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Analytical Approaches for Friction Induced Vibration and Stability Analysis

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Abstract

The traditional mass on a moving belt model without external force excitation is considered. The displacement and velocity amplitudes and the period of the friction induced vibrations can be predicted using a friction force modelled by the mean of friction characteristics. A more precise look at the non-smooth transition points of the trajectories reveals that an extended friction model is looked-for. In present job, two so-called polynomial and exponential friction functions are investigated. Both of these friction laws describe a friction force that first drops off and then raises with relative interface velocity. An analytical approximation is applied in order to derive relations for the vibration amplitudes and base frequency and in parallel a stability analysis is performed. Moreover, results and phase plots are illustrated for both analytical and numerical approaches.

Keywords: Stick-Slip, Nonlinear Oscillations, Stability.

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1. Introduction

In order to modeling and describing friction induced vibrations, the conventional mass on a moving belt is under investigation. We take this model for granted without discuss the validity for further applications. In driven systems friction often causes stick-slip vibrations. This non-linear effect appears in a wide range of engineering systems. Out coming sound of the violin string, brake squeal and creaking doors are of well-known stick-slip examples. For friction induced vibrations, because of unpredictable properties of contacting surfaces, there are always different valid equations.

For the usual friction laws which are going to be discussed here, the kinetic friction coefficient first decreases and then increases smoothly with siding speed and also there is a small but finite difference between static and kinetic friction coefficients. Hence, in what will follow, the system manner in a full cycle is analyzed. For the case, a simple but handy analytical approximation, so-called perturbation approach, is utilized in order to derive approximate expressions for vibration amplitudes, motion period and stability conditions. In addition, the dynamical behavior of the system under self-excitation will be numerically plotted and compared with experimental data.

A wide range of researches is performed in literature. Table 1 represents a tiny summary of investigations in available data bases.

Table 1. A Compact Literature Review

Author	Approach	Performed work
Panovko and Gubanova [1]	Analytical	• Introducing a polynomial expression for friction law • Self-excited oscillations occurs when the belt velocity is lower than a critical velocity-v_m
Tondl [2], Nayefeh and Mook [3]	Analytical	• Approximate expressions for amplitudes for the case without sticking
Ibrahim [4], McMillan [5]	Analytical Numerical	• Discuss basic mechanics of friction and friction model
Popp [6]	Numerical Experimental	• Presentation of models for system similar to mass on moving belt
Armstrong and Helouvy [7]	Analytical	• perturbation analysis for system with Stribeck friction characteristics • onset of robot arm
Gao et al [8]	Analytical	• Expression for change in position during stick phase in system with linearized friction law
Elmer[9]	Analytical	• Discusses stick-slip in mass-on-belt for different friction laws • Provides expressions for stick-to-slip transition • Sketch typical local and global bifurcation scenarios
Thomsen[10]	Analytical	• Setup approximate expressions for stick-slip oscillations for a friction slider
Hinrichs et al [11]	Numerical	• Suggest an advanced extended friction model

2. System Model

A simple so-called mass-on-moving belt system is considered as reference model. A mass which is equipped by a linear spring and a dash-pot is on a belt moving at constant velocity- V_b, named *excitation velocity* (Fig. 1). [10]

The motion of mass M is described in the way that as the belt is moving by a constant velocity, mass first *sticks* to the belt and moves by the same speed. This is so until the resultant spring and damper restoring force exceeds the maximum static friction force between mass and belt interface. At this moment, mass starts to *slip* on the belt. In the slipping period of motion, friction force followed by a velocity dependant law which is the subject of next section.

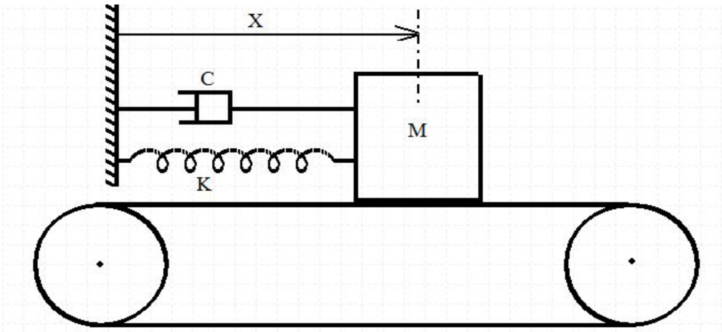

Figure 1. Model of the System under Study

The dimensionless equations of motion can be simply derived as follows.

$$\ddot{u} + 2\beta\dot{u} + u + \gamma^2\mu(\dot{u} - v_b) = 0 \quad \text{(Slip phase)} \quad (1)$$

$$\ddot{u} = 0,\ 2\beta v_b + u \leq \mu_s \quad \text{(Stick phase)} \quad (2)$$

In which u is the dimensionless displacement and μ is the dimensionless friction force or friction efficient. This friction coefficient is a function of relative sliding velocity, governed by a friction law going to be introduced in the following section.

3. Friction Functions

The primer and of course the simplest work on dry friction is attributed to Coulomb in 1785. Despite many years of research, the mathematical description of dry friction phenomenon is not yet fully developed. This is due to the unpredictability of microscopic characteristics of contacting surfaces. Among a variety of approaches to this problem, the most applicable ones are introduced in following. It is pointed out that these particular forms of friction functions are not overly restricted; they resemble the friction characteristics for common use.

3.1 Consideration of Stribeck Effect; Polynomial Description

The most useful and of course moderately simple friction law is the function which relates the kinetic friction coefficient to the relative sliding velocity. This relation is the so-called Stribeck effect. There are two well-known expressions which are directly regarded to this effect: polynomial expression and exponential expression.

The polynomial expression is first suggested by Panovko and Gubanova [1] and Ibrahim [4] in the terms of relative sliding velocity and also friction coefficients:

$$\mu(v_r) = \mu_s sgn(v_r) - k_1 v_r + k_3 {v_r}^3 \tag{3}$$

Where μ_s is the static friction coefficient and k_1 and k_3 are constant introduced by

$$k_1 = \frac{3}{2}(\mu_s - \mu_m)/v_m \tag{4}$$

$$k_3 = \frac{1}{2}(\mu_s - \mu_m)/{v_m}^3 \tag{5}$$

Fig. (2) depicts the behaviour of this function vs. relative velocity for typical parameter values. Apparently we see that μ reaches a minimum, μ_m, while relative velocity increases. The corresponding velocity of this minimum is indicated by v_m.

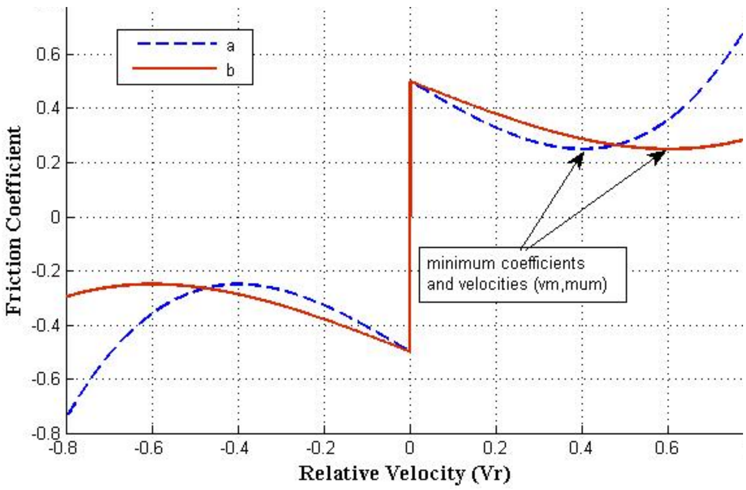

Figure 2. Variation of friction coefficient vs. relative sliding velocity. (a) $v_m = 0.4$ (b) $v_m = 0.6$

3.2 Consideration of Stribeck Effect; Exponential Description

In some simulation approaches [13] the Stribeck effect is modeled using a Gaussian distribution function in order to describe discontinuous nature of transition step from stick to slip. It is argued that the restraining force is a composition of all micro-actions across the interface of containing surfaces and their asperities. The Gaussian model introduced by Eq. (6) is a reasonable continuous approximation for this state change:

$$\mu(v_r) = sgn(v_r)\left\{\mu_k + (\mu_s - \mu_k)exp\left[-\left(\frac{v_r}{v_s}\right)^2\right]\right\} \tag{6}$$

This friction law has an almost similar behavior to polynomial expression. (Fig. (3))

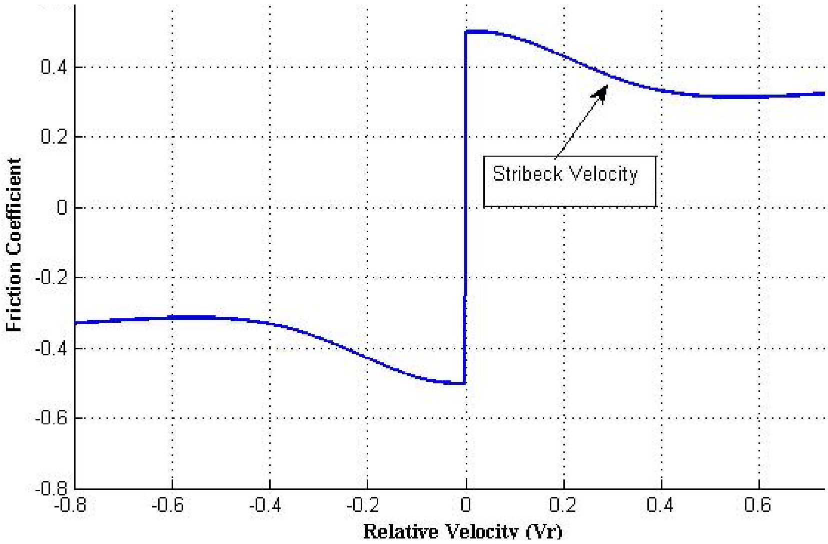

Figure 3. Variation of friction coefficient vs. relative sliding velocity based on an exponential law. At Stribeck velocity concavity of curve changes.

4. Solution Approaches

As far as it expressed in previous section, the friction model are quite complex and of course nonlinear. Thus, exact analytical solutions are not available for such problems. A majority of authors [4,5 &11] have employed the numerical approaches but a few used analytical approximation approaches. This section in devoted to deliberate some of these analytical approximate solutions [10] which are applicable for each model. In parallel for each solution a stability analysis [14] is performed.

4.1 Solution for the case of polynomial expression

For this case firstly the general response is plotted by the mean of MATLAB employing ode45 syntax which is numerically integrated. This plot indicates that in absence of external excitation the self-excited vibrations grow up and stabilize. As will be shown below, for small damping ratios, the stability will attain when the belt velocity is lower than the velocity of the minimum friction coefficient. This should be acceptable, since for greater values of belt velocity, the friction force will increase with velocity and so should the mass move faster than i.e., it will be met by opposing forces. The phase trajectory and velocity response show that the there are some intervals that mass sticks to belt, each interval lasts until the resultant of restoring force of spring and damping force exceeds the maximum attainable friction force.

When there is no harmonic excitation, the motion can be divided into three zones; static equilibrium, pure-slip periodic motion and stick-slip oscillations. In order to initiate the analysis, the Poincare strategy in stability is such followed that we shift our equation of motion to a static equilibrium point of Eq. (1). In this point the static equilibrium conditions are satisfied in the way $\ddot{u} = \dot{u} = 0$.

We may name this point $\bar{u}$:

$$u = \bar{u} = -\gamma^2\mu(-v_r) = \gamma^2\mu(v_r) \quad (7)$$

Now to shift the origin to this equilibrium point, we introduce a new variable which is the displacement from equilibrium $z = u - \bar{u}$.

Inserting this into Eqs. (1&3) we have:

$$\ddot{z} + z + \varepsilon h(\dot{z}) = 0 \quad (8)$$

Where

$$\varepsilon h(\dot{z}) = 2\beta\dot{z} + \gamma^2(\mu(\dot{z} - v_b) - \mu(-v_b)) = h_1\dot{z} + h_2\dot{z}^2 + h_3\dot{z}^3 \tag{9}$$

In which

$$h_1 = 2\beta - \eta(1 - (v_b/v_m)^2) \tag{10-a}$$
$$h_2 = -\eta(v_b/v_m{}^2) \tag{10-b}$$
$$h_3 = \frac{1}{3}\eta/v_m{}^2 \tag{10-c}$$

Here,$\varepsilon \ll 0$. This indicates that damping and friction terms are small compared to stiffness and inertia in the other words, the viscous damping parameter- β and friction difference parameter- η are considered small quantities.

$$\eta = \frac{3}{2}\gamma^2(\mu_s - \mu_m)/v_m \tag{11}$$

4.1.1 Pure-slip periodic motion

In the case of pure-slip oscillations, $\dot{z} < 0$ all the time; i.e. mass will never overhauls the belt velocity and in stick-slip oscillations, $\dot{z} \leq 0$; i.e. mass temporarily sticks to the belt. So when mass purely slips on the belt, we can take advantages of averaging method, initialized by a Van der Pol transformation. It is considered that the periodic solution is in the form of $z = A\sin(\psi)$ in which A and φ are the unknown functions of time are to be determined. Using the rules of this transformation, these two first order differential equations are obtained and can be merely solved [12]:

$$\dot{A} = -\varepsilon h(A\cos\psi)\cos\psi = 0 \tag{12-a}$$
$$A\dot{\theta} = \varepsilon h(A\cos\psi)\sin\psi = 0 \tag{12-b}$$

Integrating and then solve the equation with respect to A represents the nontrivial solution:

$$A(\tau) = \sqrt{-\frac{3}{4}\frac{h_1}{h_3}} = A_0 \tag{13}$$

After obtaining the solution, stability analysis can be performed. After rewriting the Lagrangian form of motion equations in the Hamilton's form, introducing state variables, the characteristic equation of motion which the roots are the eigenvalues of the equilibrium matrix, is obtained. The eigenvalues are:

$$\begin{matrix}\lambda_1 \\ \lambda_2\end{matrix} = \frac{1}{2}\left[-h_1 \pm \sqrt{h_1{}^2 - 4}\right] \tag{14}$$

Based on above equation, the static equilibrium is unstable when $h_1 < 0$. For such unstable equilibrium, there will be a stable periodic solution (stability due to presence of a limit cycle). The amplitude of this solution can be simply obtained by inserting Eq. (10-a & 10-c) into Eq. (13). Thus:

$$A_0 = 2v_m\sqrt{1 - (v_b/v_m)^2 - 2\beta/\eta} = 2\sqrt{v_{b1}{}^2 - v_b{}^2} \tag{15}$$

In which $v_{b1} \equiv v_m\sqrt{1 - 2\beta/\eta}$.

As far as Eq. (15) assumes the pure-slip, no stick will occur. Hence, the increase in amplitude for decreasing v_b will cease when mass sticks to the belt, i.e., when $\max(\dot{u}) = v_b$ and according to the assumed solutionmax$(\dot{u}) = A_0$. So it is found that stick will start when$v_b < v_{b0} = \sqrt{4/3}\,v_{b1}$. Hence, we can claim that the Eq. (15) will valid just for a small portion of belt velocity such that$v_{b0} < v_b < v_{b1}$. So, for belt velocities beyond this range, stick-stick oscillation will take place.

4.1.2 Stick-slip periodic motion

According to the previous discussion, there are three portions of belt velocities; a) $0 < v_b < v_{b0}$ in which stick-slip oscillations will occur, b) $v_{b0} < v_b < v_{b1}$ where pure-slip periodic motion takes place and c) $v_b > v_{b1}$ such that there will be a static equilibrium in this portion.

In the moment when$v_b = v_{b0}$, stick-slip periodic motion just started and in this moment, vibration amplitude will reaches to a maximum: $A_{0\,max} = A_0(v_b = v_{b0}) = v_{b0}$.

In what follows, we start the analysis of stick-slip by assuming after a period of slip, mass just started to stick at time$\tau = \tau_0$. This eventuates that $\dot{u}(\tau_0) = v_b$. As long as mass sticks to the belt, it moves with a constant velocity v_b, so far for sticking portion: $u(\tau) = u(\tau_0) + (\tau - \tau_0)v_b$. This equation is valid until stick stops and mass starts to slip. Consequently, in stick portion, $\ddot{u} = 0$ and $\dot{u} = v_b$. So the equation of motion will get the form $u = -2\beta v_b - \gamma^2\mu(0)$ (see Eq. (1)). Defining time τ_1, in which stick ceases and mass starts slipping again on belt, represents that $u(\tau_1) = -2\beta v_b + \gamma^2\mu_s$. Here motion during slip has a fundamental difference with pure-slip periodic motion which is discussed in advanced. We can logically claim that the oscillations ensue around the equilibrium position given by Eq. (7). Hence a periodic displacement about this position can be written in the form

$$u = A_1 \sin(\tau + \varphi) + \bar{u} \qquad (16)$$

$$\dot{u} = A_1 \cos(\tau + \varphi) \qquad (17)$$

Which in valid for the time $\tau_1 < \tau < \tau_2$ where τ_2 is the time in which slip ends. Now in this point letting $\tau = \tau_1$, considering the equilibrium position in τ_1, then adding squares of Eqs. (16&17) to find A_1, we obtain

$$A_1 = \sqrt{{v_b}^2 + \left(\gamma^2\left(\mu_s - \mu(v_b)\right) - 2\beta v_b\right)^2} = v_b\sqrt{1 + \left(\eta\left(1 - \frac{1}{3}\left(\frac{v_b}{v_m}\right)^2\right) - 2\beta\right)^2} \qquad (18)$$

And finally we can find the frequency of stick-slip vibrations. Based on definition of frequency in the form $\omega = 2\pi/T$, we could have $\omega_{ss} = 2\pi/(\tau_2 - \tau_0)$.

4.1.3 Illustrations

Despite an analytical approximation of the vibration amplitudes, numerical integration of governing ODE of system can return a better perceptive of problem. Fig. (4) which is plotted by the mean of MATLAB®, using Rung-Kutta 4th order numerical integration methods, depicts the behaviour of system in stick-slip oscillations portion. For the case of pure-slip oscillations, after take a look at the equation of motion, we may have Fig. (4), the system response and of course the limit cycle trajectory of system which indicates the stable solution but unstable equilibrium at this condition.

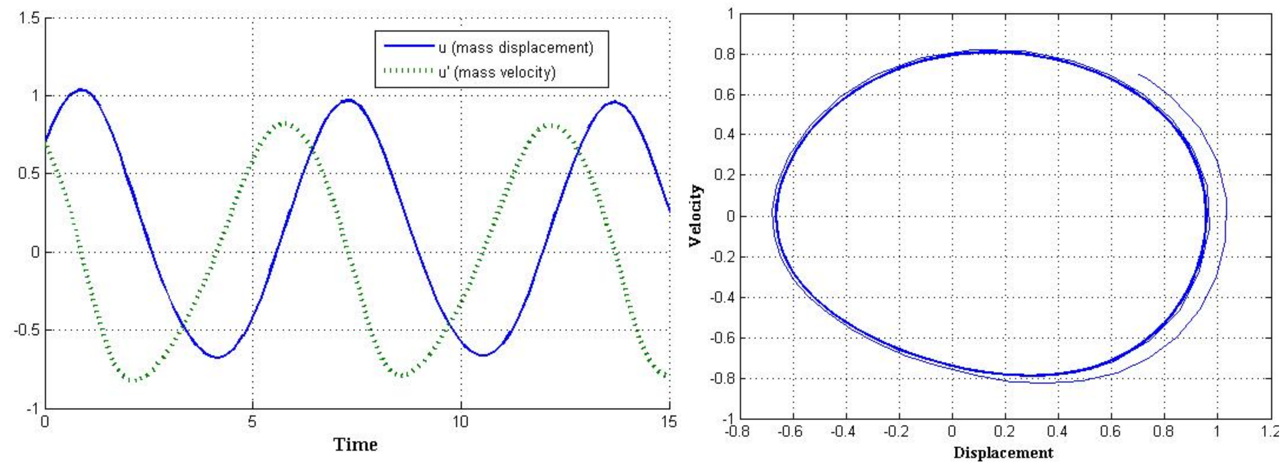

Figure 4. Pure-slip time response and phase plane trajectory for the values:
$\varepsilon = 1, v_m = 0.5, \mu_s = 0.4, \mu_m = 0.25, \beta = 0.05, v_b = 0.4$

Furthermore for the case of stick-slip oscillations, system will behaves quite differently. This is depicted in the most proper view in Fig. (5) and finally the phase plane portrait has been illustrated for this case.

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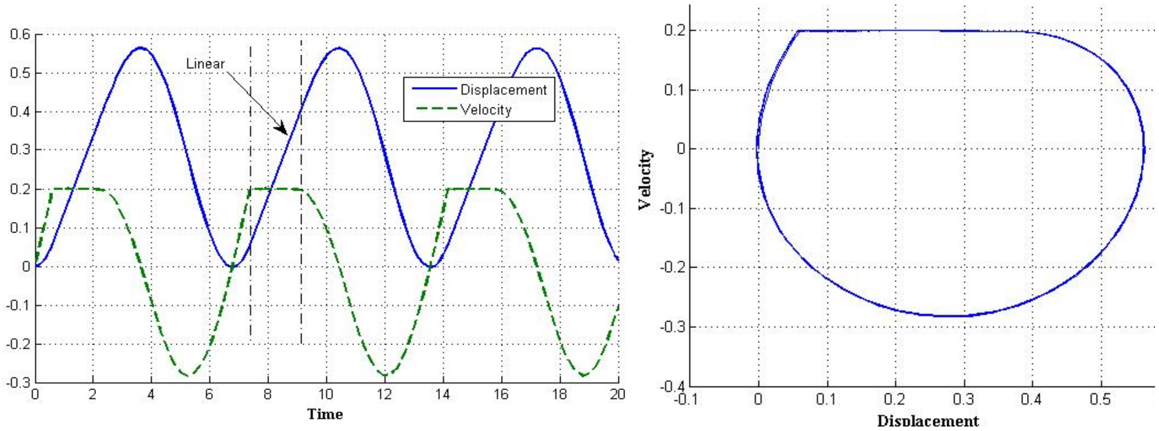

Figure 5. Stick-Slip time responses and phase plane trajectory for the same values but $\boldsymbol{v_b = 0.2}$

Based on obtained analytical approximations for amplitudes, we may see the three distinct portions of motion. First stick-slip oscillations for the range of belt velocity as $0 < v_b < v_{b0}$, then a small transition zone in which $v_{b0} < v_b < v_{b1}$ and for the greater belt velocities, $v_b > v_{b1}$, static equilibrium will be held.

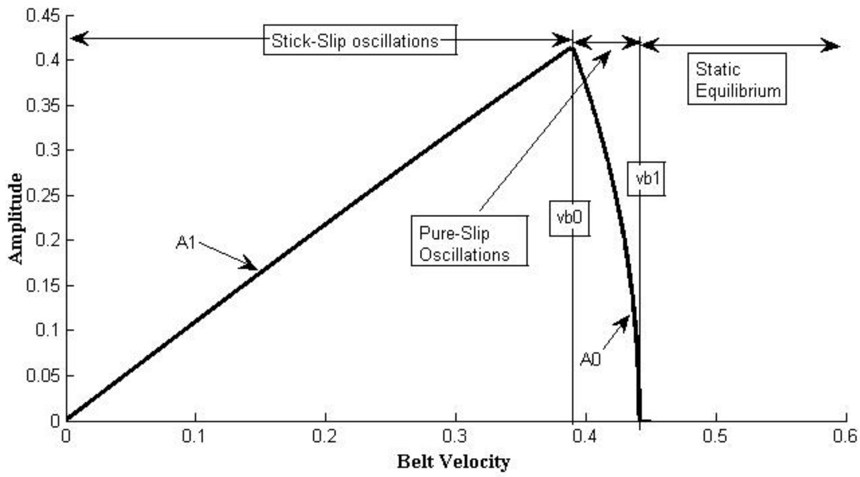

Figure 6. The Amplitude of Response. Three distinct zones are apparent

4.2 Numerical Integration for the Case of Exponential Friction Function

Exponential friction function which is based on the stiffness of asperities of contacting surfaces is a more accurate formulation of friction behaviour. But a significant difficulty arouses during the analytical procedure and that is the integrals are not analytically solvable. So again we return to a numerical integration. For the sake of brevity, self-excited response of system is illustrated and analytical calculations are covered.

Fig. (7) depicts the stick-slip oscillations of the mass. Response similarities are clear in this figure for identical values of polynomial friction function expression.

5. Conclusion

This work introduced a theoretical basis for understanding the nature of friction induced vibrations. For conventional mass-on-belt model, simple analytical expressions are derived to predict the zone in which stick-slip vibrations occur and or the zone in which mass purely slips on the belt and beyond these portions, mass is in stable equilibrium position. Further, a more accurate friction model, exponential, is introduced which would have a more detailed description of friction behaviour. As far as the relations mostly based on analytical approaches the can surely used in designing and laboratory experiments concerning the phenomenon under investigation.

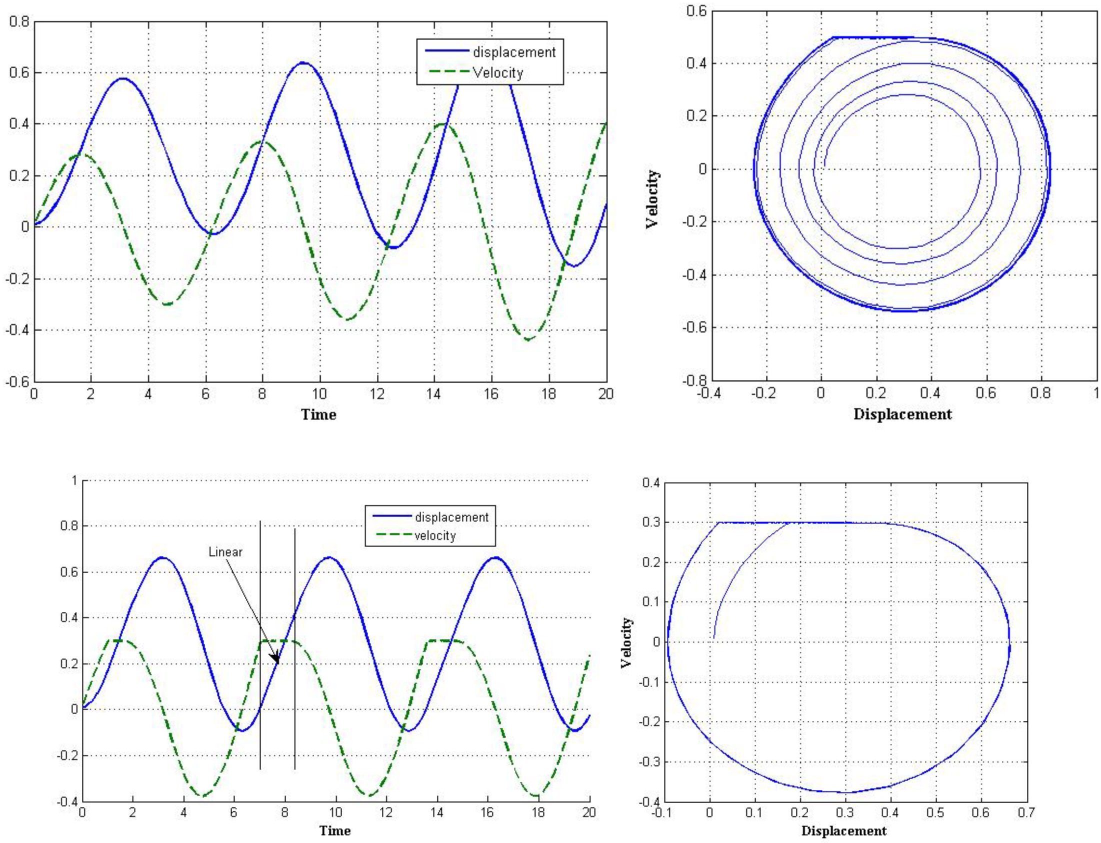

Figure 7. Numerical Response for exponential friction law

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