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Chaos Synchronization of Fractional-Order Lorenz System with Unscented Kalman Filter

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Abstract: *In this paper we numerically investigate the chaotic behaviours of the fractional-order Lorenz system and its synchronization. For the first time, a fractional chaotic synchronization using an Unscented Kalman Filter (UKF) is presented. The chaotic synchronization is implemented by the UKF design in the presence of process noise and measurement noise. To illustrate the effectiveness of the synchronization with the UKF method, a numerical example based on the fractional-order Lorenz dynamical system is presented and the results are compared to the Extended Kalman Filter (EKF) method.*

Keywords: Fractional-order system, Chaos synchronization, Unscented Kalman Filter, Lorenz.

1. Introduction

Although fractional calculus has a long history (300-year-old), it was not used in physics and engineering for many years. However, the dynamics of fractional-order systems have attracted increasing attentions in recent years [1,2]. It was found that many systems in interdisciplinary fields can be described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics and within this research field, some fractional-order systems have been shown to demonstrate chaotic behavior [3], such as the fractional Chua system, the fractional Duffing one, the fractional Chen one, the fractional-order Lü system, the fractional-order jerk model, the fractional-order cellular neural network, the fractional-order neural network and so on.

In the past two decades, a new direction of chaos research has emerged to address the more challenging problem of chaos synchronization due to its potential applications in laser physics, chemical reactor, secure communication, biomedical and so on [4, 5, 6].

Pecora and Carroll [7] in their pioneering work addressed the synchronization of chaotic system using a drive-response conception. The idea is to use the output of the driving system to control the response system so that the trajectories of the response's outputs can synchronize those of drive system and they oscillate in a synchronized manner. Recently, many efforts have been made to show that the synchronization problem of chaotic systems could be solved through observer design approach [8–11], in which only the input and output information of drive system are used to construct part or all of the state information of drive system, and many beneficial methods have been developed. For example, several kinds of nonlinear observer design methods are summarized and their adaptations to chaotic synchronizations are discussed in [8] and in [12] a sliding-mode adaptive observer synchronization method for chaotic system is developed.

As a brief introductory and historical background, Extended Kalman Filter (EKF) as an optimal observer is a stochastic estimation scheme for estimating of nonlinear state and tracking applications [13]. In this method, Kalman filtering [14] is used to linearize the nonlinear function. The first order Taylor series expansions is applied to linearization.

Application of EKF to synchronization of chaotic systems is studied in [15], and synchronization is obtained of transmitter and receiver dynamics in case the receiver is given via an Extended Kalman Filter driven by a noisy drive signal from the transmitter. However, a major drawback of EKF is the error in function approximation due to employing first order Taylor series for approximating the nonlinearities. So, large errors may be happened when it is used to systems with higher order nonlinearities. For overcoming the drawbacks associated with the approximation errors, many alternatives to EKF have been offered. UKF, as recently proposed by Julier and Uhlman [16], could in theory improve upon EKF for state estimation since linearization is avoided by an unscented transformation and at least second order accuracy is provided. This point is achieved by carefully choosing a set of sigma points, which captures the true mean and covariance of a given distribution and then passing the means and covariances of estimated states through a nonlinear transformation. As a result, UKF is capable of estimating the posterior mean and covariances accurately to a high order [17].

Recently, synchronization of chaotic fractional differential systems starts to attract increasing attention due to its potential applications in control processing and secure communication [18]. In Ref. [19], Deng and Li firstly investigated the synchronization for the chaotic fractional Lu¨ system. Afterwards, they studied chaos synchronization of the Chen system with a fractional order in a different manner.

In this paper, we study the fractional-order Lorenz system and its synchronization with using of the UKF.

The rest of this paper is organized as follows. In Section 2 Nonlinear fractional systems are introduced. In Section 3 Numerical solution algorithm of fractional differential equations is described. Principles and algorithms of UKF is presented in section 4. In Section 5 numerical simulations of synchronization scheme for fractional-order Lorenz based on UKF are presented. In Section 6 conclusions are drawn.

2. Nonlinear Fractional Systems

Fractional calculus is a generalization of integration and differentiation of the noninteger order fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation. There are many definitions for the fractional integrals and derivatives. The two definitions generally used for the fractional integral are the Grunwald–Letnikov (GL) and the Riemann–Liouville (RL) definitions for discrete systems and continuous systems, respectively [20]. The RL integral definition is:

$$I^{(\alpha)}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{1-\alpha}f(\tau)d\tau, \qquad \alpha,t>0 \tag{1}$$

With this definition of integral, the two equations, the Riemann–Liouville and Caputo fractional derivatives can be defined as (2) and (3), respectively.

$$D^{(\alpha)}f(t)=\frac{d^m}{dt^m}(I^{(m-\alpha)}f(t)) \tag{2}$$

$$D^{(\alpha)}f(t)=I^{(m-\alpha)}\left(\frac{d^m}{dt^m}f(t)\right) \tag{3}$$

m is a positive integer variable and $m-1<\alpha\le m$.

Lorenz's system is a nonlinear chaotic one, recently found in the process of chaotifying continuous systems, described by

$$\begin{cases}\dot{x}=-\sigma(x-y)\\ \dot{y}=-xz+\rho x-y\\ \dot{z}=xy-\beta z\end{cases} \tag{4}$$

in which $(\sigma,\rho,\beta)\in\mathbb{R}^3$. When $(\sigma,\rho,\beta)=(10,28,8/3)$, (4) exists a chaotic attractor.

The fractional-order Lorenz system is described as follows:

$$\begin{cases}\dfrac{d^{(q_1)}x}{dt^{q_1}}=-\sigma(x-y)\\ \dfrac{d^{(q_2)}y}{dt^{q_2}}=-xz+\rho x-y\\ \dfrac{d^{(q_3)}y}{dt^{q_3}}=xy-\beta z\end{cases} \tag{5}$$

When $(q_1,q_2,q_3)=(0.96,0.98,1.1)$ system (5) behaves chaotically.

3. Numerical solution algorithm of fractional differential equations

The standard definitions of fractional differential equations do not allow direct implementation of the fractional operators in time domain simulations. The numerical simulation of a fractional differential equation - Unlike the numerical solving of an integer differential equation - is not straightforward. To solve a fractional differential equation numerically, two approximation methods, namely, frequency domain approximation and Adams–Bashforth–Moulton algorithm, have been proposed. The first method is based on the approximation of the fractional-order system behaviour in the frequency domain. In [21], an algorithm has been proposed to calculate linear transfer function approximations of $1/s^\alpha$ where, the Laplace transform of the fractional derivative is

$$L\{{}_0D_t^\alpha f(t)\}=s^\alpha F(s)-\sum_{k=0}^{m-1}s^k[{}_0D_t^{\alpha-k-1}f(t)]_{t=0} \tag{6}$$

In zero initial condition the Laplace transform of fractional derivative is

$$L\left\{\frac{d^\alpha f(t)}{dt^\alpha}\right\}=s^\alpha L\{f(t)\} \tag{7}$$

According to the Riemann–Liouville fractional integral definition, we can define this operator as follows:

$$I^{(\alpha)}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t) \tag{8}$$

Where

$$L\left\{t^{\alpha-1}\right\}=\Gamma(\alpha)s^{-\alpha} \tag{9}$$

so, the Laplace transform of the fractional integral is

$$L\left\{I^{(\alpha)}f(t)\right\}=s^{-\alpha}F(s) \tag{10}$$

The aim is to find zeros and poles of a transfer function that has a similar amplitude diagram as $1/s^\alpha$ in a given frequency range. By utilizing frequency domain techniques based on Bode diagrams, we can obtain a linear approximation of the fractional order integrator with any desired accuracy over any frequency band. The order of this linear approximation system depends on the desired bandwidth and accuracy between the actual and the approximate magnitudes of the corresponding Bode diagrams. This approximation method has sufficient accuracy for time-domain implementations [20].

The second method is an improved version of the Adams–Bashforth–Moulton algorithm [20] and is proposed based on the predictor-correctors scheme for this system. To illustrate the method, we consider the following differential equations:

$$D_t^q y(t) = f(t, y(t)), \quad 0 \le t \le T, \\ y^{(k)}(0) = y_0^{(0)}, \quad k = 0,1,\dots,m-1 \tag{11}$$

This differential equation is equivalent to the Volterra integral equation [20]:

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds \tag{12}$$

Set $h = T/N$, $t_n = nh \quad (n = 0,1,\dots,N \in Z^+)$. Then (12) can be discredited as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{t+1})) \\ + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_h(t_j)), \tag{13}$$

Where

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, 1 \le j \le n \\ 1, \quad j = n+1 \end{cases}$$

and

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_h(t_j))$$

where

$$b_{j,n+1} = \frac{h^q}{q} ((n+1-j)^q - (n-j)^q)$$

Applying the above method, the numerical solution of a fractional order system (5) with initial condition (x_0, y_0, z_0) can be determined as follows:

$$\begin{cases} x_{n+1} = x_0 + \frac{h^{q_1}}{\Gamma(q_1+2)} [-\sigma(x_{n+1}^p - y_{n+1}^p) + \sum_{j=0}^{n} a_{1,j,n+1}.(-\sigma(x_j - y_j))] \\ y_{n+1} = y_0 + \frac{h^{q_2}}{\Gamma(q_2+2)} [(-x_{n+1}^p z_{n+1}^p + \rho x_{n+1}^p - y_{n+1}^p) + \sum_{j=0}^{n} a_{2,j,n+1}.(-x_j z_j + \rho x_j - y_j)] \\ z_{n+1} = z_0 + \frac{h^{q_3}}{\Gamma(q_3+2)} [x_{n+1}^p y_{n+1}^p - \beta z_{n+1}^p) + \sum_{j=0}^{n} a_{3,j,n+1}.(x_j y_j - \beta z_j)] \end{cases}$$

and

$$\begin{cases} x_{n+1}^p = x_0 + \frac{1}{\Gamma(q_1)} \sum_{j=0}^{n} b_{1,j,n+1}.(-\sigma(x_j - y_j)) \\ y_{n+1}^p = y_0 + \frac{1}{\Gamma(q_2)} \sum_{j=0}^{n} b_{2,j,n+1}.(-x_j z_j + \rho x_j - y_j) \\ z_{n+1}^p = z_0 + \frac{1}{\Gamma(q_3)} \sum_{j=0}^{n} b_{3,j,n+1}.(x_j y_j - \beta z_j) \end{cases}$$

where

$$a_{i,j,n+1} = \begin{cases} n^{q_i+1} - (n-q_i)(n+1)^{q_i}, & j = 0 \\ (n-j+2)^{q_i+1} + (n-j)^{q_i+1} - 2(n-j+1)^{q_i+1}, 1 \le j \le n \\ 1, \quad j = n+1 \end{cases}$$

and

$$b_{i,j,n+1} = \frac{h^{q_i}}{q_i} ((n+1-j)^{q_i} - (n-j)^{q_i}), 0 \le j \le n \quad \& \quad i = 1,2,3$$

Simulation results by proposed Adams–Bashforth–Moulton algorithm are more reliable than simulation results of the first method, due to specificity of the error estimation bound in this method. So, this method was selected for our simulations. The next section describes the unscented Kalman filter, which is used in the estimation of fractional system states.

4. Unscented Kalman Filter

The Kalman Filter (KF) is a recursive filtering tool which has been developed for estimating the trajectory of a system from a series of noisy and/or incomplete observations of the system's state which has the following specifications: the estimation process is formulated in the system's state space; the solution is obtained by recursive computation and it uses an adaptive algorithm, which can be directly applied to stationary and non-stationary environment. In the Kalman filtering algorithm, every new estimate of the state is retrieved from the previous one and the new input in a way that only the previous estimated result need to be stored. Thus, the Kalman filter is more effective in computation than those which use all or considerable amount of the previous data directly in each estimation [22].

If the system is nonlinear, the Kalman filter cannot be applied directly, but two nonlinear Kalman filtering methods, namely, EKF and UKF are appropriate for stochastic nonlinear system estimation.

The Extended Kalman Filter (EKF) is a set of mathematical equations, which makes an estimate of the current state of a system using an underlying process model and then corrects the estimate using any available sensor measurements. Using this predictor-corrector mechanism, it approximates an optimal estimate due to the linearization of the process and measurement models [23].

The problem of propagating Gaussian random variables through a nonlinear function can also be approached using another technique, namely the unscented transform (UT). Instead of linearization required by the EKF, a new approximate method UT is used in the Unscented Kalman Filter (UKF) [16].

A set of weighted sigma points is deterministically chosen for matching the sample mean and sample covariance of these points with those of a priori distribution. The nonlinear function is applied to each of these points sequently to yield transformed samples, and the predicted mean and covariance are calculated from the transformed samples as shown in Fig. 1. This strategy typically both decrease the computational complexity, and at the same time increase estimate accuracy, yielding faster, more accurate results.

The fundamental difference between EKF and UKF lies in the way in which Gaussian random variables (GRV) are represented in the process of propagating through the system dynamics. Basically, the UKF captures the posterior mean and covariance of GRV accurately to the third order (in terms of Taylor series expansion) for any form of nonlinearity, whereas the EKF only achieves first-order accuracy. Moreover, since no explicit Jacobian or Hession calculations are necessary in the UKF algorithm, the computational complexity of UKF is comparable to EKF.

The algorithm for implementing the UKF can be summarized as follows [24]:
Consider the nonlinear discrete-time system represented by

$$\begin{aligned} x_{k+1} &= f(x_k) + w_k \\ y_k &= H_k x_k + v_k \end{aligned} \tag{14}$$

$k \in \mathbb{N}$ is discrete time, and denotes the set of natural numbers. $x_k \in \mathbb{R}^{L\times1}$ is the state, and $y_k \in \mathbb{R}^{M\times1}$ is the measurement. $f(\cdot)$ is nonlinear mapping and is assumed to be continuously differentiable with respect to x_k and H_k is a measurement matrix. Moreover, $w_k \in \mathbb{R}^{L\times1}$ and $v_k \in \mathbb{R}^{M\times1}$ are uncorrelated zero-mean Gaussian white sequences and have the following characteristics:

$$E\left[w_k w_j^T\right] = Q_k\delta_{kj},\ E\left[v_k v_j^T\right] = R_k\delta_{kj},\ E\left[w_k v_j^T\right] = 0 \tag{15}$$

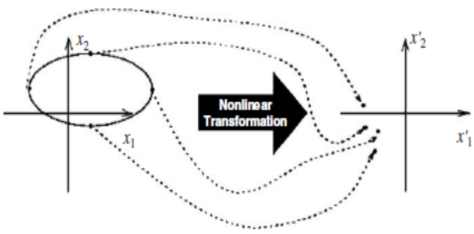

Fig 1: The principle of the unscented transform

***Step* 1:** The L-dimensional random variable x_{k-1} with mean $\hat{x}_{k-1}$ and covariance $\hat{P}_{k-1}$ is approximated by sigma points which are computed with the following equations:

$$\begin{cases} \chi_{i,k-1} = \hat{x}_{k-1}, & i = 0 \\ \chi_{i,k-1} = \hat{x}_{k-1} + \left(a\sqrt{L\hat{P}_{k-1}}\right)_i, & i = 1,2,\ldots,L \\ \chi_{i,k-1} = \hat{x}_{k-1} - \left(a\sqrt{L\hat{P}_{k-1}}\right)_{i-L}, & i = L+1,\ldots,2L \end{cases} \tag{16}$$

where $a \in R$ is a tuning parameter denoting the spread of the sigma points around $\hat{x}_{k-1}$ and $(a\sqrt{L\hat{P}_{k-1}})_i$ is the ith column of the matrix square root of $L\hat{P}_{k-1}$. The parameter is often set to a small positive value.

***Step* 2:** *Prediction*. Each point is instantiated through the process model to yield a set of transformed samples as (17).

$$\chi_{i,k|k-1} = f(\chi_{i,k-1}),\quad i = 0,1,\ldots,2L \tag{17}$$

The predicted mean and covariance are computed as

$$\hat{x}_{k|k-1} = \sum_{i=0}^{2L} w_i \chi_{i,k|k-1} \tag{18}$$

$$\sum_{i=0}^{2L}\left[w_i(\chi_{i,k|k-1} - \hat{x}_{k|k-1})(\chi_{i,k|k-1} - \hat{x}_{k|k-1})^T\right] + Q_k \tag{19}$$

where

$$\begin{cases} w_i = 1 - \dfrac{1}{a^2}, & i = 0 \\ w_i = \dfrac{1}{2La^2}, & i = 1,\ldots,2L \end{cases} \tag{20}$$

***Step* 3:** *Update*. As the measurement equation is linear, measurement update can be performed with the same equations as the classical Kalman filter as (21).

$$\begin{aligned} \hat{y}_k &= H_k \hat{x}_{k|k-1} \\ \hat{P}_{yy} &= H_k \hat{P}_{k|k-1} H_k^T + R_k \\ \hat{P}_{xy} &= \hat{P}_{k|k-1} H_k^T \\ K &= \hat{P}_{xy} \hat{P}_{yy}^{-1} \end{aligned} \tag{21}$$

***Step* 4:** Repeat steps 1 to 3 for the next sample. Clearly, the implementation of the UKF is extremely convenient, because Jacobian matrix is not needed to be evaluated which is necessary in the EKF.

5. Simulation Resullts

In this section, simulation results for chaotic synchronization of fractional-order Lorenz system (5) are presented to investigate the performance of the UKF in comparison with EKF. As mentioned in Section 3, we have implemented the improved Adams–Bashforth–Moulton algorithm for numerical simulation in MATLAB. For the simulation problem, process noise has been added to chaotic system states and the first chaotic state is employed for synchronization. Therefore, the output measurement matrix can be represented by

$$H_k = [1 \quad 0 \quad 0] \tag{22}$$

The initial conditions for the chaotic system, the EKF and UKF are

$$\begin{cases} (x_0, y_0, z_0) = (-1.0032 \quad 2.3545 \quad -0.087)^T \\ (\hat{x}_0, \hat{y}_0, \hat{z}_0) = (20 \quad 15 \quad 15)^T \end{cases} \tag{23}$$

The variances of the process and measurement noise used in the EKF and UKF are

$$R = 0.2, \qquad Q = \begin{bmatrix} 0.18 & 0 & 0 \\ 0 & 0.18 & 0 \\ 0 & 0 & 0.18 \end{bmatrix} \tag{24}$$

In Fig. 2, the attractor of the fractional Lorenz system that is described by (5), using the aforementioned parameters can be seen.

The following results are obtained for chaotic synchronization of the fractional-order Lorenz system, using EKF and UKF methods. Fig. 3 – Fig. 5 represent x and its estimate, y and its estimate and z and its estimate respectively using UKF. These figures show that UKF is capable of achieving synchronization for the system and the synchronization is done with very low error and high speed.

In order to illustrate the superior performance of UKF over EKF, for this simulation, we have calculated the mean squared error (MSE) for the three states using UKF and EKF (TABLE I). The MSE in state estimation is

$$MSE = \frac{1}{N}\sum_{i=0}^{N}\left(x_k(i) - \hat{x}_k(i)\right)^2, \qquad k = 1,2,3 \tag{25}$$

Where $x_k(i)$ and $\hat{x}_k(i)$ are the k th state variable and its estimate at instant of i respectively. As it can be observed in Table I, the UKF method has more accuracy than EKF. Then the UKF performance shows a clear superior results. The UKF outperform all results and proffer possibility to use synchronization under noisy channels for communication applications.

TABLE I: MSE (0-10 SEC) FOR THREE STATE VARIABLES

	Synchronization method	
	EKF	UKF
x	0.7730	0.5862
y	6.3926	3.4060
z	2.4991	1.0989

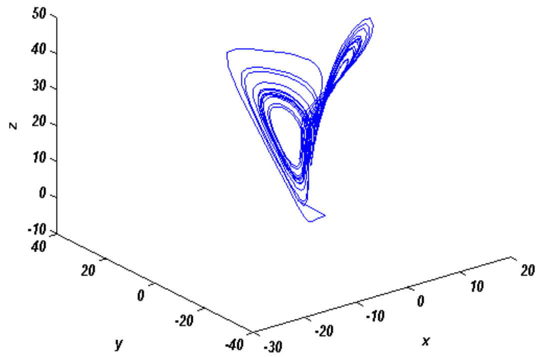

Fig 2: Phase plot of fractional-order Lorenz system in the x-y-z space

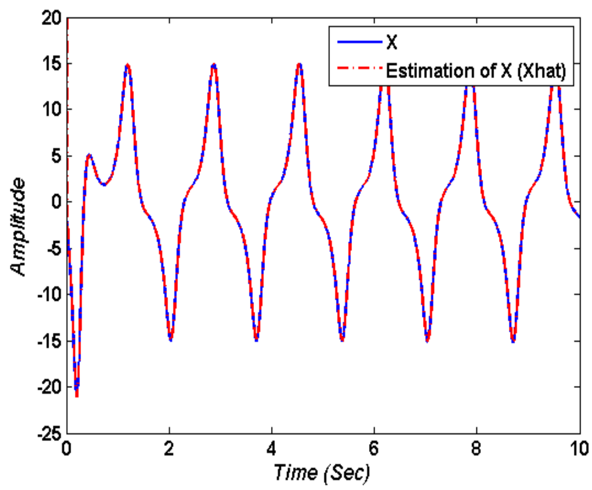

Fig 3: First state of fractional-order Lorenz system and its estimate

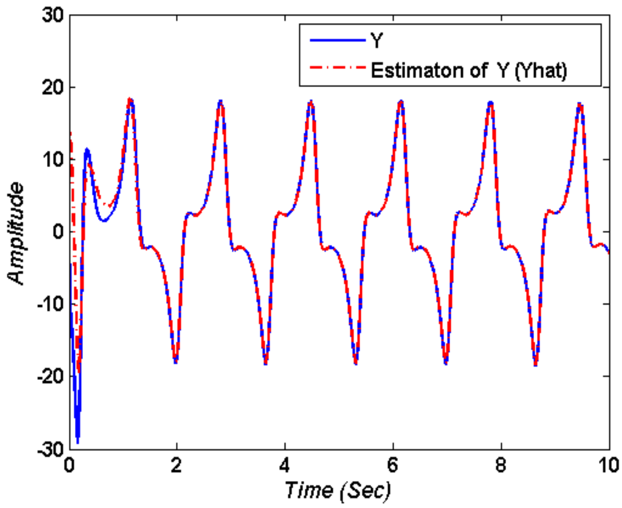

Fig 4: Second state of fractional-order Lorenz system and its estimate

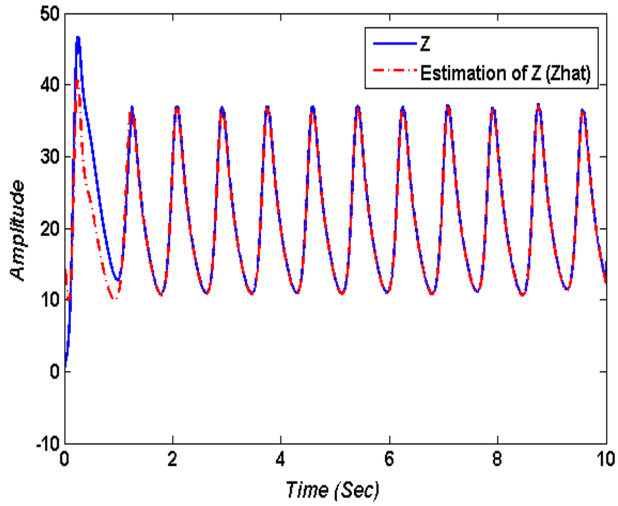

Fig 5: Third state of fractional-order Lorenz system and its estimate

6. Conclusion

This paper showed the synchronization of noisy fractional-order Lorenz chaotic system using the UKF. We have implemented the improved Adams–Bashforth–Moulton algorithm for numerical simulation of fractional-order Lorenz system in MATLAB. The synchronization of the state variables has been done with high accuracy and high speed. The UKF method has been compared with the EKF method to show the improvement of synchronization act and its growth in the performance in regard to accuracy in decreasing state variable estimation error. To more illustrate, the mean square error (MSE) for two methods has been compared. The results of simulation indicate that the UKF method is more accurate than EKF because of the lower MSE in the UKF method.

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