On the structure of groups whose exterior or tensor square is a p-group

Mohsen Parvizi
Department of Pure Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran
Peyman Niroomand
School of Mathematics and Computer Science, Damghan University, Damghan, Iran

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Table of contents

Abstract

It is well known that if G is a nilpotent (infinite) p-group of bounded exponent, then $G \otimes G$ (resp. $G \wedge G$) is also an (infinite) p-group. We study the converse under some restrictions.

Non-abelian tensor square

Definition

- The tensor square $G \otimes G$ of G is the special case of the non-abelian tensor product of two groups G and H when $G = H$, and G acts on itself by conjugation.

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- The tensor square $G \otimes G$ of G is the special case of the non-abelian tensor product of two groups G and H when $G = H$, and G acts on itself by conjugation.
- The exterior square $G \wedge G$ is obtained by imposing the additional relation $g \otimes g = 1_{\otimes}$ on $G \otimes G$.

Some known results

The tensor square and exterior square of G inherit many properties from G; for example, if G is finite, a p-group, nilpotent, solvable, polycyclic, or locally finite, then so are $G \otimes G$ and $G \wedge G$ [5, 3, 4, 5, 2].

Question

When $G \otimes G$ (resp. $G \wedge G$) is a p-group what are the possible structures for G?

Partial answer

- We prove that for a group G with finitely generated abelianization, $G \otimes G$ is a p-group if and only if G is a p-group. We show that the condition that G^{ab} be finitely generated is essential and cannot be removed.

Partial answer

- We prove that for a group G with finitely generated abelianization, $G \otimes G$ is a p-group if and only if G is a p-group. We show that the condition that G^{ab} be finitely generated is essential and cannot be removed.
- The structure of a group whose exterior square is a p-group, is a semidirect product of a p-group by a cyclic one, but the conditions under which we can deduce this fact are more than that for the tensor square.

Exterior squares: Finite case

We need this...

- Lemma: Let G be a finite group whose commutator subgroup is a p-group for some prime p. Then G is a semidirect product of P, the unique Sylow p-subgroup of G, by H in which $|H|$ and p are coprime.

Exterior squares: Finite case

We recall a result of Tahara [6, Corollary 2.2.6]:

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- Lemma: Let G be the semidirect product of N by H, where $|N|$ and $|H|$ are coprime. Then

$$\mathcal{M}(G) \cong \mathcal{M}(N)^H \oplus \mathcal{M}(H),$$

where $\mathcal{M}(N)^H$ is the H-stable subgroup of $\mathcal{M}(N)$.

Exterior squares: Finite case

Theorem

- Let G be a finite group. Then $G \wedge G$ is a p-group if and only if G is a semidirect product of a p-group by a cyclic group of order coprime to p.

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The last theorem shows that in finite case, if $G \wedge G$ is a p-group, then G need not to be a p-group. In this case even in general, $G/Z^{\wedge}(G)$ is not a p-group.
The following theorem states conditions for a finite group G to conclude $G/Z^{\wedge}(G)$ being a p-group.

Exterior squares: Finite case

Theorem

- Let G be a finite group such that $G \wedge G$ is a p-group. Then $G/Z^{\wedge}(G)$ is p-group if and only if G is nilpotent.

Exterior squares: Infinite case

The infinite case is not as straightforward as the finite case. In the following Lemma and Theorem we assume that $G \wedge G$ is a p-group and try to describe the structure of G. Of course we need to impose some restrictions on G, however we will show these restrictions are essential. First for the abelian case we have:

Exterior squares: Infinite case

Lemma

Let G be a finitely generated abelian group such that $G \wedge G$ is a p-group, then G is the direct sum of a finite p-group and a cyclic group either of infinite order or of finite order coprime to p.

Exterior squares: Infinite case

Lemma

Let G be a finitely generated abelian group such that $G \wedge G$ is a p-group, then G is the direct sum of a finite p-group and a cyclic group either of infinite order or of finite order coprime to p.

Theorem

Let G be a group such that G^{ab} is finitely generated. If $G \wedge G$ is a p-group, then G is the semidirect product of a finite p-group by a cyclic group either of infinite order or of finite order coprime to p.

Exterior squares: Infinite case

The following example shows that the condition on G^{ab} to be finitely generated is essential.

Exterior squares: Infinite case

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Example

Let $G = \mathbb{Z}(q^\infty) \times \mathbb{Z}_p \times \mathbb{Z}_p$ in which p and q are distinct primes and $\mathbb{Z}(q^\infty)$ is a quasicyclic q-group. Since $\mathbb{Z}(q^\infty)$ is a divisible torsion group, we have $\mathbb{Z}(q^\infty) \otimes \mathbb{Z}_p = \mathbb{Z}(q^\infty) \otimes \mathbb{Z}(q^\infty) = 0$ so $G \wedge G \cong \mathbb{Z}_p$ is a p-group, but G is not as the form introduced in the last theorem.

Exterior squares: Infinite case

To prove a converse of the previous theorem, we need to put some extra restrictions on groups. Recall from [1] the relative Schur multiplier of a pair of groups (G, N) is denoted by $\mathcal{M}(G, N)$. In the next contribution we need the following lemma, whose proof can be found in [1].

Exterior squares: Infinite case

To prove a converse of the previous theorem, we need to put some extra restrictions on groups. Recall from [1] the relative Schur multiplier of a pair of groups (G, N) is denoted by $\mathcal{M}(G, N)$. In the next contribution we need the following lemma, whose proof can be found in [1].

Lemma

Let G be the semidirect product of N by Q. Then

(i) $\mathcal{M}(G) \cong \mathcal{M}(G, N) \oplus \mathcal{M}(Q)$

(ii) $\mathcal{M}(G, N) \cong \ker\left(\mu : \mathcal{M}(G) \twoheadrightarrow \mathcal{M}(Q)\right)$

Exterior squares: Infinite case

We also need the following Theorem which comes from [4, Proposition 2.8].

Exterior squares: Infinite case

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Theorem

Let G be a group and N a locally finite normal subgroup of G. If exponent of N is n, then the exponent of $\mathcal{M}(G, N)$ is n-bounded.

Exterior squares: Infinite case

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Theorem

Let G be the semidirect product of a p-group P by an abelian group C. If $\mathcal{M}(C) = 0$, P/G' is of finite exponent, and G' is locally finite of finite exponent, then $G \wedge G$ is a p-group.

Exterior squares: Infinite case

Now we can obtain the following.

Exterior squares: Infinite case

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Theorem

Let G be a group with the following properties:

(i) G' is of finite exponent and locally finite;

(ii) G/G' is finite.

Then $G \wedge G$ is a p-group if and only if $G = C \ltimes P$, where P is a finite p-group and C is a cyclic group of infinite order or of order coprime to p.

Tensor squares: Abelian case

The analogous theorem for $G \otimes G$ gives a more restrictive structure for G. Again we start with abelian groups.

Tensor squares: Abelian case

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Lemma

If G is a finitely generated abelian group and $G \otimes G$ is a p-group, then G is a finite p-group.

Tensor squares: General case

We know that $G \otimes G$ is an (infinite) p-group, when G is a nilpotent (infinite) p-group of bounded exponent. The following theorem shows the converse is true in some cases.

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Theorem

Let G be a group. If G^{ab} is finitely generated and $G \otimes G$ is a p-group, then so is G.

Tensor squares: General case

The following example shows the condition on G^{ab} to be finitely generated is essential and cannot be removed.

Tensor squares: General case

The following example shows the condition on G^{ab} to be finitely generated is essential and cannot be removed.

Example

Let $G = \mathbb{Z}(q^\infty) \times \mathbb{Z}_p$ in which p and q are distinct primes and $\mathbb{Z}(q^\infty)$ is a quasicyclic q-group. Since $\mathbb{Z}(q^\infty)$ is a divisible torsion group, we have $\mathbb{Z}(q^\infty) \otimes \mathbb{Z}_p = \mathbb{Z}(q^\infty) \otimes \mathbb{Z}(q^\infty) = 0$, so $G \otimes G \cong \mathbb{Z}_p$ is a p-group but G is not a p-group.

References

- F. R. Beyl, U. Felgner and P. Schmid, On groups occurring as center factor groups. *J. Algebra* **61** (1979), 161-177.
- R. Brown, D. L. Johnson, and E. F. Robertson, ome computations of non-abelian tensor products of groups. *J. Algebra* **111** (1987), 177-202.
- R. Brown and J.-L. Loday, Van Kampen theorems for diagrams of spaces. *Topology* **26** (1987), 311-335.
- H. Dietrich and P. Moravec, On the autocommutator subgroup and absolute centre of a group. *J. Algebra* **341** (2011), 150-157.
- G. Ellis, The nonabelian tensor product of finite groups is finite. *J. Algebra* **111** (1987), 203-205.
- G. Ellis, On the capability of groups *Proc. Edinburgh Math. Soc.* **(2) 41** (1998), no. 3, 487-495

References

- G. Ellis, The Schur multiplier of a pair of groups. *Appl. Categ. Structures* **6** (1998), no. 3, 355-371.
- I. M. Isaacs, Finite group theory. Graduate Studies in Mathematics, vol. 92, American Mathematical Society, Providence, RI, 2008.
- P. Moravec, The nonabelian tensor product of polycyclic groups is polycyclic. *J. Group Theory* **10** (2007), no. 6, 795-798.
- P. Moravec, The exponents of nonabelian tensor products of groups. *J. Pure Appl. Algebra* **212** (2008), 1840-1848.
- I. Nakaoka, Nonabelian tensor product of solvable groups. *J. Group Theory* **3** (2000), 157-167.
- G. Karpilovsky, The Schur multiplier. London Math. Soc. Monographs, New Series no.2, (1987).

References

M. Parvizi and P. Niroomand, On the structure of groups whose exterior or tensor square is a p-group. *J. Algebra* **352(1)**, (2012), 347-353.

M. P. Visscher, On the nilpotency class and solvability length of nonabelian tensor product of groups. *Arch. Math. (Basel)* **73** (1999), 161-171.