UNIQUELY REMOTAL SETS IN c_0-SUMS AND ℓ^∞-SUMS OF FUZZY NORMED SPACES

A. K. MIRMOSTAFAEE AND M. MIRZAVAZIRI

Abstract. Let (X, N) be a fuzzy normed space and A be a fuzzy bounded subset of X. We define fuzzy ℓ^∞-sums and fuzzy c_0-sums of fuzzy normed spaces. Then we will show that in these spaces, all fuzzy uniquely remotal sets are singletons.

1. Introduction

A famous problem in approximation theory is whether or not sets having unique farthest point property are singletons. Klee [14] proved that any positive answer to this problem in Hilbert spaces implies that all Chebyshev sets are convex. There are many cases in which this problem can be answered affirmatively, such as in finite dimensional spaces [1], in compact sets [14] and in the case that the farthest point map is continuous [6], etc. We refer the reader to [4, 5, 9, 12, 18, 19, 20] and [21] for further results in this direction.

The celebrated paper of Zadeh [23], motivated some authors to develop fuzzy set theory to different branches of pure and applied mathematics [7, 11, 16, 22]. The concept of fuzzy norm was introduced by Katsaras [13] in 1984. Felbin [10] introduced an alternative definition of a fuzzy norm on a linear space. In 2003, following [8], Bag and Samanta in [2] and [3], introduced and studied an idea of a fuzzy norm on a linear space in such a manner that its corresponding fuzzy metric is of Kramosil and Michalek type [15].

In [17], we introduced the notion of fuzzy remotal sets and discussed closability of farthest point maps in fuzzy normed spaces. Given a family of fuzzy normed linear spaces $\{(X_\gamma, N_\gamma)\}_{\gamma\in\Gamma}$, in Section 3, we use an idea from [18] to define fuzzy c_0-sum and fuzzy ℓ^∞-sum of the family $\{(X_\gamma, N_\gamma)\}_{\gamma\in\Gamma}$. Then we will show in these fuzzy normed spaces that every set with unique fuzzy farthest point property must be a singleton.

2. Preliminaries and Some Elementary Results

In this section, we will define and discuss some ideas which will be used in the sequel. To start with, following [2] and [3], we will give the notion of a fuzzy normed space.

Received: May 2011; Revised: December 2011; Accepted: January 2011
Key words and phrases: Fuzzy normed spaces, Fuzzy remotal set.

Definition 2.1. Let X be a complex linear space. By a fuzzy norm on X, we mean a fuzzy subset of $X \times [0, \infty)$ such that the following conditions hold for all $x, y \in X$ and scalars c, s, t:

(N1) $N(x, 0) = 0$ for each $x \neq 0$;
(N2) $x = 0$ if and only if $N(x, t) = 1$ for all $t \geq 0$;
(N3) $N(cx, t) = N(x, \frac{t}{|c|})$, whenever $c \neq 0$;
(N4) $N(x + y, s + t) \geq \min\{N(x, s),\ N(y, t)\}$ (the triangle inequality);
(N5) $\lim_{t\to\infty} N(x, t) = 1$.

A linear space X with a fuzzy norm N, will be denoted by (X, N) and is called a fuzzy normed space. It follows from (N2) and (N4) that $N(x, .)$ is an increasing function for each $x \in X$. In fact, if $x \in X$ and $0 < s < t$, then

$$N(x, t) \geq \min\{N(x, s), N(0, t - s)\} = N(x, s).$$

Example 2.2. Let $(X, \|.\|)$ be a normed linear space. It is easy to verify that

$$N(x, t) = \begin{cases} \frac{t}{t+\|x\|} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

defines a fuzzy norm on X.

Throughout the rest of this section, unless otherwise stated, we will assume that $\alpha \in (0, 1)$ and (X, N) is a fuzzy normed space.

Definition 2.3. A subset A of X is said to be fuzzy α-bounded if there is a real number m such that $N(a, m) \geq \alpha$ for all $a \in A$. It is called fuzzy bounded, if there is some $m > 0$ such that for each $\alpha \in (0, 1)$, $N(a, m) \geq \alpha$ for all $a \in A$.

Example 2.4. In Example 2.2, each bounded subset of $(X, \|.\|)$, as in usual sense, is a fuzzy α-bounded subset of (X, N), but (X, N) has no fuzzy bounded subset, except the set $\{0\}$. Note that even nonzero singletons are not fuzzy bounded in this case.

Definition 2.5. Let A be a fuzzy α-bounded subset of X. For $x \in X$, we define $Q_\alpha(A, x)$ to be

$$\{a \in A : \text{ if } 0 < s < t \text{ then } N(x - a, s) \geq \alpha \text{ implies } N(x - b, t) \geq \alpha \text{ for all } b \in A\}.$$

If there is no danger of ambiguity, we denote $Q_\alpha(A, x)$ simply by $Q_\alpha(x)$. Each element $a \in Q_\alpha(x)$ is called a fuzzy α-farthest point of A from x and the map $x \mapsto Q_\alpha(x)$ is called the α-farthest point map associated with A. The set A is said to have the fuzzy α-farthest point property (or to be fuzzy α-remotal) in X if for each $x \in X$, $Q_\alpha(x)$ is nonempty. A is called α-singleton if for each $a, b \in A$ and each $t > 0$, the relation $N(a - b, t) \geq \alpha$ holds. Clearly, A is singleton if and only if it is α-singleton for all $\alpha \in (0, 1)$. If $Q_\alpha(x)$ is α-singleton then we say that x admits an α-unique α-farthest point in X. A is said to have the fuzzy α-unique α-farthest point property (or to be fuzzy α-uniquely remotal) in X if each $x \in X$ admits an α-unique α-farthest point in A.

Remark 2.6. If for some $x \in X$, $Q_\alpha(x)$ is not empty, then A is fuzzy α-bounded. To see this, let $a \in Q_\alpha(x)$. By (N5) there is an $s_0 > 0$ such that $N(x-a, s_0) \geq \alpha$. Thus for $t_0 = s_0 + 1$ and each $b \in A$ we have $N(x-b, t_0) \geq \alpha$. Moreover, we can find some $t_1 > 0$ such that $N(x, t_1) \geq \alpha$. Thus for $m = t_0 + t_1$ we have $N(b, m) \geq \min\{N(b-x, t_0), N(x, t_1)\} \geq \alpha$ for each $b \in A$. This shows that A is fuzzy α-bounded. So we logically will assume that A is α-bounded in our discussion.

Lemma 2.7. *A fuzzy α-uniquely remotal subset A of X is α-singleton if and only if $a \in Q_\alpha(a)$ for some $a \in A$.*

Proof. Let $a \in Q_\alpha(a)$. Thus for each $b \in A$ and $t > 0$ we have $N(a-b, t) \geq \alpha$, since $N(a-a, s) = 1 \geq \alpha$ for each s. Therefore, we have

$$N(b_1 - b_2, t) \geq \min\{N(b_1 - a, \frac{t}{2}), N(a - b_2, \frac{t}{2})\} \geq \alpha,$$

for each $b_1, b_2 \in A$.

The converse is obvious. □

We need the following result.

Proposition 2.8. *Let A be a fuzzy α-uniquely remotal subset of X. Then A is α-singleton if and only if for each $t > 0$,*

$$\max\{N(x - q_\alpha(x), t), N(y - q_\alpha(y), t)\} < \alpha \tag{1}$$

implies that $N(q_\alpha(x) - q_\alpha(y), t) \geq \alpha$.

Proof. Let $x, y \in X$, $q_\alpha(x) \in Q_\alpha(x)$ and $q_\alpha(y) \in Q_\alpha(y)$. If the inequality (1) holds for each $t > 0$, then by our assumption,

$$N(q_\alpha(x) - q_\alpha(y), t) \geq \alpha,$$

and hence, by the definition, A is α-singleton.

Conversely, assume that the inequality (1) implies that $N(q_\alpha(x) - q_\alpha(y), t) \geq \alpha$ and let $t > 0$ be arbitrary. We will show that $N(q_\alpha(x) - q_\alpha(y), t) \geq \alpha$. This will prove our result. If the inequality (1) does not hold for t and for each $c > 0$, $N(x - q_\alpha(x), \frac{t}{1+c}) \geq \alpha$, then for each $0 < r < t$, if we put $c = \frac{t-r}{r}$, the relation $N(x - q_\alpha(x), r = \frac{t}{1+c}) \geq \alpha$ holds. It follows that

$$N(x - q_\alpha(x), r) \geq \alpha, \quad \forall r > 0.$$

Hence by Lemma 2.7, A is α-singleton. Similar argument shows that if for each $c > 0$, $N(y - q_\alpha(y), \frac{t}{1+c}) \geq \alpha$, then A is α-singleton. Hence, we can assume that there are positive real numbers c_1 and c_2 such that

$$N(x - q_\alpha(x), \frac{t}{1+c_1}) < \alpha, \; N(y - q_\alpha(y), \frac{t}{1+c_2}) < \alpha. \tag{2}$$

Let $c = \max\{c_1, c_2\}$. By (2),

$$N(x - q_\alpha(x), \frac{t}{1+c}) < \alpha, \; N(y - q_\alpha(y), \frac{t}{1+c}) < \alpha.$$

Therefore, by our assumption,

$$N(q_\alpha(x) - q_\alpha(y), t) \geq N(q_\alpha(x) - q_\alpha(y), \frac{t}{1+c}) \geq \alpha.$$

□

Lemma 2.9. *Let A be a subset of X and Q_α denote the α-farthest point map associated with A. Let $a \in A$ and for some $q_\alpha(x) \in Q_\alpha(x)$,*

$$N(a - q_\alpha(x), t) \geq \alpha$$

for all $t > 0$. Then $a \in Q_\alpha(x)$.

Proof. Let $s < t$ and $N(x - a, s) \geq \alpha$. Then for $\varepsilon = t - s$,

$$\begin{aligned} N(x - q_\alpha(x), s + \frac{\varepsilon}{2}) &\geq \min\{N(x-a,s), N(a - q_\alpha(x), \frac{\varepsilon}{2})\} \\ &\geq \min\{\alpha, \alpha\} \\ &= \alpha. \end{aligned}$$

Let $b \in A$. Since $t = s + \varepsilon > s + \frac{\varepsilon}{2}$, the definition of $Q_\alpha(x)$ implies that $N(x - b, t) \geq \alpha$. Thus $a \in Q_\alpha(x)$. □

Corollary 2.10. *Let A be a fuzzy α-uniquely remotal subset of X. If $Q_\alpha(x) \cap Q_\alpha(y) \neq \phi$, then $Q_\alpha(x) = Q_\alpha(y)$.*

Proof. Let $a \in Q_\alpha(x) \cap Q_\alpha(y)$. Then for each $q_\alpha(x) \in Q_\alpha(x)$, $q_\alpha(y) \in Q_\alpha(y)$ and $t > 0$, we have

$$N(a - q_\alpha(x), \frac{t}{2}) \geq \alpha \text{ and } N(a - q_\alpha(y), \frac{t}{2}) \geq \alpha.$$

Therefore

$$N(q_\alpha(x) - q_\alpha(y), t) \geq \min\{N(a - q_\alpha(x), \frac{t}{2}),\ N(a - q_\alpha(y), \frac{t}{2})\} \geq \alpha.$$

Therefore, by Lemma 2.9, $Q_\alpha(x) = Q_\alpha(y)$. □

3. Fuzzy c_0-sums and Fuzzy ℓ^∞-sums of Fuzzy Normed Spaces

Given a family of fuzzy normed spaces, we will define fuzzy ℓ^∞-sums and fuzzy c_0-sums of the family. Then we will show that in this spaces, all fuzzy uniquely remotal sets are singleton.

Definition 3.1. Let $\{(X_\gamma, N_\gamma)\}_{\gamma \in \Gamma}$ be a family of fuzzy normed spaces. Let $X = \Pi_{\gamma \in \Gamma} X_\gamma$. Define

$$N(x, t) = \inf\{N_\gamma(x_\gamma, t) : \gamma \in \Gamma\}, \tag{3}$$

where $x = (x_\gamma)_{\gamma \in \Gamma} \in X$. Then the c_0-sum of the family $\{(X_\gamma, N_\gamma)\}_{\gamma \in \Gamma}$, is defined by

$$\begin{aligned} & c_0(X_\gamma, N_\gamma)_{\gamma \in \Gamma} \\ = \ & \{(x_\gamma)_{\gamma \in \Gamma} \in X : \ \{\gamma : N_\gamma(x_\gamma, \varepsilon) < \alpha\} \text{ is finite for each } \alpha \in (0,1) \text{ and } \varepsilon > 0\}. \end{aligned}$$

We also define the ℓ^∞-sum of the family by

$$\ell^\infty(X_\gamma, N_\gamma)_{\gamma \in \Gamma} = \{x = (x_\gamma)_{\gamma \in \Gamma} \in X : \lim_{t \to \infty} N(x, t) = 1\}.$$

Proposition 3.2. *Let $\{(X_\gamma, N_\gamma)\}_{\gamma\in\Gamma}$ be a family of fuzzy normed spaces. Let $X = c_0(X_\gamma, N_\gamma)_{\gamma\in\Gamma}$ and $Y = \ell^\infty(X_\gamma, N_\gamma)_{\gamma\in\Gamma}$. Then (X, N) and (Y, N) are fuzzy normed spaces, where N is the fuzzy set defined by (3).*

Proof. It follows from the definition that (N1) is satisfied. Hence, we have to show that the conditions (N2)-(N5) of Definition 2.1 hold.

(N2) If $x = (x_\gamma) = 0$, then for each $\gamma \in \Gamma$, $x_\gamma = 0$. Therefore $N_\gamma(x_\gamma, t) = 1$ for each $t \geq 0$. It follows that

$$N(x,t) = \inf\{N_\gamma(x_\gamma, t) : \gamma \in \Gamma\} = 1, \ \forall t \geq 0.$$

Conversely, if $N(x, t) = 1$ for each $t \geq 0$, then for each $\gamma \in \Gamma$ and $t \geq 0$, $N_\gamma(x_\gamma, t) = 1$. Therefore $x_\gamma = 0$ for each $\gamma \in \Gamma$ and hence $x = 0$.

(N3) Let c be a nonzero scalar. Then

$$\begin{aligned} N(cx,t) &= \inf\{N_\gamma(cx_\gamma, t) : \gamma \in \Gamma\} \\ &= \inf\{N_\gamma(x_\gamma, \frac{t}{|c|}) : \gamma \in \Gamma\} \\ &= N(x, \frac{t}{|c|}). \end{aligned}$$

(N4) If x, y in X or Y and $s, t > 0$, then for each $\gamma \in \Gamma$,

$$\begin{aligned} N_\gamma(x_\gamma + y_\gamma, s+t) &\geq \min\{N_\gamma(x_\gamma, s), \ N_\gamma(y_\gamma, t)\} \\ &\geq \min\{N(x,s), N(y,t)\}. \end{aligned}$$

Therefore

$$N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}.$$

(N5) Let $x = (x_\gamma) \in X$. Given $\alpha \in (0,1)$, by definition, there is a finite subset Γ_1 of Γ, such that

$$N_\gamma(x_\gamma, 1) \geq \alpha, \ \forall \gamma \notin \Gamma_1.$$

For each γ in the finite set Γ_1, there is a t_γ such that $N(x_\gamma, t_\gamma) \geq \alpha$. Put $t = \max\{t_\gamma : \gamma \in \Gamma_1\}$. Thus

$$N(x,t) \geq \alpha, \ \forall \gamma \in \Gamma.$$

Since $\alpha \in (0,1)$ is arbitrary, $\lim_{t\to\infty} N(x,t) = 1$ for all $x \in X$. This proves (N5) for (X, N). It follows from definition that (N5) holds for (Y, N). □

In the reminder of this section, assume that α is a fixed element in $(0, 1)$.

Lemma 3.3. *For each $u = (u_\gamma) \in c_0(X_\gamma, N_\gamma)_{\gamma\in\Gamma}$ there is a $\gamma_0 \in \Gamma$ such that for all $\gamma \in \Gamma$,*

$$N_{\gamma_0}(u_{\gamma_0}, s) < \alpha \ \text{or} \ N_\gamma(u_\gamma, t) \geq \alpha \tag{4}$$

if $0 < s < t$.

Proof. By definition, for each $\varepsilon > 0$ the set

$$U_\varepsilon = \{\gamma : N_\gamma(u_\gamma, \varepsilon) < \alpha\}$$

is finite. If U_ε is empty for all $\varepsilon > 0$, then there is nothing to prove. Let U_ε is non-empty for some $\varepsilon > 0$. Fix such an $\varepsilon > 0$. First note that it is sufficient to prove that there is an element $\gamma_0 \in U_\varepsilon$, such that the relation (4) holds for each $\gamma \in U_\varepsilon$. In fact, if for each $\gamma_0 \in U_\varepsilon$, there is some $\gamma_1 \in \Gamma$ such that

$$N_{\gamma_0}(u_{\gamma_0}, s_0) \geq \alpha > N_{\gamma_1}(U_{\gamma_1}, t_0)$$

,0 them since $N_{\gamma_0}(u_{\gamma_0}, \varepsilon) < \alpha$, we have $s_0 > \varepsilon$ and therefore,

$$N_{\gamma_1}(u_{\gamma_1}, \varepsilon) < N_{\gamma_1}(u_{\gamma_1}, t_0) < \varepsilon.$$

Hence $\gamma_1 \in U_\varepsilon$.

Let for each $n \in \mathbb{N}$, $P(n)$ denote, the following statement:

For a nonempty subset $V_n \subset U_\varepsilon$, with n element, there is some $\gamma_0 \in V_n$ such that (4) holds for each $\gamma \in V_n$.

Then $P(1)$ trivially holds. Let $P(n)$ holds and V_{n+1} be a subset of U_ε with $n+1$ elements. Let γ_{n+1} be an arbitrary element of V_{n+1}. Then $V_n = V_{n+1} \setminus \{\gamma_{n+1}\}$ is a subset of U_ε with n element. By our assumption, $P(n)$ holds for some $\gamma_0 \in V_n$. If for some $0 < s_0 < t_0$,

$$N_{\gamma_{n+1}}(u_{\gamma_{n+1}}, s_0) \geq \alpha > N_{\gamma_1}(u_{\gamma_1}, t_0), \tag{5}$$

for some $\gamma_1 \in V_{n+1}$ $\gamma_1 \neq \gamma_{n+1}$, since $s_0 > t_0$. Therefore, $\gamma_1 \in V_n$ and by assumption, $N(u_{\gamma_0}, s_0) < \alpha$. We will show that (4) holds for each $\gamma \in V_{n+1}$. If $0 < s < t$, them by the assumption, (4) holds for each $\gamma \in V_n$. If $N_{\gamma_{n+1}}(u_{\gamma_{n+1}}, t) < \alpha$, then by (5) $s < t < s_0$. Hence

$$N_{\gamma_0}(u_{\gamma_0}, s) \leq N_{\gamma_0}(u_{\gamma_0}, s_0) < \alpha.$$

This proves our induction. Since U_ε is finite, the lemma follows. □

Definition 3.4. X is called α-zero if for each $x \in X$ and $t > 0$ we have $N(x, t) \geq \alpha$.

Theorem 3.5. *Let Γ be any set with $|\Gamma| > 1$, and (X, N) be a fuzzy c_0-sum of the family $\{(X_\gamma, N_\gamma)\}_{\gamma\in\Gamma}$ of fuzzy α-nonzero normed spaces. Then every α-uniquely remotal subset A of X is an α-singleton.*

Proof. Let $x, y \in A$. Applying the above lemma for $u = x - q_\alpha(x)$ and $u = y - q_\alpha(y)$, respectively, we can find $\gamma_1, \gamma_2 \in \Gamma$ such that for all $\gamma \in \Gamma$,

$$N_{\gamma_1}(x_{\gamma_1} - q_\alpha(x)_{\gamma_1}, s) < \alpha \text{ or } N_\gamma(x_\gamma - q_\alpha(x)_\gamma, t) \geq \alpha$$

and

$$N_{\gamma_2}(y_{\gamma_2} - q_\alpha(y)_{\gamma_2}, s) < \alpha \text{ or } N_\gamma(y_\gamma - q_\alpha(y)_\gamma, t) \geq \alpha$$

if $0 < s < t$.

Two cases may happen:

(a) It is possible to choose $\gamma_1 \neq \gamma_2$.

In this case let $z = (z_\gamma)$ be defined as

$$z_\gamma = \begin{cases} x_\gamma & \gamma \neq \gamma_2 \\ y_\gamma & \gamma = \gamma_2 \end{cases}$$

We claim that $q_\alpha(x), q_\alpha(y) \in Q_\alpha(z)$. Indeed, we have to show that for all $b \in A$,

$$N(z - q_\alpha(x), s) < \alpha \text{ or } N(z - b, t) \geq \alpha$$

and

$$N(z - q_\alpha(y), s) < \alpha \text{ or } N(z - b, t) \geq \alpha$$

if $0 < s < t$.

Let $t > 0$ be arbitrary. By Proposition 2.8, we may assume that

$$\max\{N(x - q_\alpha(x), t), N(y - q_\alpha(y), t)\} < \alpha.$$

We will show that $N(z - q_\alpha(x), s) < \alpha$ for all $0 < s < t$. This proves that $q_\alpha(x) \in Q_\alpha(z)$. On the contrary, suppose that for some $0 < s < t$, $N(z - q_\alpha(x), s) \geq \alpha$. Then

$$N_{\gamma_1}(x_{\gamma_1} - q_\alpha(x)_{\gamma_1}, s) \geq N(z - q_\alpha(x), s) \geq \alpha.$$

By the property of γ_1,

$$N_\gamma(x_\gamma - q_\alpha(x)_\gamma, t) \geq \alpha \qquad \forall \gamma \in \Gamma.$$

Hence $N(x - q_\alpha(x), t) \geq \alpha$ which contradicts our hypothesis. Similar argument shows that $q_\alpha(y) \in Q_\alpha(z)$. Hence by Corollary 2.10, $Q_\alpha(x) = Q_\alpha(z) = Q_\alpha(y)$.

(b) It is not possible to choose $\gamma_1 \neq \gamma_2$.

Since $|\Gamma| > 1$, in this case, we can find some $\gamma_0 \neq \gamma_1$. Since A is α-bounded, there is an $m > 0$ such that $N(a, m) \geq \alpha$ for all $a \in A$. Since X_{γ_0} is not α-zero, there is an element $u \in X_{\gamma_0}$ with $N(u, t_0) < \alpha$ for some $t_0 > 0$. Let $w_{\gamma_0} = \frac{2m}{t_0}u$. Then $N_{\gamma_0}(w_{\gamma_0}, 2m) = N_{\gamma_0}(u, t_0) < \alpha$. Define $w = (w_\gamma)$ as

$$w_\gamma = \begin{cases} w_{\gamma_0} & \gamma = \gamma_0 \\ 0 & \text{otherwise} \end{cases}$$

We claim that

$$N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, s) < \alpha \text{ or } N_\gamma(w_\gamma - q_\alpha(w)_\gamma, t) \geq \alpha$$

if $0 < s < t$. To see this let $s < m$. Then

$$\begin{aligned} \alpha &> N_{\gamma_0}(w_{\gamma_0}, 2m) \\ &\geq \min\{N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, m), N_{\gamma_0}(q_\alpha(w)_{\gamma_0}, m)\} \\ &\geq \min\{N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, m), \alpha\}. \end{aligned}$$

Thus $N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, s) < \alpha$.

On the other hand, if $s \geq m$ and $\gamma \neq \gamma_0$, then

$$\begin{aligned} N_\gamma(w_\gamma - q_\alpha(w)_\gamma, t) &= N_\gamma(0 - q_\alpha(w)_\gamma, t) \\ &\geq N_\gamma(q_\alpha(w)_\gamma, m) \\ &\geq \alpha. \end{aligned}$$

Therefore

$$N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, s) < \alpha \text{ or } N_{\gamma_0}(w_{\gamma_0} - q_\alpha(w)_{\gamma_0}, t) \geq \alpha,$$

for all $0 < s < t$. By Case (a), $Q_\alpha(x) = Q_\alpha(w) = Q_\alpha(y)$. □

Lemma 3.6. *Let $u = (u_\gamma) \in \ell^\infty(X_\gamma, N_\gamma)_{\gamma \in \Gamma}$. Then there is a sequence $\{\gamma_n\}$ in Γ such that for each $0 < s < t$, there is some n_0, such that for each $n \geq n_0$,*

$$N_{\gamma_n}(u_{\gamma_n}, s) < \alpha \text{ or } N_\gamma(u_\gamma, t) \geq \alpha \qquad \forall \gamma \in \Gamma. \tag{6}$$

Proof. If $N(u,t) \geq \alpha$ for each $t > 0$, then for each $\gamma \in \Gamma$, $N_\gamma(u_\gamma, t) \geq \alpha$. Otherwise, by (N5) we can find some $t_0 > 0$ such that

$$N(u,s) < \alpha \text{ and } N(u,t) \geq \alpha \tag{7}$$

for each $0 < s < t_0 < t$. Hence for each $n \in \mathbb{N}$, the set $U_n = \{\gamma \in \Gamma : N_\gamma(u_\gamma, t_0 - \frac{1}{n}) < \alpha\}$ is not empty. Let γ_n be an arbitrary element of this set. We will show that for each $0 < s < t$, there is some $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, (6) holds. On the contrary, suppose that there are $\gamma_0 \in \Gamma$ and $0 < s < t$ such that for each $n_0 \in \mathbb{N}$, there is some $n \geq n_0$ with

$$N_{\gamma_n}(u_{\gamma_n}, s) \geq \alpha > N_{\gamma_0}(u_{\gamma_0}, t).$$

Therefore $t_0 - \frac{1}{n} < s$ for infinitely many n. Hence $t_0 \leq s < t$. By (7), $N(u,t) \geq \alpha$. Therefore $N_\gamma(u_\gamma, t) \geq \alpha$, for each $\gamma \in \Gamma$, which contradicts the choice of γ_0. □

Theorem 3.7. *Let Γ be any set with $|\Gamma| > 1$, and (X, N) be a fuzzy ℓ^∞-sum of the family $\{(X_\gamma, N_\gamma)\}_{\gamma\in\Gamma}$ of fuzzy α-nonzero normed spaces. Then every α-uniquely remotal subset A of X is α-singleton.*

Proof. Let $x, y \in A$. By Lemma 3.6, there are sequences $\{\gamma_n\}$ and $\{\gamma'_n\}$ in Γ such that

$$N_{\gamma_n}(x_{\gamma_n} - q_\alpha(x)_{\gamma_n}, s) < \alpha \text{ or } N_\gamma(x_\gamma - q_\alpha(x)_\gamma, t) \geq \alpha$$

and

$$N_{\gamma'_n}(y_{\gamma'_n} - q_\alpha(y)_{\gamma'_n}, s) < \alpha \text{ or } N_\gamma(y_\gamma - q_\alpha(y)_\gamma, t) \geq \alpha$$

for all $\gamma \in \Gamma$, $0 < s < t$ and large enough n.
Two cases may happen:
(a) It is possible to choose $\gamma_n \neq \gamma'_m$ for each $n, m \in \mathbb{N}$.
In this case let $z = (z_\gamma)$ be defined by

$$z_\gamma = \begin{cases} x_\gamma & \gamma \notin \{\gamma'_m\}_{m\in\mathbb{N}} \\ y_\gamma & \gamma \in \{\gamma'_m\}_{m\in\mathbb{N}} \end{cases}$$

Let $t > 0$. We will show that

$$N(z - q_\alpha(x), s) < \alpha \text{ and } N(z - q_\alpha(y), s) < \alpha$$

if $0 < s < t$. This proves that $Q_\alpha(x) = Q_\alpha(z) = Q_\alpha(y)$. By Proposition 2.8, we may assume that

$$\max\{N(x - q_\alpha(x), t), N(y - q_\alpha(y), t)\} < \alpha,$$

so that for all $0 < s < t$ and large enough n,

$$\max\{N_{\gamma_n}(x_{\gamma_n} - q_\alpha(x)_{\gamma_n}, s), N_{\gamma'_n}(y_{\gamma'_n} - q_\alpha(y)_{\gamma'_n}, s)\} < \alpha.$$

It follows from the definition of z that for large enough n,

$$\begin{aligned} &\max\{N(z - q_\alpha(x), s), N(z - q_\alpha(y), s)\} \\ \leq\ &\max\{N_{\gamma_n}(x_{\gamma_n} - q_\alpha(x)_{\gamma_n}, s), N_{\gamma'_n}(y_{\gamma'_n} - q_\alpha(y)_{\gamma'_n}, s)\} < \alpha. \end{aligned}$$

This proves our result in this case.

(b) $\gamma_n = \gamma'_m$ for all $n, m \in \mathbb{N}$. It is easy to see that in this case there exists some $\gamma_0 \in \Gamma$ such that for all $\gamma \in \Gamma$,

$$N_{\gamma_0}(x_{\gamma_0} - q_\alpha(x)_{\gamma_0}, s) < \alpha \text{ or } N_\gamma(x_\gamma - q_\alpha(x)_\gamma, t) \geq \alpha$$

and

$$N_{\gamma_0}(y_{\gamma_0} - q_\alpha(y)_{\gamma_0}, s) < \alpha \text{ or } N_\gamma(y_\gamma - q_\alpha(y)_\gamma, t) \geq \alpha$$

if $0 < s < t$. The rest of the proof is similar to the Case (b) of Theorem 3.5, hence it is omitted. □

Acknowledgements. The authors are grateful to the referee for careful reading the manuscript and useful commends. This research was supported by a grant from Ferdowsi University of Mashhad No. MP90246MIZ.

References

[1] E. Asplund, *Sets with unique farthest points*, Israel J. Math., **5** (1967), 201-209.

[2] T. Bag and S. K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., **11(3)** (2003), 687-705.

[3] T. Bag and S. K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, **151** (2005), 513-547.

[4] M. V. Balashov and G. E. Ivanov, *On farthest points of sets*, Mathematical Notes, **80(1-2)** (2006), 159-166.

[5] M. Baronti and P. L. Papini, *Remotal sets revisited*, Taiwanese J. Math., **5(2)** (2001), 367-373.

[6] J. Blatter, *Weiteste punkte und nachste punkte*, Ren. Poum. Math. Pures Appl., **14** (1969), 615-621.

[7] R. A. Borzooei and M. Bakhshi, *T-fuzzy congruences and T-fuzzy filters of a BL-algebras,* Iranian Journal of Fuzzy Systems, **6(4)** (2009), 37-47.

[8] S. C. Chang and J. N. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Cal. Math. Soc., **86** (2004), 429-436.

[9] S. Cobzas, *Geometric properties of Banach spaces and the existence of nearest and farthest points*, Abstr. Appl. Anal., **3** (2005), 259-285.

[10] C. Feblin, *The completion of a fuzzy normed space*, J. Math. Anal. Appl., **174** (1993), 428-440.

[11] S. B. Hosseini, D. Oregan and R. Saadati, *Some results on intuitonistic fuzzy spaces,* Iranian Journal of Fuzzy Systems, **4(1)** (2007), 53-64.

[12] G. E. Ivanov, *Farthest points and the strong convexity of sets,* (Russian) Mat. Zametki, **87(3)** (2010), 382-395.

[13] A. K. Katsaras, *Fuzzy topological vector space II* , Fuzzy sets and Systems, **12** (1984), 143-154.

[14] L. Klee, *Convexity of Chebyshev sets*, Math. Ann., **142** (1961), 292-304.

[15] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, **11** (1975), 326-334.

[16] A. K. Mirmostafaee, *Perturbation of generalized derivations in fuzzy Menger normed algebras,* Fuzzy sets and systems, doi:10.1016/j.fss.2011.10.015.

[17] A. K. Mirmostafaee and M. Mirzavaziri, *Closability of farthest point maps in fuzzy normed spaces*, Bull. Math. Anal., **2(4)** (2010), 140-145.

[18] A. K. Mirmostafaee and A. Niknam, *A remark on uniquely remotal sets*, Indian J. Pure Appl. Math., **29(8)** (1998), 849-854.

[19] T. D. Narang, *A study of farthest points*, Nieuw Arch. Wisk., **25** (1977), 54-79.

[20] T. D. Narang, *On farthest points in metric spaces*, J. Korea Soc. Math. Educ. Ser B Pure Appl. Math., **9** (2002), 1-7.

[21] T. D. Narang, *Uniquely remotal sets are sigletons*, Nieuw Arch. Wisk., **4(9)** (1991), 1-12.

[22] S. M. Vaezpour and F. Karimi, *t-best approximation in fuzzy normed spaces*, Iranian Journal of Fuzzy Systems, **5(2)** (2008), 93-99.
[23] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338-353.

Alireza Kamel Mirmostafaee*, Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
E-mail address: mirmostafaei@ferdowsi.um.ac.ir

Madjid Mirzavaziri, Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran
E-mail address: mirzavaziri@gmail.com

*Corresponding author