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A global linearization approach to solve nonlinear nonsmooth constrained programming problems

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Abstract. In this paper we introduce a new approach to solve constrained nonlinear non-smooth programming problems with any desirable accuracy even when the objective function is a non-smooth one. In this approach for any given desirable accuracy, all the nonlinear functions of original problem (in objective function and in constraints) are approximated by a piecewise linear functions. We then represent an efficient algorithm to find the global solution of the later problem. The obtained solution has desirable accuracy and the error is completely controllable. One of the main advantages of our approach is that the approach can be extended to problems with non-smooth structure by introducing a novel definition of Global Weak Differentiation in the sense of L_1 norm. Finally some numerical examples are given to show the efficiency of the proposed approach to solve approximately constraints nonlinear non-smooth programming problems.

Mathematical subject classification: 90C30, 49M37, 49M25.

Key words: nonlinear programming problem, non-smooth analysis, equicontinuity, uniform continuity.

1 Introduction

Frequently practitioners need to solve global optimization problem in many fields such as engineering design, molecular biology, neural network training and social science. So that the global optimization becomes a popular computational

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task for researchers and practitioners. There are some interesting recent papers for solving nonlinear programming problems [3], non-smooth global optimization [6, 10], solving a class of non-differentiable programming based on neural network method [11] and controllability for time-varying systems [4].

One of the efficient approaches for solving nonlinear programming problems is to linearize the nonlinear functions when the domain of the function is partitioned to very small sub-domains. However many realistic problems cannot be adequately linearized. So throughout its domain efforts to approximate nonlinear problems efficiently is the focused of the new researcher. Two other aspects that should be considered are non-convexity and non-smooth dynamics due to our ability to obtain the global solution of nonlinear, non-convex and non-smooth problems(when they exist) is still limited. So an efficient approach which is applicable in the presence of non-convex and non-smooth functions should be investigated (see [1, 2, 5]).

In this paper we introduce a new approach to solve approximately nonlinear non-smooth programming problems which don't have any limitation upon convexity and smoothness of the nonlinear functions. In this approach any given nonlinear function is approximated by a piecewise linear function with controlled error. In this manner, the difference between global solution of the approximated problem and the main problem is less than or equal a desirable upper bound which is shown by $\varepsilon > 0$. Also we represent an efficient algorithm to find global solution of approximated problem. One of the main advantages of our approach is that it can be extended to problems with non-smooth functions by introducing a novel definition of Global Weak Differentiation in the sense of L_1-norm. The paper is organized as follow:

In section two we explain our approach for one dimensional nonlinear programming problem. In the third section we deal with the extension of our approach for n dimensional nonlinear programming problems. In section four the approach was extended for non-smooth nonlinear programming problems by introducing the definition of global weak differentiation. In the fifth sections some illustrative examples are given to show the effectiveness of the proposed approach. Some suggestions and Conclusions are included in Section 6.

2 Proposed approach for one dimensional problem

Consider the following non-constrained nonlinear minimization problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{subject to} \quad x \in [a, b] \end{aligned} \tag{1}$$

where $f : [a, b] \longrightarrow R$; is a nonlinear smooth function. We may approximate the nonlinear function $f(x)$ by a piecewise linear function defined on $[a, b]$. Let us mention the following definitions.

Definition 2.1. Let $P_n([a, b])$ be a partition of the interval $[a, b]$ as the form:

$$P_n([a, b]) = \left\{a = x_0, x_1, \cdots, x_n = b\right\}$$

where $h = \frac{b-a}{n}$ and $x_i = x_0 + ih$. The norm of partition defined by:

$$\| P_n([a, b])\| = \max_{1 \le i \le n} \left\{x_i - x_{i-1}\right\}. \tag{2}$$

It is easy to show that $\| P_n([a, b])\| \to 0$ as $n \to \infty$.

Definition 2.2. The function $f_i(x, s_i)$ is defined as follows:

$$f_i(x, s_i) \triangleq f'(s_i)x + f(s_i) - s_i f'(s_i); \; x \in [x_{i-1}, x_i] \qquad i = 1, \cdots, n \tag{3}$$

where $s_i \in (x_{i-1}, x_i)$ is an arbitrary point. *The function $f_i(x, s_i)$ is called the linear parametric approximation of $f(x)$ on $[x_{i-1}, x_i]$ at the point $s_i \in (x_{i-1}, x_i)$. (In usual linear expansion the point s_i is fixed, but here we assume s_i is a free point in $[x_{i-1}, x_i]$).*

Now, we define $g_n(x)$ as the parametric linear approximation of $f(x)$ on $[a, b]$, associated with the partition P_n as follows:

$$g_n(x) = \sum_{i=1}^{n} \left[f_i(x, s_i)\chi_{[x_{i-1}, x_i](x)}\right] \tag{4}$$

where χ_A is the characteristic function and defined as below:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A. \end{cases}$$

The following theorems are shown that $g_n(x)$ is convergence uniformly to the original nonlinear function $f(x)$ when $\|P_n([a,b])\| \to 0$. In the other word we show that

$$g_n \to f \quad \text{uniformly on} \quad [a,b] \quad \text{as} \quad \|P_n([a,b])\| \to 0$$

Lemma 2.3. *Let $P_n([a,b])$ be an arbitrary regular partition of $[a,b]$. If $f(x)$ is continuous function on $[a,b]$ and $x, s \in [x_{i-1}, x_i]$ are an arbitrary points then*

$$\lim_{\|P_n([a,b])\| \to 0} f_i(x, s_i) = f(x_i).$$

Proof. The proof is an immediate consequence of the definition.

This lemma shown that $g_n \to f$ point-wise on $[a,b]$.

Definition 2.4. A family F of complex functions f defined on a set A in a metric space X, is said to be equicontinuous on A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $d(x,y) < \delta, x \in A, y \in A, f \in F$. Here $d(x,y)$ denotes the metric of A (see [7]).

Since $\{g_n(x)\}$ is a sequence of linear functions it is trivial that this sequence is equicontinuous.

Theorem 2.1. *Let $\{f_n\}$ is an equicontinuous sequence of function on a compact set A and $\{f_n\}$ converges point-wise on A. Then $\{f_n\}$ converges uniformly on A.*

Proof. Since $\{f_n\}$ is a sequence of equicontinuous function on A then:

$$\forall\, \varepsilon > 0 \;\exists\, \delta > 0 \;\; s.t$$

$$d(x,y) < \delta \to |f_n(x) - f_n(y)| < \varepsilon \quad x, y \in A\,; \quad n = 1, 2, \cdots .$$

For each $x \in A$ there exists $\delta > 0$ such that $A \subseteq \bigcup_{x \in A} N(x, \delta)$. Since A is a compact, this open covering of A has a finite sub-covering. Thus, there exists a finite number of points such as $x_1, x_2, \ldots, x_r$ in A such that $A \subseteq \bigcup_{i=1}^{r} N(x_i, \delta)$. Therefore for each $x \in A$ there exists $x_i \in A \;\; i = 1, 2, \ldots, r$; such that $d(x, x_i) < \delta$.

We know f_n is point-wise convergent sequence then there exists a natural number N such that for each $n \geq N, m \geq N$ we have:

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_m(x) - f_m(x_i) + f_m(x_i) - f_n(x_i) + f_n(x_i) - f_n(x)| \\ &\leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \\ &\leq 3\epsilon. \end{aligned}$$

Then according to the Theorem 7.8 in [7] the sequence $\{f_n\}$ is uniformly continuous on A and the proof is completed.

Theorem 2.2. *Let $g_n(x)$ is a piecewise linear approximation of $f(x)$ on $[a, b]$ as* (4). *Then:*

$$g_n \to f \text{ uniformly on } [a, b].$$

Proof. The proof is an immediate consequence of Lemma 2.3 and Theorem 2.1.

Now, we introduce a novel definition of global error for approximated $f(x)$ with linear parametric function $g_n(x)$ in the sense of L_1-norm which is a suitable criterion to show the goodness of fitting.

Definition 2.5. Let $f(x)$ be a nonlinear smooth function defined on $[a, b]$ and let $g_n(x)$ defined in (4) be a parametric linear approximation of $f(x)$. Let the global error for approximation of the function $f(x)$ with function $g_n(x)$ in the sense of L_1-norm is defined as follows:

$$E_n = \int_a^b |f(x) - g_n(x)|dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(x) - f_i(x)|dx. \tag{5}$$

It is easy to show that E_n tends to zero uniformly when $\|P_n([a, b])\| \to 0$.

This definition is used to make the fine partition which is matched with a desirable accuracy. These partitions can be obtained according to the following iterative algorithm.

Step 1. Let select an acceptable upper bound for desirable global error of approximation which called U_ε and set $n = 1$.

Step 2. n is substituted by $2n$ and then determine E_n as in (5).

Step 3. If $E_n > U_\varepsilon$ go to 2 and else end the process.

The value of n which is achieved in the above algorithm indicates the number of points in the suitable partition which is matched with the desirable accuracy. Let $f(x)$ in the problem (1) is replaced with its piecewise linear approximation $g_n(x)$. So, we will have the following minimization problem:

$$\begin{aligned} \text{Minimize} \quad & g_n(x) \\ \text{subject to} \quad & x \in [a, b]. \end{aligned} \tag{6}$$

Where its solution is an approximation for the solution of the problem (1) we want this approximated solution have a given desirable accuracy. For this mean the partition should be chosen enough fine. But we don't know how fine the partition should be chosen? In the next section this question will be answered.

2.1 Error analysis for one dimensional problem

Assume that global optimum solution of (6) and (1) are happened at $x = \alpha$ and $x = \beta$ respectively. It means that:

$$g_n(\alpha) \leq g_n(x) \quad \forall\, x \in [a, b] \quad \text{and} \quad f(\beta) \leq f(x) \quad \forall\, x \in [a, b].$$

Now it is desirable to find an appropriate partition such that for any given $\varepsilon > 0$ the following inequality is hold:

$$|g_n(\alpha) - f(\beta)| < \varepsilon. \tag{7}$$

The following theorems are proved to show the achievement to the above goal.

Theorem 2.3. *Consider nonlinear real function $f(x)$ and it's piecewise linear approximation $g_n(x)$ defined in* (4). *Then, for each $x \in [a, b]$ and $\varepsilon > 0$ such that $\varepsilon \ll b - a$, we have:*

$$g_n(x) - \frac{E_n}{\varepsilon} \leq f(x) \leq \frac{E_n}{\varepsilon} + g_n(x),$$

where E_n is a global error of fitting defined in (5).

Proof. We know that $[a, b] = [a, b) \bigcup \{b\}$. Thus the above inequality is proved separately for $[a, b)$ and $\{b\}$ as follows:

Let $[a, b)$ is considered then for each $x \in [a, b)$ there exist $\varepsilon_1 > 0$ such that $[x, x + \varepsilon_1] \subseteq [a, b]$. Therefore we have:

$$\int_x^{x+\varepsilon_1} |f(x) - g_n(x)| dx \leq \int_a^b |f(x) - g_n(x)| dx.$$

According to (5) the right hand side of the above inequality is E_n. Additionally, if ε_1 is chosen such that $\varepsilon_1 \ll b - a$ the left hand side of the above inequality is calculated approximately using the rectangular role. Therefore we have:

$$|f(x) - g_n(x)| \times \varepsilon_1 \leq E_n$$

$$|f(x) - g_n(x)| \leq \frac{E_n}{\varepsilon_1}.$$

Let $\{b\}$ is considered then for $x = b$ there exist $\varepsilon_2 > 0$ such that $[x - \varepsilon_2, x] \subseteq [a, b]$. Therefore we have:

$$\int_{x-\varepsilon_2}^{x} |f(x) - g_n(x)| dx \leq \int_a^b |f(x) - g_n(x)| dx.$$

If ε_2 be chosen such that $\varepsilon_2 \ll b - a$ the left hand side of the above inequality is calculated approximately in the same manner which yield:

$$|f(x) - g_n(x)| \times \varepsilon_2 \leq E_n$$

$$|f(x) - g_n(x)| \leq \frac{E_n}{\varepsilon_2}.$$

Let $\varepsilon \leq \{\varepsilon_1, \varepsilon_2\}$. According to the above discussion for any $x \in [a, b) \bigcup \{b\} = [a, b]$ there exists $\varepsilon > 0$ such that:

$$|f(x) - g_n(x)| \leq \frac{E_n}{\varepsilon}$$

or

$$g_n(x) - \frac{E_n}{\varepsilon} \leq f(x) \leq \frac{E_n}{\varepsilon} + g_n(x).$$

Thus the proof is completed.

Theorem 2.4. *Let $f(x)$ is a nonlinear function if for each $\varepsilon > 0$ we have $E_n \leq \varepsilon^2$ then* (7) *is satisfied.*

Proof. Let $\varepsilon \ll b-a$ according to the Theorem 2.3 we have:

$$g_n(x) - \frac{E_n}{\varepsilon} \leq f(x) \leq \frac{E_n}{\varepsilon} + g_n(x) \quad \forall x \in [a,b].$$

First, consider the right inequality i.e.:

$$f(x) \leq \frac{E_n}{\varepsilon} + g_n(x) \quad \forall x \in [a,b].$$

According to the definition of $f(\beta)$ we have:

$$f(\beta) \leq \frac{E_n}{\varepsilon} + g_n(x) \quad \forall x \in [a,b].$$

Let $x = \alpha$, so we have:

$$f(\beta) \leq \frac{E_n}{\varepsilon} + g_n(\alpha) \quad \forall x \in [a,b]$$

or

$$f(\beta) - g_n(\alpha) \leq \frac{E_n}{\varepsilon} \quad \forall x \in [a,b].$$

Now consider the left inequality i.e:

$$g_n(x) - \frac{E_n}{\varepsilon} \leq f(x) \quad \forall x \in [a,b]$$

or

$$g_n(x) \leq f(x) + \frac{E_n}{\varepsilon} \quad \forall x \in [a,b].$$

According to the definition of $g_n(\alpha)$ we have:

$$g_n(\alpha) \leq f(x) + \frac{E_n}{\varepsilon} \quad \forall x \in [a,b].$$

Setting $x = \beta$,we have:

$$g_n(\alpha) \leq f(\beta) + \frac{E_n}{\varepsilon} \quad \forall x \in [a,b]$$

or

$$g_n(\alpha) - f(\beta) \leq \frac{E_n}{\varepsilon} \quad \forall x \in [a,b].$$

Let n is chosen such that $E_n \leq \varepsilon^2$. Then the above inequality is transformed to the following ones:

$$|g_n(\alpha) - f(\beta)| < \varepsilon$$

and the proof is complete.

2.2 Described algorithm for one dimensional problem

According to the previous section in the first step of our algorithm for finding the optimum solution of nonlinear constrained programming problem with a desirable accuracy ε we must find an appropriate partition of $[a, b]$. Then the function $f(x)$ must be approximated by the parametric linear function $g_n(x)$.

At the next step the global optimum solution of the problem (6) must be calculated which is an accurate approximation for the global optimum solution of the problem (1). Here an efficient algorithm to solve the problem (6) is represented.

In each sub-interval of the form $[x_{i-1}, x_i]$ we have the following optimization problem:

$$\begin{array}{ll} \text{Minimize}_{1\leq i\leq n} & f_i(x) \\ \text{subject to} & x \in [x_{i-1}, x_i] \end{array} \tag{8}$$

where $f_i(x)$ is a parametric linear approximation of $f(x)$ which is defined in (3). Since $f_i(x)$ has an affine form such as $a_i x + b_i$ ($a_i = f'(s_i)$ and $b_i = f(s_i) - s_i f'(s_i)$) based on the sign of a_i the global minimum of $f_i(x)$ is happened at extreme points of its validity domain or equivalently on $\{x_{i-1}, x_i\}$. Thus the optimization problem (8) is transferred to the following ones:

$$\begin{array}{ll} \text{Minimize}_{1\leq i\leq n} & f_i(x) \\ \text{subject to} & x \in \{x_{i-1}, x_i\}. \end{array} \tag{9}$$

Here we define $\alpha_i \;\; i = 1, \cdots, n$ as the global solution of problem (9). So α_i can be formulated as follows:

$$\alpha_i = \begin{cases} f_i(x_{i-1}) & \alpha_i > 0 \\ f_i(x_i) & \alpha_i < 0. \end{cases}$$

Therefore the optimization problem (6) is converted to the following ones:

$$\text{Minimize}_{1\leq i\leq n} \;\; \alpha_i.$$

3 Extension of the proposed approach for n dimensional problems

Consider the following nonlinear minimization problem:

$$\begin{array}{ll} \text{Minimize} & f(x) \\ \text{subject to} & x \in A \end{array} \tag{10}$$

where $A = \prod_{i=1}^{n}[a_i, b_i] \subseteq R^n$ and $f(.) : A \to R$ is nonlinear smooth function. Here we introduce a piecewise linear parametric approximation for $f(x)$ which is the extension of Definition 2.2.

Definition 3.1. Consider the nonlinear smooth function $f(.) : A \to R$ where $A = \prod_{i=1}^{n}[a_i, b_i]$. Also consider $P_n([a_i, b_i])$ as a regular partition of $[a_i, b_i]$, $i = 1, \ldots, n$ as follows:

$$P_n([a_i, b_i]) = \{a_i = x_i^0, \ldots, x_i^{k_i}, \ldots, x_i^{n_i} = b_i\}$$

where $k_i = 0, 1, \ldots, n_i$ and $i = 1, \ldots, n$.

Therefore A is partitioned to N cells where $N = n_1 \times \cdots \times n_n$. Let us show the k^{th} cell by E_k, $k = 1, \ldots, N$. Let $s_k = (s_k^1, \ldots, s_k^n)$ be an arbitrary point of E_k. Now $f_k(x)$ is defined as a linear parametric approximation of $f(x)$ for $x \in E_k$ as follows:

$$f_k(x) = \nabla f(x)|_{x=s_k}.(x - s_k) + f(s_k) \tag{11}$$

where $x \in E_k$, $k = 1, \ldots, N$.

Now $g_N(x)$ is defined as a piecewise linear approximation of $f(x)$ as follows:

$$g_N(x) = \sum_{k=1}^{N}[f_k(x) \times \chi_{E_k}(x)].$$

we have $\lim_{\|P_n\| \to 0} g_N(x) = f(x)$ or equivalently $\lim_{N \to \infty} g_N(x) = f(x)$.

Now a definition of global error of approximation nonlinear function $f(x)$ and it's piecewise linear approximation $g_n(x)$ in the sense of L_1-norm is introduced which is the extension of Definition 2.5.

Definition 3.2. Consider the nonlinear smooth function $f(x)$ and it's piecewise linear approximation $g_N(x)$. We define a global error of approximation in the sense of L_1-norm to be E_N as follows:

$$E_N = \int_A |f(x) - g_N(x)|dx = \sum_{k=1}^{N} \int_{E_k} |f(x) - f_k(x)|dx. \tag{12}$$

Remark 3.3. The iterative algorithm which is presented in Section 2 can be used to find the appropriate number of partitions. According to that manner this number increases until the approximation is achieved with a desirable accuracy.

Therefore the following minimization problem must be solved:

$$\begin{aligned} &\text{Minimize} \quad g_N(x) \\ &\text{subject to} \quad x \in \prod_{i=1}^{n} [a_i, b_i]. \end{aligned}$$

The solution of this optimization problem is an approximated solution of the original problem (10). Since we want to achieve to a given desirable accuracy the partition should also be chosen enough fine. Therefore the method which has been explained in Section 2.1 is extended.

3.1 Error analysis for n dimensional problems

Assume that the global minimum of $g_N(x)$ and $f(x)$ on $A = \prod_{i=1}^{n} [a_i, b_i]$ are happened at $x = \alpha$ and $x = \beta$ respectively. So the approximated partition must be found such that:

$$|g_n(\alpha) - f(\beta)| < \varepsilon$$

where ε is a given desirable error.

Since the above inequality must be satisfied thus the manner which has been represented in Section 2.1 should be repeated in n dimensions. Then, we find N such that we have $E_N \leq \varepsilon^{n+1}$. ($E_n$ is defined in (12)).

3.2 Description of the algorithm for n dimensional problems

According to the above manner which is explained in the previous sections the following independent linear optimization problem are defined:

$$\begin{aligned} &\text{Minimize} \quad f_k(x) \\ &\text{subject to} \quad x = (x_1, \cdots, x_n) \in E_k \ ; \ k = 1, \cdots, N. \end{aligned} \tag{13}$$

Where $f_k(x)$ is a linear parametric approximation of $f(x)$ on E_k which is defined in (11). Since $f_k(x)$ has an affine form similar to

$$a_k.x + b_k(a_k = \nabla f(x)|_{x=s_k} \quad \text{and} \quad b_k = f(s_k) - \nabla f(x)|_{x=s_k}.s_k)$$

based on the sign of a_k the global minimum of $f_k(x)$ is happened at 2^n extreme points of its validity domain E_k.

Therefore the optimization problem (13) is transferred to the following ones:

$$\begin{aligned} &\text{Minimize} \quad f_k(x) \\ &\text{subject to} \quad x \in \left\{2^n \text{ distinct extreme point of } E_k\right\}; \; k = 1, \cdots, N. \end{aligned} \tag{14}$$

Here we define α_k, $k = 1, \ldots, N$ as the global solution of problem (14). Thus the optimization problem (13) is converted to the following simpler ones:

$$\text{Minimize}_{1 \leq k \leq N} \quad \alpha_k.$$

4 Extension to nonlinear non-smooth problems

In general it is reasonable to assume that the objective function is a non-smooth ones. Therefore we define a kind of generalized differentiation for non-smooth functions in the sense of L_1-norm. This kind of differentiation is coincideing with usual differentiation for smooth functions. Therefore the following theorem is represented.

Theorem 4.1. *Consider the nonlinear smooth function $f : A \to R$ where $A = \prod_{i=1}^{n}[a_i, b_i]$. Then the optimal solution of the following optimization problem is $f'(x)$.*

$$\text{Minimize}_{p(.)} \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} |f(x) - (f(s) + p(s).(x - s))| dx_1 \cdots dx_n \tag{15}$$

where $s = (s_1, s_2, \cdots, s_n) \in A$ is an arbitrary point and $p(.) = (p_1(.), \ldots, p_n(.))$ is a vector.

Proof. See [9].

Now based on Theorem 4.1 the following definition can be stated for non-smooth functions.

Definition 4.1. Let $f : A \to R$ is a non-smooth function where $A = \prod_{i=1}^{n}[a_i, b_i]$. The global weak differentiation with respect to x in the sense of L_1-norm is defined as the $p(.)$ the optimal solution of the minimization problem which is shown in (15).

5 Examples

In the current section we apply the performance of our method on some examples.

Example 5.1. Consider the following nonlinear minimization problem:

$$\begin{aligned} &\text{Minimize} \quad f(x) = x \sin \frac{1}{x} \\ &\text{subject to} \quad x \in [0.1, 1]. \end{aligned}$$

Which is desirable to be solved with accuracy more than $\varepsilon = 10^{-3}$.

Based on our proposed approach we approximate $f(x)$ with a piecewise linear function with global error less than $(10^{-3})^2$. An appropriate number of partitions which is matched with desirable accuracy is obtained as $n = 128$. Figure 1 shows $f(x)$ and its accurate enough piecewise linear approximation.

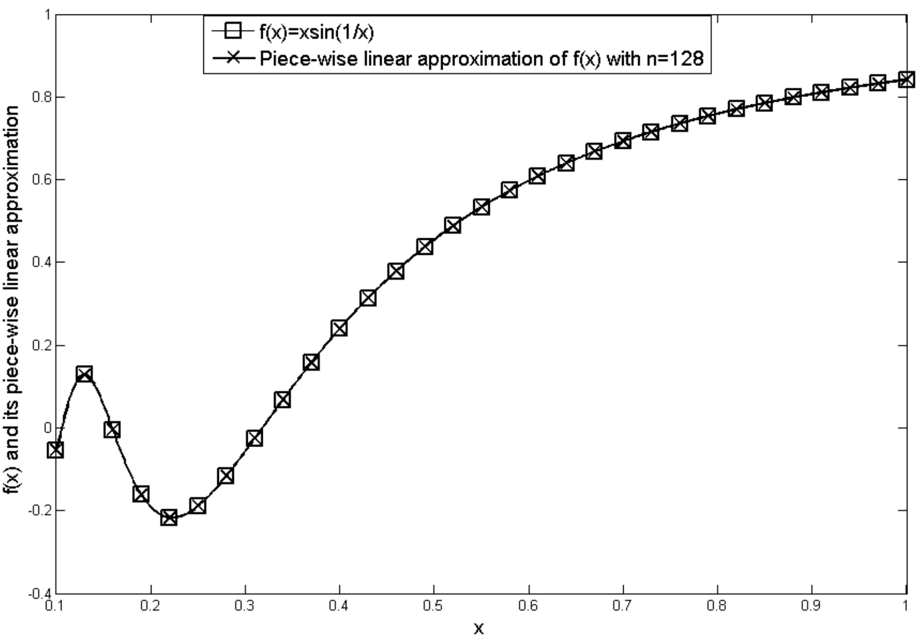

Figure 1 – Nonlinear function $f(x) = x \sin \frac{1}{x}$ and it's piecewise linear approximation.

Table 5.1 compares approximated solution and exact solution of this example. Comparison results show the effectiveness of the proposed approach to solve this problem with desirable accuracy.

Nonlinear function	Global error	Exact solution	Approximated solution
$f(x) = x \sin \frac{1}{x}$	5.6527×10^{-8}	–0.2172336	–0.2174230

Table 5.1 – Numerical results of example 5.1.

Example 5.2. Consider the following minimization problem (see Schuldt [8]):

$$\begin{aligned} \text{Minimize} \quad & f(x, y) = y + 10^{-5}(y - x)^2 \\ \text{subject to} \quad & -1 \le x \le 1 \\ & 0 \le y \le 1. \end{aligned}$$

Here it is desirable to solve above problem with accuracy more than $\varepsilon = 10^{-3}$. Thus we approximate $f(x, y)$ with a piecewise linear function with global error less than $(10^{-3})^3$. Table 5.2 compares the solution which is obtained by our proposed approach and exact solution of this problem. It can be shown that proposed approach is effective to solve the problem with desirable accuracy.

Nonlinear function	Global error	Exact solution	Approximated solution
$f(x, y) = y + 10^{-5}(y - x)^2$	2.6615×10^{-11}	0	-5.625×10^{-6}

Table 5.2 – Numerical results of example 5.2.

Example 5.3. In this example we consider a nonlinear non-smooth function as follows:

$$\begin{aligned} \text{Minimize} \quad & f(x) = |x|e^{-|x|} \\ \text{subject to} \quad & x \in [-1, 1]. \end{aligned}$$

It is desirable to solve with accuracy more than $\varepsilon = 10^{-5}$

Since objective function is non-smooth function we find the global weak differentiation of $f(x) = |x|e^{-|x|}$; $x \in [-1, 1]$ which is the optimal solution of the following optimization problem:

$$\text{Minimize}_{p(.)} \int_{-1}^{1} \left||x|e^{-|x|} - |s|e^{-|s|} - p(s).(x - s)\right| dx$$

The optimal solution is shown in Figure 2.

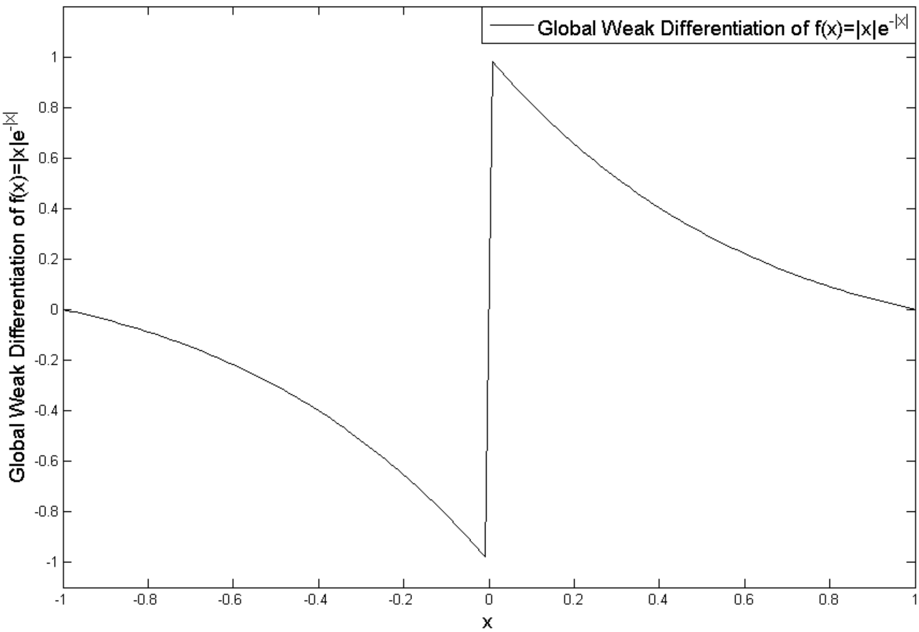

Figure 2 – Global Weak Differentiation of nonlinear non-smooth function $f(x) = |x|e^{-|x|}$.

Now we find a piecewise linear approximation for non-smooth function $f(x) = |x|e^{-|x|}$ on $[-1, 1]$ with the global error less than $(10^{-5})^2$. Therefore number of partitions should be chosen as $n \geq 512$. Figure 3 shows $f(x)$and its accurate enough piecewise linear approximation with $n = 512$.

Table 5.3 compares approximated and exact solution of last example. Comparison results show the effectiveness of the proposed approach in the presence of non-smooth functions.

Nonlinear function	Global error	Exact solution	Approximated solution
$f(x) = |x|e^{-|x|}$	1.7697×10^{-12}	0	3.8073×10^{-6}

Table 5.3 – Numerical results of example 5.3.

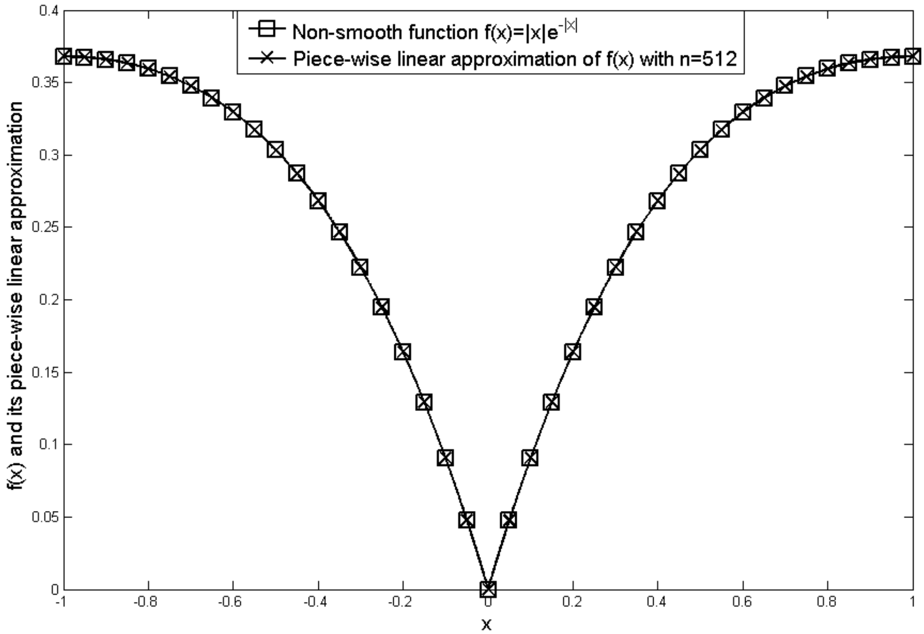

Figure 3 – Nonlinear function $f(x) = |x|e^{-|x|}$ and it's piecewise linear approximation.

6 Conclusion

In this paper we introduce a new approach to solve approximately wide class of constrained nonlinear programming problems. The main advantage of this approach is that we obtained an approximation for the optimum solution of the problem with any desirable accuracy. Also the approach can be extended for problems with non-smooth dynamics by introducing a novel definition of global weak differentiation in the sense of L_1 and L_p norms. In this paper we assume f be a non-smooth function, so it may have a finite or infinite points where the gradient of f does not exist. It is very interesting that we may not know these point (where are located) and also the set of points where the functions are non-smooth may be an infinite set.

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