity $\Phi(M)$ by solving the following optimization problem:

$$\max \ \|\boldsymbol{x}_0\|_2^2 \quad \text{s.t.} \quad E(\boldsymbol{x}_0) < \delta. \quad (23)$$

Note that the integral in Eq. (21) is analytically computable, and thus the constraint in Eq. (23) is a nonlinear inequality in variable $\boldsymbol{x}_0$. Therefore, the optimization problem (23) can be solved numerically by various methods such as sequential quadratic programming (SQP), penalty methods, and barrier methods (Bazaraa *et al.*, 2006). In this work, we use the SQP, which is one of the most popular and robust algorithms for solving nonlinear continuous optimization problems. A number of packages including MATLAB and MAPLE can be used to run the SQP.

## 5 Suboptimal control design strategy

In this section, we obtain an accurate enough suboptimal trajectory-control pair for the infinite horizon nonlinear OCP (1). For any $\boldsymbol{x}_0\in\Phi(M)$, the $M$th order suboptimal trajectory-control pair can be obtained as follows:

$$\boldsymbol{x}^{(M)}(t) = \sum_{i=1}^{M}\boldsymbol{g}_i(t), \quad \boldsymbol{u}^{(M)}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{M}\boldsymbol{h}_i(t). \quad (24)$$

The integer $M$ in Eq. (24) is generally determined according to a concrete control precision. For example, the $M$th order suboptimal trajectory-control pair in Eq. (24) has the desirable accuracy if for a given positive constant $e>0$, the following condition holds:

$$\left|\frac{J^{(M)} - J^{(M-1)}}{J^{(M)}}\right| < e, \quad (25)$$

where

$$J^{(M)} = \frac{1}{2}\int_{t_0}^{\infty}\left((\boldsymbol{x}^{(M)}(t))^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{x}^{(M)}(t) + (\boldsymbol{u}^{(M)}(t))^{\mathrm{T}}\boldsymbol{R}\boldsymbol{u}^{(M)}(t)\right)\mathrm{d}t. \quad (26)$$

To obtain an accurate enough suboptimal trajectory-control pair, we present an iterative algorithm which has low computational complexity and a fast convergence rate. Therefore, only a few iterations are required to reach a desirable accuracy. This fact reduces the size of computations effectively.

**Algorithm 1** Suboptimal control design

Step 1: Let $i$=1.

Step 2: Calculate the $i$th order terms $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ from the presented sequence of linear time-invariant TPBVPs in Eqs. (14a)–(14c).

Step 3: Let $M$=$i$ and obtain $\boldsymbol{x}^{(M)}(t)$ and $\boldsymbol{u}^{(M)}(t)$ from Eq. (24). Then calculate $J^{(M)}$ according to Eq. (26).

Step 4: If Eq. (25) holds for the given small enough constant $e$>0, go to Step 5; else, replace $i$ by $i$+1 and go to Step 2.

Step 5: Determine $\Phi(M)$. If $\boldsymbol{x}_0\in\Phi(M)$, then $\boldsymbol{x}^{(M)}(t)$-$\boldsymbol{u}^{(M)}(t)$ is the desirable suboptimal trajectory-control pair.

## 6 Numerical example

In this section, the effectiveness of the proposed approach is verified by solving a numerical example. Here, we consider the optimal manoeuvres of a rigid asymmetric spacecraft (Junkins and Turner, 1986). Euler's equations for the angular velocities of a spacecraft are given by

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\underbrace{\begin{bmatrix}-\frac{I_3-I_2}{I_1}x_2(t)x_3(t)\\ -\frac{I_1-I_3}{I_2}x_1(t)x_3(t)\\ -\frac{I_2-I_1}{I_3}x_1(t)x_2(t)\end{bmatrix}}_{\boldsymbol{F}(\boldsymbol{x}(t))} \quad (27)$$
$$+\underbrace{\begin{bmatrix}1/I_1 & 0 & 0\\ 0 & 1/I_2 & 0\\ 0 & 0 & 1/I_3\end{bmatrix}}_{\boldsymbol{B}}\underbrace{\begin{bmatrix}u_1(t)\\ u_2(t)\\ u_3(t)\end{bmatrix}}_{\boldsymbol{u}(t)},$$

where $x_1$, $x_2$, and $x_3$ are the angular velocities of the spacecraft, $u_1$, $u_2$, and $u_3$ are control torques, and $I_1$=86.24 kg·m$^2$, $I_2$=85.07 kg·m$^2$, and $I_3$=113.59 kg·m$^2$ are the spacecraft's principle inertia.

The infinite horizon quadratic cost functional to be minimized is given by

$$J=\frac{1}{2}\int_0^{\infty}(\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{x}(t)+\boldsymbol{u}^{\mathrm{T}}(t)\boldsymbol{R}\boldsymbol{u}(t))\,\mathrm{d}t, \quad (28)$$

where $\boldsymbol{Q}$=$\boldsymbol{R}$=$\boldsymbol{I}_{3\times3}$. In addition, the initial conditions are

$$x_1(0)=0.01 \text{ r/s}, x_2(0)=0.005 \text{ r/s}, x_3(0)=0.001 \text{ r/s}. \quad (29)$$

According to Pontryagin's maximum principle, the following nonlinear TPBVP is obtained:

$$\dot{\boldsymbol{x}}(t)=\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\end{bmatrix}=\underbrace{-\begin{bmatrix}\frac{1}{I_1^2}\lambda_1(t)\\ \frac{1}{I_2^2}\lambda_2(t)\\ \frac{1}{I_3^2}\lambda_3(t)\end{bmatrix}}_{-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t)}+\underbrace{\begin{bmatrix}-\frac{I_3-I_2}{I_1}x_2(t)x_3(t)\\ -\frac{I_1-I_3}{I_2}x_1(t)x_3(t)\\ -\frac{I_2-I_1}{I_3}x_1(t)x_2(t)\end{bmatrix}}_{\boldsymbol{F}(\boldsymbol{x}(t))}, \quad (30a)$$

$$\dot{\boldsymbol{\lambda}}(t)=\begin{bmatrix}\dot{\lambda}_1(t)\\ \dot{\lambda}_2(t)\\ \dot{\lambda}_3(t)\end{bmatrix}=\underbrace{-\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}}_{-\boldsymbol{Q}\boldsymbol{x}(t)}$$
$$\underbrace{-\begin{bmatrix}-\frac{I_1-I_3}{I_2}x_3(t)\lambda_2(t)-\frac{I_2-I_1}{I_3}x_2(t)\lambda_3(t)\\ -\frac{I_3-I_2}{I_1}x_3(t)\lambda_1(t)-\frac{I_2-I_1}{I_3}x_1(t)\lambda_3(t)\\ -\frac{I_3-I_2}{I_1}x_2(t)\lambda_1(t)-\frac{I_1-I_3}{I_2}x_1(t)\lambda_2(t)\end{bmatrix}}_{-\left(\frac{\partial \boldsymbol{F}(\boldsymbol{x}(t))}{\partial \boldsymbol{x}(t)}\right)^{\mathrm{T}}\boldsymbol{\lambda}(t)}, \quad (30b)$$

$$\boldsymbol{x}(0)=\begin{bmatrix}x_1(0)\\ x_2(0)\\ x_3(0)\end{bmatrix}=\begin{bmatrix}0.01\\ 0.005\\ 0.001\end{bmatrix}\text{ r/s},\ \boldsymbol{\lambda}(\infty)=\begin{bmatrix}\lambda_1(\infty)\\ \lambda_2(\infty)\\ \lambda_3(\infty)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}, \quad (30c)$$

and the optimal control law is

$$\boldsymbol{u}^*(t)=\begin{bmatrix}u_1^*(t)\\ u_2^*(t)\\ u_3^*(t)\end{bmatrix}=\underbrace{-\begin{bmatrix}\lambda_1(t)/I_1\\ \lambda_2(t)/I_2\\ \lambda_3(t)/I_3\end{bmatrix}}_{-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{\lambda}(t)},\quad t>0. \quad (31)$$

Following the procedure proposed in Section 3, we solve recursively the presented sequence of linear time-invariant TPBVPs in Eqs. (14a)–(14c). In accordance with Eq. (14a), we obtain the following homogeneous linear time-invariant TPBVP:

$$\dot{\boldsymbol{g}}_1(t)=\begin{bmatrix}\dot{g}_{1,1}(t)\\ \dot{g}_{1,2}(t)\\ \dot{g}_{1,3}(t)\end{bmatrix}=\underbrace{-\begin{bmatrix}h_{1,1}(t)/I_1^2\\ h_{1,2}(t)/I_2^2\\ h_{1,3}(t)/I_3^2\end{bmatrix}}_{-\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{h}_1(t)}, \tag{32a}$$

$$\dot{\boldsymbol{h}}_1(t)=\begin{bmatrix}\dot{h}_{1,1}(t)\\ \dot{h}_{1,2}(t)\\ \dot{h}_{1,3}(t)\end{bmatrix}=\underbrace{-\begin{bmatrix}g_{1,1}(t)\\ g_{1,2}(t)\\ g_{1,3}(t)\end{bmatrix}}_{-\boldsymbol{Q}\boldsymbol{g}_1(t)}, \tag{32b}$$

$$\boldsymbol{g}_1(0)=\begin{bmatrix}g_{1,1}(0)\\ g_{1,2}(0)\\ g_{1,3}(0)\end{bmatrix}=\begin{bmatrix}0.01\\ 0.005\\ 0.001\end{bmatrix},\ \boldsymbol{h}_1(\infty)=\begin{bmatrix}h_{1,1}(\infty)\\ h_{1,2}(\infty)\\ h_{1,3}(\infty)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{32c}$$

where $g_{1,j}(t)$ and $h_{1,j}(t)$ are the $j$th elements of vectors $\boldsymbol{g}_1(t)$ and $\boldsymbol{h}_1(t)$, respectively. Solving Eqs. (32a)–(32c), $\boldsymbol{g}_1(t)$ and $\boldsymbol{h}_1(t)$ are obtained as

$$g_{1,1}(t)=0.01\mathrm{e}^{-0.01159554731t}, \tag{33a}$$
$$g_{1,2}(t)=0.005\mathrm{e}^{-0.01175502527t}, \tag{33b}$$
$$g_{1,3}(t)=0.001\mathrm{e}^{-0.008803591866t}, \tag{33c}$$
$$h_{1,1}(t)=0.8624\mathrm{e}^{-0.01159554731t}, \tag{33d}$$
$$h_{1,2}(t)=0.4253500001\mathrm{e}^{-0.01175502527t}, \tag{33e}$$
$$h_{1,3}(t)=0.11359\mathrm{e}^{-0.008803591866t}. \tag{33f}$$

Substituting $\boldsymbol{g}_1(t)$ and $\boldsymbol{h}_1(t)$ from Eqs. (33a)–(33f) into Eq. (14b), Eq. (14b) becomes the following non-homogeneous linear time-invariant TPBVP:

$$\dot{\boldsymbol{g}}_2(t)=\begin{bmatrix}\dot{g}_{2,1}(t)\\ \dot{g}_{2,2}(t)\\ \dot{g}_{2,3}(t)\end{bmatrix}=-\begin{bmatrix}h_{2,1}(t)/I_1^2\\ h_{2,2}(t)/I_2^2\\ h_{2,3}(t)/I_3^2\end{bmatrix}+\begin{bmatrix}-1.653525046\times10^{-6}\,\mathrm{e}^{-0.02055861714t}\\ 3.214999412\times10^{-6}\,\mathrm{e}^{-0.02039913918t}\\ 5.150101240\times10^{-7}\,\mathrm{e}^{-0.02335057258t}\end{bmatrix}, \tag{34a}$$

$$\dot{\boldsymbol{h}}_2(t)=\begin{bmatrix}\dot{h}_{2,1}(t)\\ \dot{h}_{2,2}(t)\\ \dot{h}_{2,3}(t)\end{bmatrix}=-\begin{bmatrix}g_{2,1}(t)\\ g_{2,2}(t)\\ g_{2,3}(t)\end{bmatrix}+\begin{bmatrix}-1.426\times10^{-4}\,\mathrm{e}^{-0.02055861714t}\\ 2.735\times10^{-4}\,\mathrm{e}^{-0.02039913918t}\\ 5.85\times10^{-5}\,\mathrm{e}^{-0.02335057258t}\end{bmatrix}, \tag{34b}$$

$$\boldsymbol{g}_2(0)=\begin{bmatrix}g_{2,1}(0)\\ g_{2,2}(0)\\ g_{2,3}(0)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},\ \boldsymbol{h}_2(\infty)=\begin{bmatrix}h_{2,1}(\infty)\\ h_{2,2}(\infty)\\ h_{2,3}(\infty)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{34c}$$

where $g_{2,j}(t)$ and $h_{2,j}(t)$ are the $j$th elements of vectors $\boldsymbol{g}_2(t)$ and $\boldsymbol{h}_2(t)$, respectively. Solving Eqs. (34a)–(34c), $\boldsymbol{g}_2(t)$ and $\boldsymbol{h}_2(t)$ are obtained as

$$g_{2,1}(t)=-0.0001844819998\mathrm{e}^{-0.01159554731t}+0.0001844819998\mathrm{e}^{-0.02055861714t}, \tag{35a}$$
$$g_{2,2}(t)=0.0003719293205\mathrm{e}^{-0.01175502527t}-0.0003719293205\mathrm{e}^{-0.02039913918t}, \tag{35b}$$
$$g_{2,3}(t)=0.00003540323138\mathrm{e}^{-0.008803591866t}-0.00003540323138\mathrm{e}^{-0.02335057258t}, \tag{35c}$$
$$h_{2,1}(t)=-0.01590972766\mathrm{e}^{-0.01159554731t}+0.01590972766\mathrm{e}^{-0.02055861714t}, \tag{35d}$$
$$h_{2,2}(t)=0.03164002730\mathrm{e}^{-0.01175502527t}-0.03164002729\mathrm{e}^{-0.02039913918t}, \tag{35e}$$
$$h_{2,3}(t)=0.004021453052\mathrm{e}^{-0.008803591866t}-0.004021453053\mathrm{e}^{-0.02335057258t}. \tag{35f}$$

Continuing as above, $\boldsymbol{g}_i(t)$ and $\boldsymbol{h}_i(t)$ for $i\geq3$ are obtained by solving only a nonhomogeneous linear time-invariant TPBVP.

To obtain an accurate enough suboptimal trajectory-control pair, we applied the proposed algorithm in Section 5 with the tolerance error bound $e=2\times10^{-4}$. In this case, convergence was achieved after three iterations, i.e., $|(J^{(3)}-J^{(2)})/J^{(3)}|=1.552263728\times10^{-4}<2\times10^{-4}$, and a minimum of $J^{(3)}=0.005432195475$ was obtained. In addition, the third-order suboptimal trajectory-control pair was obtained as

$$\begin{aligned}x_1^{(3)}(t)=\sum_{i=1}^{3}g_{i,1}(t)&=9.804739614\times10^{-3}\,\mathrm{e}^{-0.01159554731t}\\&+2.047361117\times10^{-4}\,\mathrm{e}^{-0.02055861714t}\\&-2.490004422\times10^{-6}\,\mathrm{e}^{-0.03510559785t}\\&-6.985721918\times10^{-6}\,\mathrm{e}^{-0.02920273105t},\end{aligned} \tag{36a}$$

$$\begin{aligned}x_2^{(3)}(t)=\sum_{i=1}^{3}g_{i,2}(t)&=5.378307474\times10^{-3}\,\mathrm{e}^{-0.01175502527t}\\&-3.808511240\times10^{-4}\,\mathrm{e}^{-0.02039913918t}\\&+4.907977390\times10^{-6}\,\mathrm{e}^{-0.03494611989t}\\&-2.364327034\times10^{-6}\,\mathrm{e}^{-0.02936220901t},\end{aligned} \tag{36b}$$

$$x_3^{(3)}(t)=\sum_{i=1}^{3}g_{i,3}(t)=1.036135826\times10^{-3}\,\mathrm{e}^{-0.008803591866t}-3.738360544\times10^{-5}\,\mathrm{e}^{-0.02335057258t}+1.651904484\times10^{-6}\,\mathrm{e}^{-0.03199468649t}-4.041254502\times10^{-7}\,\mathrm{e}^{-0.03231364241t},\tag{36c}$$

$$u_1^{(3)}(t)=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{3}h_{i,1}(t)=-9.804739614\times10^{-3}\,\mathrm{e}^{-0.01159554731t}-2.047361117\times10^{-4}\,\mathrm{e}^{-0.02055861714t}+2.490004421\times10^{-6}\,\mathrm{e}^{-0.03510559785t}+6.985721917\times10^{-6}\,\mathrm{e}^{-0.02920273105t},\tag{36d}$$

$$u_2^{(3)}(t)=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{3}h_{i,2}(t)=-5.378307474\times10^{-3}\,\mathrm{e}^{-0.01175502527t}+3.827746135\times10^{-4}\,\mathrm{e}^{-0.02039913918t}-4.907977391\times10^{-6}\,\mathrm{e}^{-0.03494611989t}+8.601353213\times10^{-7}\,\mathrm{e}^{-0.02936220901t},\tag{36e}$$

$$u_3^{(3)}(t)=-\boldsymbol{R}^{-1}\boldsymbol{B}^{\mathrm{T}}\sum_{i=1}^{3}h_{i,3}(t)=-1.036135827\times10^{-3}\,\mathrm{e}^{-0.008803591866t}+3.738360543\times10^{-5}\,\mathrm{e}^{-0.02335057258t}-1.651904495\times10^{-6}\,\mathrm{e}^{-0.03199468649t}+4.041254505\times10^{-7}\,\mathrm{e}^{-0.03231364241t},\tag{36f}$$

where $x_j^{(3)}(t)$ and $u_j^{(3)}(t)$ are the $j$th elements of vectors $\boldsymbol{x}^{(3)}(t)$ and $\boldsymbol{u}^{(3)}(t)$, respectively.

**Remark 1** Following the proposed procedure in Section 4 and using the SQP method for solving the optimization problem (23), the domain of validity for the third-order approximate solution with $\delta=10^{-6}$ is obtained as

$$\Phi(3)=\{\boldsymbol{x}_0=(x_{1,0},x_{2,0},x_{3,0})\in\mathbb{R}^3: |x_{1,0}|<0.127,\ |x_{2,0}|<0.0173,\ |x_{3,0}|<0.00315\}.\tag{37}$$

Obviously, the initial conditions in Eq. (29) belong to $\Phi(3)$, which confirms the validity of the obtained approximate solutions.

Simulation curves of $\boldsymbol{x}^{(3)}(t)$ and $\boldsymbol{u}^{(3)}(t)$ are shown in Figs. 1 and 2. Simulation curves have also been obtained by directly solving the nonlinear TPBVP Eqs. (30a)–(30c) using the collocation method (Ascher *et al.*, 1995). Note that, to apply the collocation method, we have to replace the boundary condition $\boldsymbol{\lambda}(\infty)=\mathbf{0}$ in Eq. (30c) by the condition $\boldsymbol{\lambda}(T)=\mathbf{0}$, where $T$ is a large enough positive constant.

Figs. 1 and 2 show that the results of our proposed procedure are nearly identical to those of the collocation method. However, compared with the collocation method, the computing procedure of our method is straightforward and can be performed without a computer. In addition, unlike the collocation method, the suggested technique presents explicit expressions for the approximate solutions. Furthermore, unlike the collocation method, the boundary

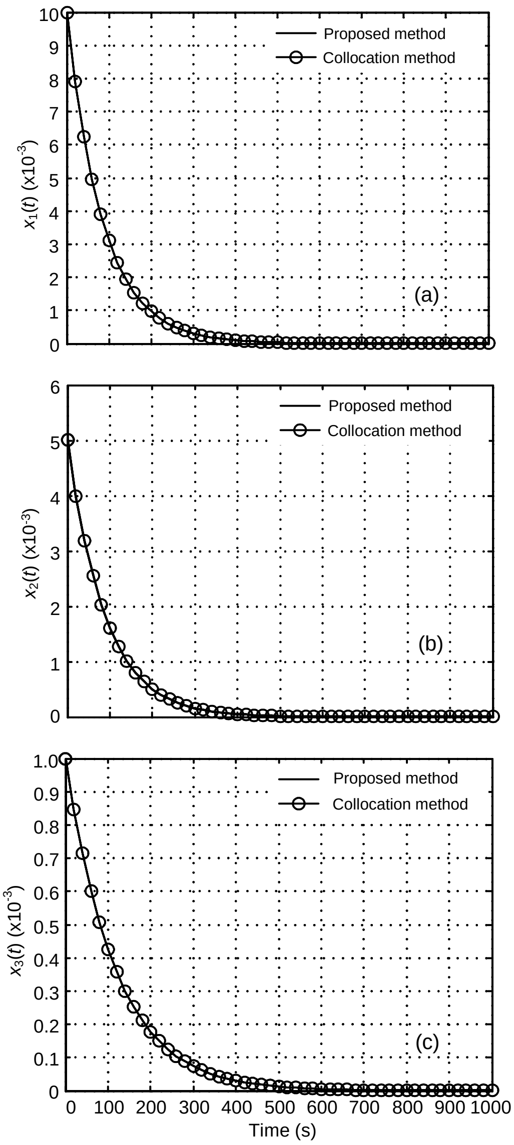


**Fig. 1 Simulation curves of the suboptimal trajectory $x_1(t)$ (a), $x_2(t)$ (b), and $x_3(t)$ (c) computed by our proposed method and the collocation method**

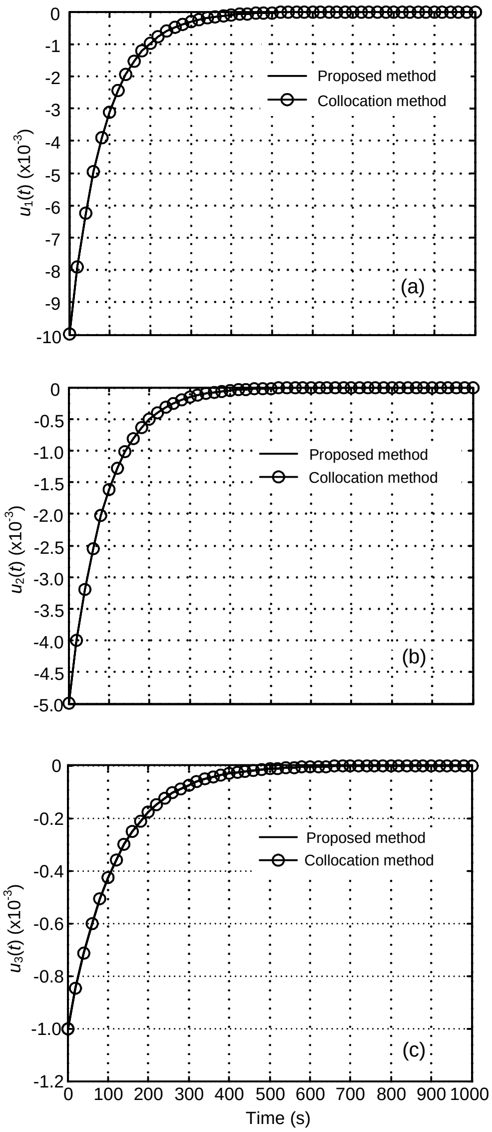


**Fig. 2 Simulation curves of the suboptimal control law $u_1(t)$ (a), $u_2(t)$ (b), and $u_3(t)$ (c) computed by our proposed method and the collocation method**

condition at $t=\infty$ for the co-state equation is handled easily in our proposed technique, since the general solution of each linear time-invariant ODE in Eqs. (14a)–(14c) is available explicitly. However, note that the proposed method may not work in as wide a range of initial conditions as the collocation method.

## 7 Conclusions and future work

This paper presents a new analytical technique, called the extended modal series method, for solving a class of infinite horizon nonlinear OCPs. The proposed technique avoids directly solving the nonlinear TPBVP or the HJB equation. Furthermore, contrary to the other approximate approaches such as SAA (Tang, 2005), ASRE (Banks and Dinesh, 2000), SDRE (Cimen, 2008), and DT (Hwang *et al.*, 2009), the suggested technique avoids solving a sequence of linear time-varying TPBVPs, a sequence of matrix Riccati differential (or algebraic) equations, or a system of nonlinear algebraic equations. Our proposed method requires solving only a sequence of linear time-invariant TPBVPs. Deriving each linear equation in the sequence is straightforward and can be performed without a computer. This property gives our method an advantage compared with the algorithms presented by Huang and Lin (1995) and Abu-Khalaf *et al*. (2006). Therefore, in terms of computational complexity, the proposed approach is more practical than the other approximate approaches. Future work can be focused on extending the modal series method for solving more general forms of nonlinear OCPs.

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