Vibration analysis of plane frames by customized stiffness and diagonal mass matrices

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The manuscript was received on 21 April 2010 and was accepted after revision for publication on 9 March 2011.

DOI: 10.1177/0954406211405435

Abstract: This article presents a formulation for the vibration analysis of plane frames. The strain gradient notation is utilized to determine the mass and stiffness matrices. The obtained matrices can easily be parameterized due to their simple structure. Both Euler-Bernoulli- and Timoshenko-beam elements are investigated in this study. The parameterized stiffness and mass matrices are optimized for accurate performance in the vibration analysis of frame structures. Some numerical examples are solved to show the advantages of the presented scheme. Results of these sample vibration problems indicate that the proposed technique increases the accuracy of analysis, when these new stiffness and diagonal mass matrices are used.

Keywords: Euler-Bernoulli-beam, finite element, mass matrix, stiffness matrix, Timoshenko-beam, vibration analysis

1 INTRODUCTION

Vibration analysis is mostly performed by utilizing the finite element method (FEM) to determine the dynamic behaviour of the frame structures. There are some sophisticated well-performed numerical methods to deal with the problem [**1**, **2**]. The process for vibration analysis needs to solve eigenproblems, which require structural mass and stiffness matrices. These matrices are assembled from the constituent element mass and stiffness matrices. The element mass and stiffness matrices thus significantly contribute to the accuracy and performance of structural analysis.

Many researchers have focused on creating FEs, characterized by their mass and stiffness matrices, with higher performance or specialized application. Their attempts have generated a number of stiffness and mass matrices. Non-conforming shape functions, mixed and hybrid formulations, reduced and selective integration, and a variety of others are some well-known approaches to develop well-behaved stiffness matrices [**3**]. Several others tried to derive well-performed mass matrices for dynamic analysis by FEs. Special attention was paid to formulate efficient diagonal mass matrices to replace conventional lumped mass (LM) matrix introduced by Duncan and Collar [**20**] and the consistent mass matrix (CM) proposed by Archer [**4**]. The CM matrix, though is sufficiently accurate, frustrates for some explicit time integration methods [**5**], for example, the well-known central-difference method.

A familiar approach for developing well-performed mass and stiffness matrices is to scale the matrix entries by some parameters. These scalable matrices may then be customized for special applications. While this scheme has been employed by different investigators, e.g. [**6**, **7**], the most general and matured one was proposed by Felippa, who introduced a new term in FE terminology denoted by 'template'.

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Templates are parameterized algebraic forms that provide a continuum of consistent and stable FEs for a given configuration. Optimal and custom elements can be found by setting appropriate values to free parameters through non-classic optimization procedures. There are some valuable articles that have surveyed the development of FE templates [**8–11**]. Furthermore, Felippa has introduced some tactics to establish FE mass and stiffness templates [**12–14**]. From analysis point of view, templates contribute to FEM technology by their generality and customability features [**11**]. It is important to note that the procedure to formulate and optimize a FE template is almost complicated and may require some innovation, especially for complex configurations.

Felippa proposed different methods to construct the mass matrix for beam elements [**15**]. He customized mass template for the two cases of vibration analysis and wave propagation. However, the authors are not aware whether these templates have been used in the frame vibration analysis or not. Archer and Whalen [**6**] presented a one-factor scalable diagonal stiffness matrix to account for the angular momentum conservation. They observed that lower frequency modes of vibration are estimated fairly well when their rotationally consistent mass (RCM) matrix, rather than the conventional LM matrix, is employed. Nevertheless, this trend ceases for higher frequency modes.

This article presents a new technique to formulate parameterized mass and stiffness matrices for the two types of Euler-Bernoulli and Timoshenko beam elements. The strain gradient notation (SGN) is utilized in this study. It should be reminded, this innovative displacement-based FE formulation, which can deal with both C0 and C1 elements, was first used by Dow [**16**]. He admirably employed the notation to detect and omit modelling errors at the element level. However, the approach could not replace well-known classical FE formulations for other applications, due to its inability to consider boundary conditions for visible degrees of freedom (DOF) at element nodes. SGN is especially characterized for its physical and microscopic view towards FE method at the element level, as well as its ability to separate rigid body motions (RBMs) from other strain states. The proposed formulation, which is based on SGN, will significantly simplify the procedures to parameterize and optimize mass and stiffness matrices.

In Section 2, an approach to generate basis functions for strain gradients (SGs) is introduced. These functions are then employed in Section 3 to calculate mass and stiffness matrices in the SG system. In Section 4, the suggested scheme is implemented for the Euler-Bernoulli-beam element. A procedure to parameterize the calculated mass and stiffness matrices will produce a simple representation for the mass and stiffness templates of the beam element. The proposed templates include Euler-Bernoulli- and Timoshenko-beam, as well. In Section 5, the mass and stiffness templates are customized for exact evaluation of natural frequencies for the first and second modes of the Euler-Bernoulli-beam with different end conditions. The same process will be repeated in Section 6 for the Timoshenko-beam element. In all cases, mass templates are customized so that the calculated mass matrices are diagonal. It is worth emphasizing that these diagonal matrices will be different from the well-known LM matrices. In Section 7, the obtained customized matrices will be utilized in some numerical experiments, to verify the efficiency of the method for the vibration analysis of plane frame structures.

2 STRAIN STATES AND BASE FUNCTIONS

In the FE method, analysts seek an unknown field of displacement, stress, or strain over the structural domain. This field is assumed to be a linear combination of some basic sub-fields. In the displacement formulation scheme, element deformation is considered as a superposition of a set of basic functions, known as the standard basis. From an FE point of view, these basic deformations are also called shape functions. Such functions display how the element deforms when subjected to a unit displacement at each DOF, while others are constrained against movement.

The following displacement function, u, is presumed to be a linear combination of the shape functions, with its coefficients representing the unknown nodal displacements

$$u = \boldsymbol{N}\boldsymbol{D} \tag{1}$$

where $\boldsymbol{N}$ is the matrix of shape functions and $\boldsymbol{D}$ the vector of nodal displacements. It should be added that the standard basis is just one of an infinite number of base functions which can be considered for the element deformation space. The standard shape functions have a vast practical use in the FE technique, since their coefficients are simply the nodal displacements. In the most convenient way, these nodal values can easily fulfill the inter-element compatibility requirements.

From another interesting viewpoint, the element deformation can be represented by a different useful set of basic functions, which is called the strain-state basis in this study. This type of basis contains RBMs, as well as, constant and higher order strain states. The RBMs might be called zero-order strain states. It should be added that each strain state is associated with a base function, displaying a corresponding deformation of the element.

The base functions may be reproduced by a novel formulation denoted by SGN. It is worth emphasizing that the mentioned strategy turns out to be a revised form of the displacement-based FE formulation. In this approach, the displacement field is extrapolated by SGs at an arbitrary point along the element, mostly at the mid-point, rather than nodal displacements. SGs refer to derivatives of the displacement field with different orders, evaluated at the mid-point. Each SG might then represent a special strain state, depending on its corresponding base function.

In the following, a revised form of the SG formulation is introduced. The displacement field is assumed to be indicated by a polynomial as given below

$$\boldsymbol{u} = \boldsymbol{P}\boldsymbol{\alpha} \tag{2}$$

where $\boldsymbol{P}$ and $\boldsymbol{\alpha}$ are the polynomial matrix and the coordinate vector, respectively. SGs are obtained by evaluating appropriate derivatives of equation (2) at the mid-point having known coordinates x_0. Any FE with n DOFs is capable of modelling n strain states or SGs, as follows

$$\begin{aligned} q_1 &= L^{(1)}\boldsymbol{P}\boldsymbol{\alpha}\big|_{x_\circ} \\ q_2 &= L^{(2)}\boldsymbol{P}\boldsymbol{\alpha}\big|_{x_\circ} \\ &\cdot \\ &\cdot \\ &\cdot \\ q_n &= L^{(n)}\boldsymbol{P}\boldsymbol{\alpha}\big|_{x_\circ} \end{aligned} \tag{3}$$

Here, $L^{(i)}$, $0 \le i \le n$ is an appropriate differential operator for the strain state i, and $q_1, \ldots, q_n$ are SGs. These relations can be rewritten in the following matrix form

$$\begin{aligned} \boldsymbol{q} &= \boldsymbol{G}_q\,\boldsymbol{\alpha} \\ \boldsymbol{G}_q &= \begin{bmatrix} L^{(1)}\boldsymbol{P}(x_\circ) \\ \vdots \\ L^{(n)}\boldsymbol{P}(x_\circ) \end{bmatrix}_{n\times n} \end{aligned} \tag{4}$$

where $\boldsymbol{G}_q$ is a transform matrix, which relates the polynomial coefficient vector $\boldsymbol{\alpha}$ to the SG vector $\boldsymbol{q}$. The parameter $\boldsymbol{\alpha}$ can thus be defined as given below

$$\boldsymbol{\alpha} = \boldsymbol{G}_q^{-1}\,\boldsymbol{q} \tag{5}$$

By substituting equation (5) into equation (2), the displacement field is obtained in terms of SGs, as follows

$$\boldsymbol{u} = \boldsymbol{P}\,\boldsymbol{G}_q^{-1}\,\boldsymbol{q} \tag{6}$$

In the following, a base matrix $\boldsymbol{N}_q$ is introduced, which extrapolates the displacement field by SGs $\boldsymbol{q}$

$$\boldsymbol{u} = \boldsymbol{N}_q\,\boldsymbol{q} \tag{7}$$

Utilizing equations (6) and (7), the base matrix will be obtained as shown below

$$\begin{aligned} \boldsymbol{N}_q &= \boldsymbol{P}\boldsymbol{G}_q^{-1} \\ \boldsymbol{N}_q &= [N_q1 \quad N_q2 \quad \cdots \quad N_qn] \end{aligned} \tag{8}$$

Each entry of this matrix denotes a base function for a strain state. In the next section, this matrix is employed to calculate the mass and stiffness matrices in terms of the basis of strain states.

3 MASS AND STIFFNESS MATRICES

From the mathematical point of view, the mass and stiffness matrices may be interpreted as representations for linear mappings which map vectors from the vector space of element displacement, velocity or acceleration into the vector space of forces. All spaces have the same dimension, which is equal to the number of DOFs for that element. Each representation for a linear mapping is associated with a basis for the two vector spaces. Moreover, the basis must meet the criteria of linear independency.

The well-known mass and stiffness matrices used for structural dynamic analysis are obtained in the standard or canonical basis, with the following base vectors

$$\begin{aligned} e_1 &= \{1 \quad 0 \quad \cdots 0\} \\ e_2 &= \{0 \quad 1 \quad \cdots 0\} \\ &\vdots \\ e_n &= \{0 \quad 0 \quad \cdots 1\} \end{aligned} \tag{9}$$

To establish the mass or stiffness matrix in the standard basis, it is a common procedure to calculate the forces developed at all DOF, when the element is subjected to a unit acceleration or displacement of a desired DOF, while all others are constrained against movement. This will develop a canonical base vector for the element. The analyst could use these matrices to write the following well-known equation of motion

$$\boldsymbol{M}\,\ddot{\boldsymbol{D}} + \boldsymbol{K}\,\boldsymbol{D} = \boldsymbol{P} \tag{10}$$

In this equation, the damping term is neglected for the sake of simplicity. On the other hand, equation (10) may alternatively be rewritten for SG basis as below

$$\boldsymbol{M}_q\,\ddot{\boldsymbol{q}} + \boldsymbol{K}_q\,\boldsymbol{q} = \boldsymbol{P}_q \tag{11}$$

where

$$\boldsymbol{M}_q = \boldsymbol{G}^T \boldsymbol{M}\, \boldsymbol{G} \tag{12}$$

$$\boldsymbol{K_q} = \boldsymbol{G}^T \boldsymbol{K}\, \boldsymbol{G} \tag{13}$$

$$\boldsymbol{q} = \boldsymbol{HD} \tag{14}$$

$$\boldsymbol{P_q} = \boldsymbol{G}^T \boldsymbol{P} \tag{15}$$

The mass and stiffness matrices for SG basis are obtained by the following relations

$$\boldsymbol{M_q} = \int_V \rho\, \boldsymbol{N_q}^T \boldsymbol{N_q}\, dV \tag{16}$$

$$\boldsymbol{K_q} = \int_V \boldsymbol{B}_q^T \boldsymbol{D}_m \boldsymbol{B}_q\, dV \tag{17}$$

In equation (17), $\boldsymbol{B}_q$ denotes the strain–SG matrix that can be calculated from the base functions by the following relation

$$\boldsymbol{B}_q = \boldsymbol{\Delta}\, \boldsymbol{N}_q \tag{18}$$

The FE compatibility relation could be utilized to obtain $\boldsymbol{\Delta}$, which is called the differential operator matrix.

4 PARAMETERIZED MASS AND STIFFNESS MATRICES FOR BEAM ELEMENT

It is worthwhile to note that the mass and stiffness matrices, which were developed in the SG basis, are either the diagonal one or the one near the diagonal and is sparse. Free parameters can then be easily replaced with their non-zero entries. Furthermore, a fundamental feature of SGN is its ability to separate RBMs from other strain states. In fact, the entries corresponding to RBMs could be simply detected in their mass and stiffness matrices developed by SGN. Since no strain energy is stored by RBMs, the corresponding entries in the SG-basis stiffness matrix are equal to zero. Whereas, the entries corresponding to constant strain states will not be parameterized to maintain convergence criteria.

The conservation of linear momentum, as well as the angular momentum, can be considered in the RBM entries of the FE SG-basis mass matrix. As the conservation of linear momentum may not be disregarded, its corresponding entry will not accept any parameterization.

The aforementioned procedure will now be followed to formulate the parameterized mass and stiffness matrices for the elastic uniform Euler–-Bernoulli-beam element. In the following formulation, displacements are assumed to be small and the equilibrium equations are established for the undeformed state of the beam element by a Lagrangian coordinate system. In other words, the geometrical non-linearity is ignored. The axial and flexural effects are supposed to be decoupled. For the sake of simplicity, axial deformation will not be considered in this study. Shear and torsion deformations are also neglected. Note that the following relations are established at the element level.

The beam element has four DOFs, which are 'a displacement' and 'a rotation' at each end node. Therefore, it can model four strain states as depicted in Fig. 1 by the following SGs

$$\begin{aligned} q_1 &= w_\circ \\ q_2 &= \theta_\circ \\ q_3 &= \kappa_{x\circ} \\ q_4 &= \kappa_{xx\circ} \end{aligned} \tag{19}$$

These SGs are related to the values of deflection, rotation, constant curvature, and linear curvature at the mid-point, respectively. The polynomial matrix $\boldsymbol{P}$ is selected as follows

$$\boldsymbol{P} = [1 \quad x \quad x^2 \quad x^3] \tag{20}$$

The first four derivatives of the polynomial matrix is given below

$$\begin{aligned} \boldsymbol{P} &= [1 \quad x \quad x^2 \quad x^3] \\ \boldsymbol{P}' &= [0 \quad 1 \quad 2x \quad 3x^2] \\ \boldsymbol{P}'' &= [0 \quad 0 \quad 2 \quad 6x] \\ \boldsymbol{P}''' &= [0 \quad 0 \quad 0 \quad 6] \end{aligned} \tag{21}$$

Replacing the mid-point coordinate $(x=0)$ will result in the following transform matrix, $\boldsymbol{G_q}$

$$\boldsymbol{G}_q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \tag{22}$$

Inverting this matrix and using equation (8), the base functions in the SG basis are obtained as given below

$$\boldsymbol{N}_q = \begin{bmatrix} 1 & \boldsymbol{x} & \frac{\boldsymbol{x}^2}{2} & \frac{\boldsymbol{x}^3}{6} \end{bmatrix} \tag{23}$$

At this stage, the transform matrix $\boldsymbol{G}$ is calculated by substituting the two end coordinates into the base functions and their first derivatives. The result is given as follows

$$\boldsymbol{G} = \begin{bmatrix} 1 & \frac{-L}{2} & \frac{L^2}{8} & \frac{-L^3}{48} \\ 0 & 1 & \frac{-L}{2} & \frac{L^2}{8} \\ 1 & \frac{L}{2} & \frac{L^2}{8} & \frac{L^3}{48} \\ 0 & 1 & \frac{L}{2} & \frac{L^2}{8} \end{bmatrix} \tag{24}$$

Columns of $\boldsymbol{G}$, considered as individual vectors, represent the strain states for the beam element, as shown in Fig. 1. The inverse of $\boldsymbol{G}$, which will be used further, has the following form

$$\boldsymbol{H}=\begin{bmatrix} \frac{1}{2} & \frac{L}{8} & \frac{1}{2} & \frac{-L}{8} \\ \frac{-3}{2L} & \frac{-1}{4} & \frac{3}{2L} & \frac{-1}{4} \\ 0 & \frac{-1}{L} & 0 & \frac{1}{L} \\ \frac{12}{L^3} & \frac{6}{L^2} & \frac{-12}{L^3} & \frac{6}{L^2} \end{bmatrix} \tag{25}$$

The differential operator $\boldsymbol{\Delta}$, for the Euler–Bernoulli-beam element, is the second derivative. By utilizing equation (18), $\boldsymbol{B}_q$ is obtained as given below

$$\boldsymbol{B}_q=\begin{bmatrix} 0 & 0 & 1 & x \end{bmatrix} \tag{26}$$

In SG basis, the mass and stiffness matrices are calculated by using equations (16) and (17). They appear as follows

$$\boldsymbol{M}_q=\begin{bmatrix} mL & & & \\ 0 & \frac{mL^3}{12} & & \\ \frac{mL^3}{24} & 0 & \frac{mL^5}{320} & \\ 0 & \frac{mL^5}{480} & 0 & \frac{mL^7}{16128} \end{bmatrix} \tag{27}$$

$$\boldsymbol{K}_q=\begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & EIL & \\ 0 & 0 & 0 & \frac{EIL^3}{12} \end{bmatrix} \tag{28}$$

The parameterized forms of the mass and stiffness matrices can now be developed by inserting free parameters in non-zero entries. In addition, the requirements for the conservation of linear momentum for the mass matrix, as well as, the convergence criteria for the stiffness matrix should always be considered [**8**, **3**]. The parameterized mass and stiffness matrices will finally have the following structure

$$\boldsymbol{M}_q(\mu_1,\ldots,\mu_5)=\begin{bmatrix} mL & & & \\ 0 & \mu_1 mL^3 & & \\ \mu_4 mL^3 & 0 & \mu_2 mL^5 & \\ 0 & \mu_5 mL^5 & 0 & \mu_3 mL^7 \end{bmatrix} \tag{29}$$

$$\boldsymbol{K}_q(\beta)=\begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & \alpha EIL & \\ 0 & 0 & 0 & \beta EIL^3 \end{bmatrix} \tag{30}$$

These two equations present a general template for mass and stiffness matrices of the beam element. In other words, they can be used in the analysis of Euler–Bernoulli- and Timoshenko-beams. For instance, the well-known LM matrix can be retrieved from the proposed mass template by assigning the following values for the free parameters

$$\mu_1=\frac{1}{4};\ \mu_2=\frac{1}{64};\ \mu_3=\frac{1}{2304};\ \mu_4=\frac{1}{8};\ \mu_5=\frac{1}{96} \tag{31}$$

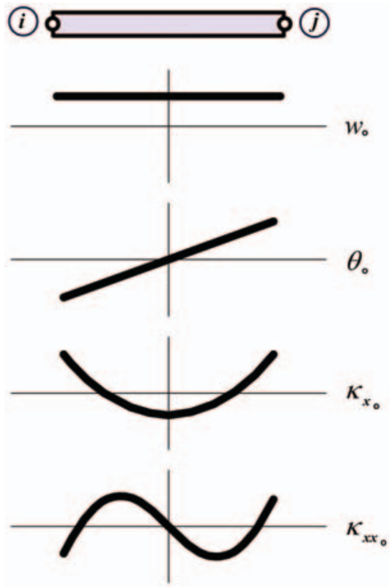

Fig 1 Strain states for Euler-Bernoulli-beam element

The RCM matrix, which satisfies the conservation law for both linear and angular momentum, may be reproduced by the free parameters given below

$$\mu_1=\frac{27}{140};\ \mu_2=\frac{3}{2240};\ \mu_3=-\frac{37}{80640};\ \mu_4=\frac{1}{8};\ \mu_5=\frac{11}{3360} \tag{32}$$

These values give the same mass matrix as obtained by Archer and Whalen [**6**]. Furthermore, the well-known CM matrix could be obtained by utilizing the following values

$$\mu_1=\frac{1}{12};\ \mu_2=\frac{1}{320};\ \mu_3=\frac{1}{16128};\ \mu_4=\frac{1}{24};\ \mu_5=\frac{1}{480} \tag{33}$$

The classical Euler–Bernoulli-beam stiffness matrix may also be determined by substituting $\alpha=1$ and $\beta=\frac{1}{12}$ into equation (30). It should be added that $\alpha=1$ is considered to preserve element convergence. However, in some cases, for instance, some versions of Timoshenko-beam, it may take other values, which will be discussed later. The procedure to find values for the free parameters needs some optimality criteria, which will be explained in the following section.

5 CUSTOMIZED EULER–BERNOULLI-BEAM ELEMENT

At this stage, after the parameterized mass and stiffness matrices are formulated, they can be customized to exactly evaluate the natural frequencies of any arbitrary beam with different end conditions. In this section, the optimization process is performed for the three end conditions: pinned-pinned or simple beam, fixed-free or cantilever beam and fixed-pinned. As diagonal mass matrices significantly simplify the analysis procedures, a diagonal instance of the mass template will be employed for all computations. This diagonal mass template is defined by the following parameters

$$\begin{aligned}\mu_1 &= \frac{3}{16} + 4\mu_2;\;\; \mu_3 = -\frac{5}{9216} + \frac{1}{16}\mu_2;\\ \mu_4 &= \frac{1}{8};\;\; \mu_5 = \frac{1}{384} + \frac{1}{2}\mu_2\end{aligned} \tag{34}$$

These relations are obtained by setting non-diagonal entries of the parameterized mass matrix in the standard basis, $\boldsymbol{M}(\mu_1, \ldots, \mu_5)$, to zero. By using the inverse form of the transformation introduced in equation (12), this template is obtained from the mass template $\boldsymbol{M}_q\ (\mu_1, \ldots, \mu_5)$ in equation (29). It is easy to notice that all parameters are dependent on a free parameter μ_2. In fact, four independent linear equations having five unknowns, $\mu_1, \ldots, \mu_5$, are obtained by setting to zero the six non-diagonal entries of the symmetric mass template. As the following matrix shows, the formulated diagonal mass template can be written in terms of only one parameter

$$\boldsymbol{M}(\mu_2) = \begin{bmatrix} \frac{mL}{2} & & & \\ 0 & \frac{1}{32}mL^3(64\mu_2 - 1) & & \\ 0 & 0 & \frac{mL}{2} & \\ 0 & 0 & 0 & \frac{1}{32}mL^3(64\mu_2 - 1) \end{bmatrix} \tag{35}$$

5.1 Pinned-pinned end condition

The parameterized mass and stiffness matrices are customized to find the exact values for the first two natural frequencies of the simple beam. By utilizing the stiffness and diagonal mass templates, respectively from equations (30) and (35), and performing a simple vibration analysis, the first two natural frequencies of the simple beam could be obtained. The results have following forms

$$\omega_1^2 = \frac{64}{64\,\mu_2 - 1}\frac{EI}{m\,L^4} \tag{36}$$

$$\omega_2^2 = \frac{2304\,\beta}{64\,\mu_2 - 1}\frac{EI}{mL^4} \tag{37}$$

On the other hand, exact values for the first two natural frequencies of the simple beam may be obtained from the analytical solution of the governing equation. They have the following amounts

$$\omega_1^2 = \pi^4\frac{EI}{mL^4} \tag{38}$$

$$\omega_2^2 = 16\pi^4\frac{EI}{mL^4} \tag{39}$$

Setting the parameterized values equivalent with those of the exact ones yield the free parameters as given below

$$\mu_2 = \frac{1}{64} + \frac{1}{\pi^4} = 0.025891 \tag{40}$$

$$\beta = \frac{4}{9} = 0.44444 \tag{41}$$

By substituting these values into the templates, the customized mass and stiffness matrices turn out to be

$$\boldsymbol{M} = \begin{bmatrix} 0.5\ mL & & & \\ 0 & 0.020532\ mL^3 & & \\ 0 & 0 & 0.5\ ml & \\ 0 & 0 & 0 & 0.020532\ mL^3 \end{bmatrix} \tag{42}$$

$$\boldsymbol{K} = \frac{EI}{L^3}\begin{bmatrix} 64 & & & \\ 32L & 17L^2 & & \\ -64 & -32L & EIL & \\ 32L & 15L^2 & -32L & 17L^2 \end{bmatrix} \tag{43}$$

It is worth emphasizing that the value obtained for the free parameter β in equation (41) will not yield the exact well-known Euler–Bernoulli-beam stiffness matrix. This fact is obvious from equation (43). The obtained mass matrix is not capable of satisfying the conservation law for angular momentum. A comment should be made that the presented optimized values for mass and stiffness matrices are just qualified for a vibration analysis of a simple beam. In fact, the formulated mass and stiffness matrices may not be efficient for other analytical applications.

In an alternative customization procedure, one may set the free parameter β equal to $\frac{1}{12}$ to restore the well-known stiffness matrix for the Euler-Bernoulli-beam element. The diagonal mass template may then be customized for exact evaluation of the first and second natural frequencies individually. To retrieve the exact natural frequency for the first mode, the procedure will yield the same value for μ_2 as given by equation (40). This value is not appropriate to estimate the frequency for the second mode. In fact, the following value for μ_2 should be used to evaluate the second mode frequency

$$\mu_2 = 0.017611 \tag{44}$$

This will lead to the diagonal mass matrix given as follows

$$\boldsymbol{M} = \begin{bmatrix} 0.5\ mL & & & \\ 0 & 0.0039721\ mL^3 & & \\ 0 & 0 & 0.5\ mL & \\ 0 & 0 & 0 & 0.0039721\ mL^3 \end{bmatrix} \tag{45}$$

This mass matrix may not be efficient to compute the first natural frequency of the simple beam.

5.2 Fixed-free end condition

In this section, the stiffness and diagonal mass templates are customized for exact evaluation of the first two natural frequencies of a cantilever beam. As a rational approach, β is set to $\frac{1}{12}$ to retrieve the Euler-Bernoulli-beam stiffness matrix, which exactly includes the bending strain energy. The related mass and stiffness matrices may then be used unconcernedly for any quasi-static or time history analysis. As far as the diagonal mass matrix is concerned, μ_2 is the only parameter which should be calculated. By setting the first natural frequency obtained from the diagonal mass template and Euler–Bernoulli-beam stiffness matrix to the exact one, μ_2 will be obtained as shown below

$$\mu_2 = -0.0728 \tag{46}$$

This will result in the following diagonal mass matrix

$$\boldsymbol{M} = \begin{bmatrix} 0.5\,\mathrm{mL} & & & \\ 0 & -0.1768\,\mathrm{mL}^3 & & \\ 0 & 0 & 0.5\,\mathrm{mL} & \\ 0 & 0 & 0 & -0.1768\,\mathrm{mL}^3 \end{bmatrix} \tag{47}$$

The negative rotational inertias appeared in the mass matrix counterbalance the overestimate caused by lumping the translational mass to the extremities of the element [**6**]. The developed mass matrix is main suited for dynamic analysis, when the first mode contribution has a considerable effect.

If the same procedure is performed for the second natural frequency, then μ_2 will be equal to the following value

$$\mu_2 = 0.0199 \tag{48}$$

Consequently, the following mass matrix is obtained

$$\boldsymbol{M} = \begin{bmatrix} 0.5\,mL & & & \\ 0 & 0.00856\,mL^3 & & \\ 0 & 0 & 0.5\,mL & \\ 0 & 0 & 0 & 0.00856\,mL^3 \end{bmatrix} \tag{49}$$

In this case, the amount of RBM does not have any significant effect on the mode shape, and the rotational inertia will not be significantly overestimated. From this point of view, it would not be a need to counterbalance the negative rotational inertia for the second mode.

5.3 Fixed-pinned end condition

In this section, the parameterized mass and stiffness matrices are customized to find an exact value for the first natural frequency of the beam having fixed-pinned end condition. The beam element with this end condition has one free rotational DOF at the pinned end. This structure has the first natural frequency with the following parameterized value

$$\omega^2 = \frac{32(36\,\beta + 1)}{64\mu_2 - 1}\frac{EI}{mL^4} \tag{50}$$

If $\beta = \frac{1}{12}$ is utilized to calculate the Euler–Bernoulli-beam stiffness matrix, an optimized diagonal mass matrix will be obtained by setting μ_2 to the following value

$$\mu_2 = 0.0240382 \tag{51}$$

The related mass matrix could be written in the following form

$$\boldsymbol{M} = \begin{bmatrix} 0.5\,mL & & & \\ 0 & 0.0168264\,mL^3 & & \\ 0 & 0 & 0.5\,mL & \\ 0 & 0 & 0 & 0.0168264\,mL^3 \end{bmatrix} \tag{52}$$

6 CUSTOMIZED TIMOSHENKO-BEAM ELEMENT

Free parameters in mass and stiffness templates can be evaluated to consider the shear as well as bending deformations. This formulation will be suitable for the analysis of Timoshenko-beam. For instance, four typical stiffness matrices for Timoshenko-beam, which were introduced by Felippa [**17**], may be represented in the following form of SG basis

$$\boldsymbol{K}_E = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & EI\,L & \\ 0 & 0 & 0 & \frac{1}{12(1+\Phi)}\,EI\,L^3 \end{bmatrix} \tag{53}$$

$$\boldsymbol{K}_R = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & EI\,L & \\ 0 & 0 & 0 & \frac{1}{12\Phi}\,EI\,L^3 \end{bmatrix} \tag{54}$$

$$\boldsymbol{K}_F = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \frac{1}{\Phi}\,EI\,L^3 \end{bmatrix} \tag{55}$$

$$\boldsymbol{K}_X = \begin{bmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 0 & \frac{1+\Phi}{\Phi} EI\,L & \\ 0 & 0 & 0 & \frac{1}{12\Phi} EI\,L^3 \end{bmatrix} \tag{56}$$

In these formulas, $\Phi = \frac{12\,EI}{\kappa GA\,L^2}$ and κ is the shear factor which depends on the shape of beam section. Some typical features of various Timoshenko-beam stiffness matrices are discussed by Felippa [**17**]. They may easily be explained by considering the diagonal non-zero entries of the proposed SG-basis matrices, which represent stiffness values for constant and linear curvature modes. For example, $\boldsymbol{K}_E$ is found to be unaffected by shear locking, as the diagonal entries are still bounded when the shear deformations diminish. Actually, if $\Phi \to 0$, the well-known Euler–Bernoulli-beam stiffness matrix is recovered from $\boldsymbol{K}_E$. The two matrices $\boldsymbol{K}_R$ and $\boldsymbol{K}_F$, though not shear locked for constant curvature state, they suffer shear locking in linear curvature mode and $\boldsymbol{K}_X$ suffers shear locking for both modes.

6.1 Pinned-pinned end condition

In this section, the parameterized mass and stiffness matrices are customized for exact estimation of the first two natural frequencies of the Timoshenko-beam having pinned-pinned end conditions. The diagonal mass template in equation (35), with a parameter μ_2, and the well-behaved stiffness matrix $\boldsymbol{K}_E$ are used for the customization procedure. In this case, the following values are selected for the free parameters α and β in the stiffness template

$$\alpha = 1 \qquad \beta = \frac{1}{12(1+\Phi)} \tag{57}$$

The parameter μ_2 is found by equaling the natural frequencies obtained from the selected stiffness matrix and mass template with those of exact ones. Karnovsky has reported the exact values for vibration frequencies [**18**]. For the exact evaluation of the first natural frequency of the simple beam, the free parameter could be written in the following form

$$\mu_2 = \frac{1}{64} + \frac{1}{\pi^4\left(1 - \frac{\pi^2}{24}\eta\Phi\right)^2} \tag{58}$$

In this equation, $\eta = 1 + \frac{\kappa G}{E}$. It is worthwhile to note that, when $\Phi \to 0$, the value for the free parameter will be the same as the one derived for Euler-Bernoulli-beam element in equation (40). The obtained value yields a customized diagonal mass matrix to estimate the first natural frequency of the simple beam as given below

$$\boldsymbol{M} = \begin{bmatrix} 0.5\,mL & & & \\ 0 & \frac{11.8264}{(\pi^2\,\eta\Phi - 24)^2}\,mL^3 & & \\ 0 & 0 & 0.5\,mL & \\ 0 & 0 & 0 & \frac{11.8264}{(\pi^2\,\eta\Phi - 24)^2}\,mL^3 \end{bmatrix} \tag{59}$$

Optimizing the template for the second frequency of the simple beam will lead to the following new value for μ_2

$$\mu_2 = \frac{1}{64} + \frac{6.75}{\pi^4(1+\Phi)(\pi^2\,\eta\Phi - 6)^2} \tag{60}$$

Once again, when $\Phi \to 0$, the value for the free parameter will be the same as the one obtained for Euler–Bernoulli-beam element in equation (44). This value yields a customized diagonal mass matrix to evaluate the second natural frequency of the simple beam as follows

$$\boldsymbol{M} = \begin{bmatrix} \frac{mL}{2} & & & \\ 0 & \frac{13.5mL^3}{\pi^4(1+\Phi)(\pi^2\,\eta\Phi-6)^2 - 108} & & \\ 0 & 0 & \frac{mL}{2} & \\ 0 & 0 & 0 & \frac{13.5mL^3}{\pi^4(1+\Phi)(\pi^2\,\eta\Phi-6)^2 - 108} \end{bmatrix} \tag{61}$$

6.2 Fixed-free end condition

For this case, the free parameters of the stiffness template are assumed as previously described by equation (57). The one-parameter diagonal mass template is again considered. The customization procedure is performed for the exact estimation of the first frequency of the cantilever beam. This will lead to μ_2, which is a very complicated function of Φ and η. These functions are too lengthy to be reported in this article.

7 NUMERICAL VERIFICATION

This study is focused on the vibration analysis of plane frames by customized stiffness and diagonal mass matrices. The objective of this verification is to validate the efficiency of the customized mass and stiffness matrices proposed so far. The selected problems are a two-span continuous beam and five frame structures. For the sake of comparison, a near-exact result is found by a fine mesh of 20 FEs for each structural member. The accuracy of the presented formulation to find the exact solutions is explored through some numerical examples. It is also notable that the eigenproblem can be solved by any well-performed solution procedure such as Lanczos and QR algorithms.

7.1 Example 1

The properties of the sample structure are shown in Fig. 2. The axial stiffness for the sections of the two members is assumed to be equal to one (EA = 1). This simple frame is characterized for its two members representing deformations near to the cantilever and simple beams. The frame can thus be considered as a typical example to verify the customized mass and stiffness matrices. In Tables 1 and 2, the results for the vibration analysis obtained by different mass and stiffness matrices are compared with the exact values.

As it can be seen from the values in Tables 1 and 2, the results found by using the LM matrix seem to be the worst among all. However, the CM matrix, RCM matrix and the customized mass and stiffness matrices present good outcomes. In case 4 of both tables, the customized mass matrix for the first mode of a cantilever beam is employed. It should be added that this mode shape resembles the configuration adopted by the frame column, for both first and second modes. Figure 3 illustrates these mode shapes.

The customization introduced in case 4 is modified in case 5 for the true vibration configuration of the frame. As the rotational stiffness of the beam does not provide enough resistance against rotation at the column end, the column may still be supposed to behave as a cantilever, with the customized mass matrix described in equation (47). However, the beam may experience a high rigid rotation due to low axial rigidity of the column. This will surely necessitate assuming negative rotational inertia to decrease overestimated rotational inertia caused by lumping the translational mass to the extremities of the member. Thus, a value −0.055 is assumed for the free parameter μ_2. As a result, the accuracy for the natural frequencies are clearly improved for both modes as indicated by Tables 1 and 2.

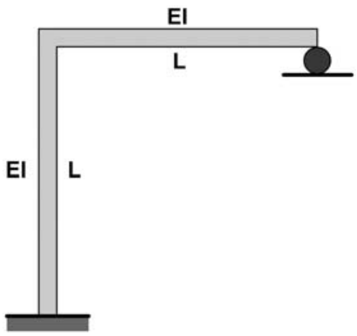

Fig. 2 Frame structure of Example 1

7.2 Example 2

A two-span continuous beam with the properties shown in Fig. 4 is considered. The two members of the structure may roughly be regarded as close to pinned-pinned and fixed-pinned elements. The mass and stiffness matrices customized for these end conditions are applied to find the frequency for the first mode of the structure. The proposed method may not be employed to find the frequency for the second mode, as customization for fixed-pinned condition is not defined for higher modes. The results obtained for the different mass and stiffness matrices are shown in Table 3.

According to Table 3, while the two lumped and RCM matrices fail to give acceptable results, the proposed mass matrix with its diagonal form leads to a better result compared with the CM matrix. In addition, Archer and Whalen [**6**] suggested that the negative frequency obtained by the RCM is caused by overestimating negative inertias. It is worth emphasizing that the same problem may occur for the proposed formulation; and the customized mass matrices may produce the artificial negative frequencies, especially when sufficient elements with negative mass entries are interconnected at the same

Table 1 First natural frequency for Example 1 obtained by different methods ($\omega^2_{1\,exact} = 1.3677\frac{EI}{mL^4}$)

No.	Member	Mass matrix	Stiffness matrix	$\omega_1^2/\frac{EI}{mL^4}$	Relative error %
1	Beam	Consistent	EB (Euler-Bernoulli)	1.2959	5.3
	Column	Consistent	EB		
2	Beam	Lumped	EB	0.1643	88
	Column	Lumped	EB		
3	Beam	Rotationally consistent diagonal	EB	1.2087	11.6
	Column	Rotationally consistent diagonal	EB		
4	Beam	From equation (42)	From equation (43)	1.2264	10.3
	Column	From equation (47)	EB		
5	Beam	$\mu_2 = -0.055$	EB	1.3672	0.05
	Column	From equation (47)	EB		
	Column	From equation (47)	EB		

Table 2 Second natural frequency for Example 1 obtained by different methods ($\omega^2_{2\ exact} = 2.8803\ \frac{EI}{mL^4}$)

No.	Member	Mass matrix	Stiffness matrix	$\omega_2^2/\frac{EI}{mL^4}$	Relative error %
1	Beam	Consistent	EB	2.3951	16.8
	Column	Consistent	EB		
2	Beam	Lumped	EB	–	–
	Column	Lumped	EB		
3	Beam	Rotationally consistent diagonal	EB	2.3073	19.9
	Column	Rotationally consistent diagonal	EB		
4	Beam	From equation (45)	From equation (43)	2.3190	19.5
	Column	From equation (47)	EB		
5	Beam	$\mu_2 = -0.055$	EB	2.7380	5
	Column	From equation (47)	EB		

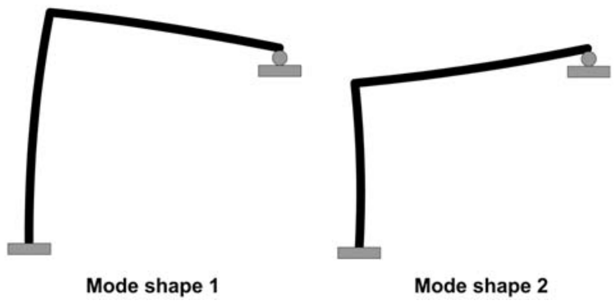

Fig. 3 First two mode shapes for Example frame 1

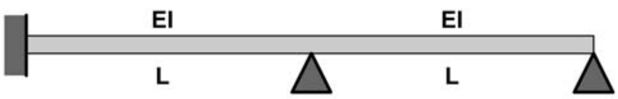

Fig. 4 Frame structure of Example 2

nodes. This will result in the negative entries in the overall structural mass matrix, which eventually leads to the negative eigenvalues. While negative masses may not affect analyses that are performed in the frequency domain, they will produce the unbounded time-history solutions [**19**]. However, the customized mass matrices may still be used for the time-domain analyses, in the typical structural applications, such as seismic analyses. Archer and Whalen [**6**] proposed a scheme to shift the eigenvalue from the negative value to positive one, which could be utilized for the customized mass matrices proposed in this study, as well.

7.3 Example 3

The proposed formulation is applied to find the first natural frequency of the portal frame shown in Fig. 5. The configuration adopted by this frame in the first mode may intuitively be considered to involve single-curvature columns and a double-curvature beam. The column members may then resemble a cantilever, while the beam member might behave as a simple beam. The diagonal mass matrix customized for the first mode of a cantilever and that customized for the second mode of a simple beam are respectively adopted for the frame columns and beam. The well-known Euler-Bernoulli-beam stiffness matrix is used for all members. The results obtained by different mass matrices are reported in Table 4. The customized mass matrix, as well as the CM matrix, seems to yield the best result. However, if the free parameter μ_2 is supposed to be 0.026 for the customized mass matrix of the beam, the near-exact value for the first natural frequency of the frame is obtained. This value is close to the one which presents the diagonal mass matrix customized for the first mode of a simple beam. It can then be concluded that the beam configuration resembles to a simple beam in the first mode having a single-curvature shape.

7.4 Example 4

The asymmetric frame structure shown in Fig. 6 is considered. IPE160-section with $A = 20.1\,\text{cm}^2$ and $I = 869\,\text{cm}^4$, and 2IPE160-section with $A = 40.2\,\text{cm}^2$ and $I = 1738\,\text{cm}^4$ are used for beams and columns, respectively. Natural frequencies for the first and second modes are calculated and reported in Tables 5 and 6. The well-known Euler–Bernoulli-beam stiffness matrix is utilized for all cases. As it is shown in the tables, the CM matrix presents near-accurate results; however, the LM matrix fails to yield good results. Unlike for the first mode, the RCM matrix is unable to accurately estimate the natural frequency for the second vibration mode. As mentioned by Archer and Whalen [**6**], the RCM matrix is mostly efficient for the first mode. In case 4, the mass matrix customized for the first mode of a fixed-pinned beam and the one customized for the

Table 3 First natural frequency for Example 2 obtained by different methods ($\omega_{1\,exact}^2 = 132.582\,\frac{EI}{mL^4}$)

No.	Member	Mass matrix	Stiffness matrix	$\omega_1^2/\frac{EI}{mL^4}$	Relative error %
1	Beam 1	Consistent	EB	177.418	33.82
	Beam 2	Consistent	EB		
2	Beam 1	Lumped	EB	–	–
	Beam 2	Lumped	EB		
3	Beam 1	Rotationally consistent diagonal	EB	Negative value!	–
	Beam 2	Rotationally consistent diagonal	EB		
4	Beam 1	From equation (51)	EB	137.372	3.61
	Beam 2	From equation (42)	From equation (43)		

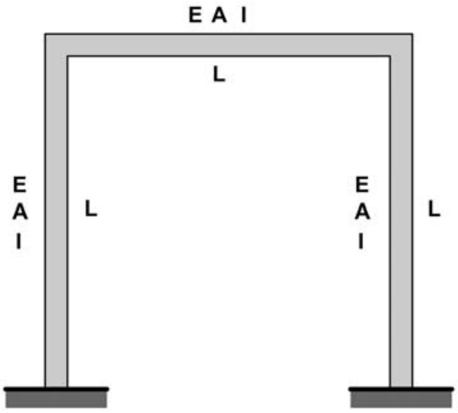

Fig. 5 Frame structure of Example 3

second mode of a simple beam are respectively used for columns and beams. In fact, a customization for the first mode of a cantilever beam applied to the columns which undergo significant drifts would be more efficient. Therefore, in case 5, mass matrices customized for the first mode of a cantilever beam are assigned to the columns. Obviously, the results are considerably improved due to this modification, especially for the first mode.

7.5 Example 5

As another example, the first natural frequency for coupled shear walls shown in Fig. 7 is discussed. This structure is modelled the same as a simple portal frame. Due to the considerable depth-to-length ratio for the members, the shear deformations may not be neglected, and the Timoshenko-beam kinematic constraints should thus be applied to the element formulation. The beam is considered to have a length of $L = 2.0$ m. Both columns are supposed to be the same with the length of $L = 3.0$ m. The widths of the beam and columns are assumed to be of 20 cm. In this example, two cases of the length-to-depth ratio are considered as: (1) $L/h = 2$ and (2) $L/h = 5$. The Timoshenko formulation is employed for L/h between 2 and 10, and usually the ratio less than 2 cannot theoretically be considered as correct. The modulus of elasticity, shear modulus, Poisson ratio, and density are respectively assumed to be 25 and 10 GPa and 0.25 and 2400 Kg/m^3. The shear factor is assumed to be $\kappa = \frac{2}{3}$. Table 7 shows the values obtained by different mass and stiffness matrices for the first natural frequency of the structure with $L/h = 2$.

Based on the behaviour of the shear wall structure, in the first vibration mode, the columns of the frame deform with double-curvature. This behaviour opposes the case in Example 3, where columns experience single-curvature shape due to the small depth-to-length ratio. The columns deformed in the first mode may thus closely imitate a simple beam in its second mode, rather than a cantilever beam in its first mode. The mass matrix customized for the second mode of a simple beam is then used for the estimation of the first natural frequency of the coupled walls. As it can be seen in Table 7, the result related to case 4 signifies an improvement in accuracy compared with the three previous cases. The same customized mass matrix is also applied to the beam member; however, the vibration analysis turns out to be slightly sensitive to the beam mass matrix in this case. As implied by Table 7, the RCM matrix may not be considered as a reliable choice for this case. It should be added, if μ_2 is equal to 0.028, the exact value for the first natural frequency is acquired.

In Fig. 8, the values for the first six vibration frequencies of the coupled walls obtained by different mass matrices are demonstrated. The abbreviations CM, LM, and RCM stand for the consistent mass matrix, the lumped mass matrix and the rotationally consistent diagonal mass matrix, respectively. The two other choices appeared on the legend point to the customized mass matrices introduced as cases 4 and 5 in Table 7. Obviously, these mass matrices which are customized for the first natural frequency may not reliably be used for other vibration modes.

Table 4 First natural frequency for Example 3 obtained by different methods ($\omega^2_{1\ exact} = 2.8657\frac{EI}{mL^4}$)

No.	Member	Mass matrix	Stiffness matrix	$\omega_1^2/\frac{EI}{mL^4}$	Relative error %
1	Left column	Consistent	EB	3.1504	9.04
	Beam	Consistent	EB		
	Right column	Consistent	EB		
2	Left column	Lumped	EB	4.2204	32.10
	Beam	Lumped	EB		
	Right column	Lumped	EB		
3	Left column	Rotationally consistent diagonal	EB	2.4251	18.17
	Beam	Rotationally consistent diagonal	EB		
	Right column	Rotationally consistent diagonal	EB		
4	Left column	From equation (49)	EB	2.9485	2.9
	Beam	From equation (45)	EB		
	Right column	From equation (49)	EB		

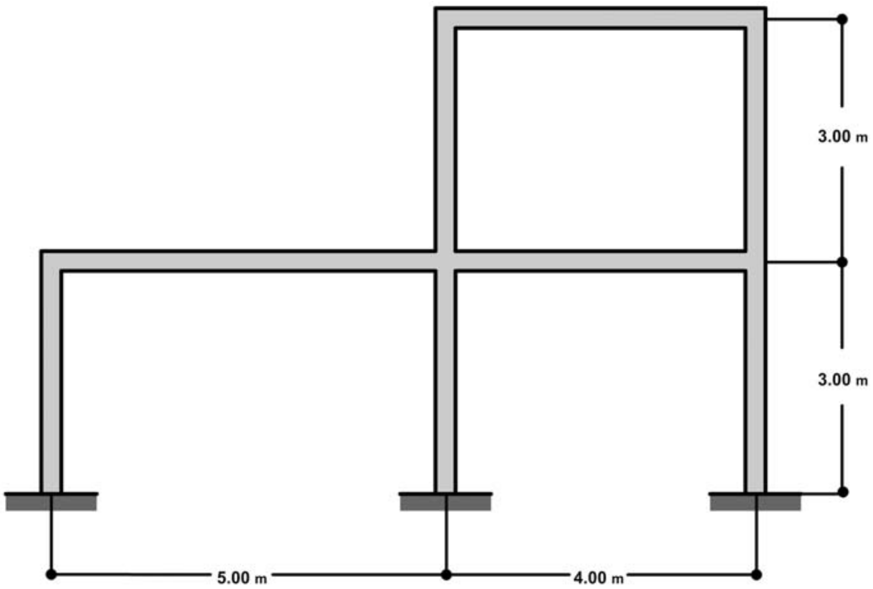

Fig. 6 Frame structure of Example 4

Table 5 First natural frequency for Example 4 obtained by different methods ($f_{1\ exact} = 8.9871\ Hz$)

No.	Member	Mass matrix	Stiffness matrix	Frequency (Hz)	Relative error %
1	Columns	Consistent	EB	9.5607	6.38
	Beams	Consistent	EB		
2	Columns	Lumped	EB	6.4696	28.01
	Beams	Lumped	EB		
3	Columns	Rotationally consistent diagonal	EB	8.7707	2.41
	Beams	Rotationally consistent diagonal	EB		
4	Columns	From equation (52)	EB	8.547	4.50
	Beams	From equation (45)	EB		
5	Columns	From equation (47)	EB	9.0466	0.66
	Beams	From equation (45)	EB		

Table 6 Second natural frequency for Example 4 obtained by different methods ($f_{1\,exact} = 26.0078\ Hz$)

No.	Member	Mass matrix	Stiffness matrix	Frequency (Hz)	Relative error %
1	Columns	Consistent	EB	25.2552	2.89
	Beams	Consistent	EB		
2	Columns	Lumped	EB	17.1292	34.14
	Beams	Lumped	EB		
3	Columns	Rotationally consistent diagonal	EB	23.2778	10.50
	Beams	Rotationally consistent diagonal	EB		
4	Columns	From equation (52)	EB	21.8455	16.00
	Beams	From equation (45)	EB		
5	Columns	From equation (47)	EB	24.5445	5.63

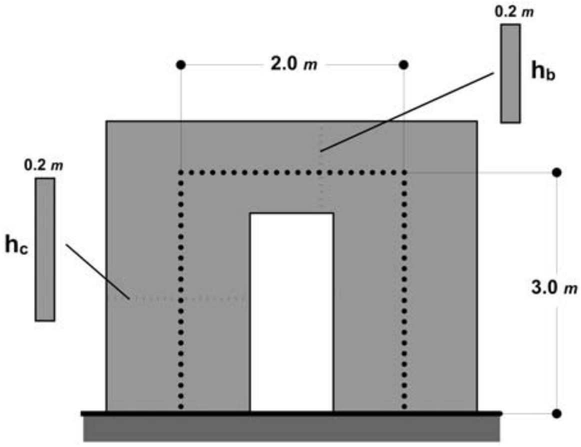

Fig. 7 Frame structure of Example 5

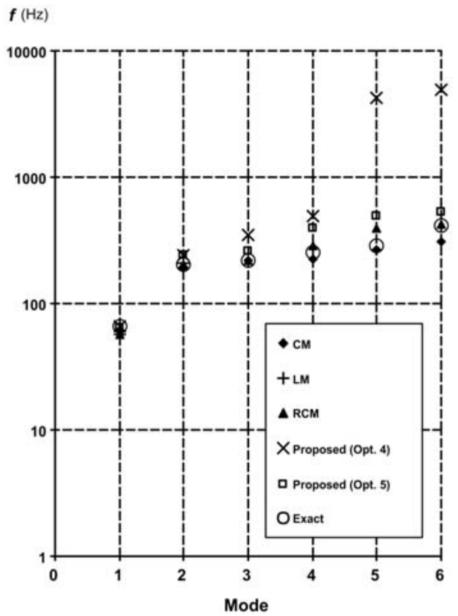

Fig. 8 First six natural frequencies for Example frame 5 by different methods

As shown in Fig. 8, the results offered by customized mass matrices deteriorate for modes other than 1. The CM matrix and RCM matrix can generally be considered as good choices to perform structural dynamic analysis, which deal with different modes.

Table 8 shows the results for the second case ($L/h = 5$) obtained by different mass and stiffness matrices for the first natural frequency of the structure. The CM matrix offers a near-exact estimation. It is worth emphasizing that among the three mass matrices with diagonal form, the proposed method presents an approximate answer with less error.

7.6 Example 6

To examine this fact that the suggested method is effective not only for small academic case studies but for the real large-scale structures, a ten-story four-bay planar frame in Fig. 9 is analysed. In this example, few assumptions are considered for the mass and stiffness distributions of this structure. As reported in Table 9, some vibration analyses are performed, and the natural period for the first mode of the frame is found. The near-exact natural period is assumed to be the converged response resulted from a sequence of mesh refinements. As a result, the related period of the frame is equal to $T_{1\,\text{exact}} = 339.955\, L^2 \sqrt{m_0/\text{EI}_0}$ s. According to Table 9, the response obtained by employing the CM matrix suffers the highest relative error, especially in comparison with the lumped and RCM matrices. Interestingly, the LM matrix presents the best outcome among the first three. By reviewing the structural mode shape resulted from vibration analysis using the CM or LM matrices, two customized options are considered. First, No. 4 in Table 9, which approximates mode shapes for columns as the one for the first mode of the cantilever beam, and the second mode of the simple beam for the beams. Second, more sophisticated choice is No. 5 in which the columns of the stories other than the first one are approximated by the second mode of a simple beam. The latter seems to be the best assumption among all. It is notable that the exact value for the natural frequency of the first mode is obtained by

Table 7 First natural frequency for Example 5 obtained by different methods ($L/h = 2$) ($f_{1\ exact} = 64.81\ Hz$)

No	Member	Mass matrix	Stiffness matrix	Frequency (Hz)	Relative error %
1	Columns	Consistent	T	61	5.9
	Beam	Consistent	T		
2	Columns	Lumped	T	57.2	11.7
	Beam	Lumped	T		
3	Columns	Rotationally consistent diagonal	T	56.6	12.7
	Beam	Rotationally consistent diagonal	T		
4	Columns	$\mu_2 = 0.0157$	T	66.3	2.3
	Beam	$\mu_2 = 0.0157$	T		
5	Columns	$\mu_2 = 0.028$	T	64.8	0.015

Table 8 First natural frequency for Example 5 obtained by different methods ($L/h = 2$) ($f_{1\ exact} = 33.0\ Hz$)

No.	Member	Mass matrix	Stiffness matrix	Frequency (Hz)	Relative error %
1	Columns	Consistent	T	33	0.01
	Beam	Consistent	T		
2	Columns	Lumped	T	19.8	40
	Beam	Lumped	T		
3	Columns	Rotationally consistent diagonal	T	26.2	20.6
	Beam	Rotationally consistent diagonal	T		
4	Columns	$\mu_2 = 0.0192$	T	27.7	16
	Beam	$\mu_2 = 0.0192$	T		

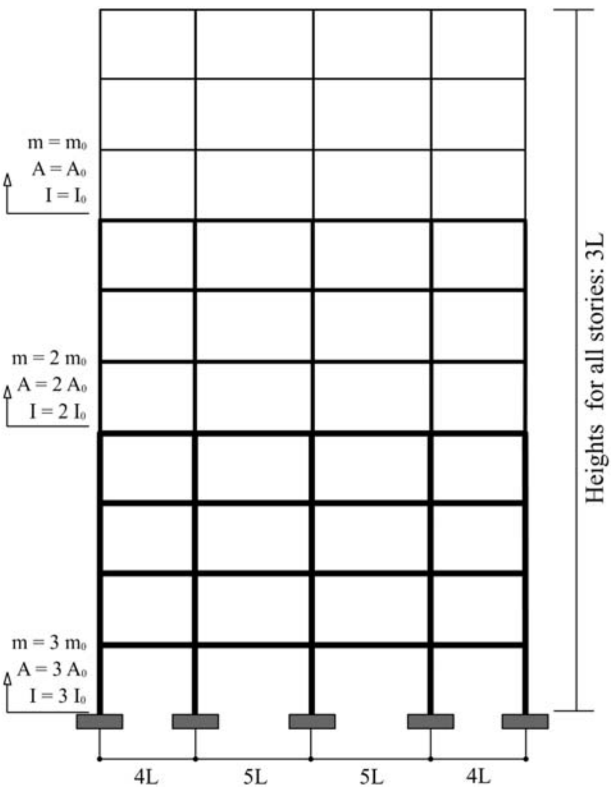

Fig. 9 Frame structure of Example 6

considering $\mu_2 = 2.05$. This results the rotational component of the diagonal mass matrix equal to $4.07mL^2$, which is nearly a large value, when compared with other diagonal mass matrices, similar to the LM or RCM matrices. This outcome may be related to the huge amount of the rigid body rotations experienced by the columns, especially at the higher stories.

8 CONCLUSION

This article presents a new formulation to establish general parameterized mass and stiffness matrices. The proposed technique calculates these matrices by utilizing SG basis. To outline the merit of this procedure, the following features for the obtained matrices are stated:

1. They benefit simple representations having typically sparse or diagonal matrix arrangements. Consequently, this will lead to a simpler parameterizing and customizing procedures.
2. The calculated entries of the mass and stiffness matrices in SG-basis can physically be interpreted, and accordingly, might be utilized in the analysis.

In this investigation, the parameterized diagonal mass matrices are customized for exact estimation of the natural frequencies for the beam elements with different end conditions. The customized matrices are then applied to vibration analysis of some sample structures. Based on the numerical findings, concludingly, the following results are presented:

1. Generally, the mass and stiffness matrices obtained for the exact evaluation of the beam natural frequencies, having various end conditions, provide adequately accurate results in the vibration analysis of frame structures, especially when

Table 9 First natural frequency for Example 6 obtained by different methods

No	Member	Mass matrix	Stiffness matrix	Period (s)	Relative error %
1	Columns	Consistent	EB	296.893	12.67
	Beams	Consistent	EB		
2	Columns	Lumped	EB	307.013	9.69
	Beams	Lumped	EB		
3	Columns	Rotationally consistent diagonal	EB	306.556	9.82
	Beams	Rotationally consistent diagonal	EB		
4	Columns	$\mu_2 = -0.0728$	EB	306.722	9.78
	Beams	$\mu_2 = 0.017611$	EB		
5	First story Columns	$\mu_2 = -0.0728$	EB	307.342	9.59
	Other Columns	$\mu_2 = 0.017611$	EB		
	Beams	$\mu_2 = 0.017611$	EB		

accurate values for fundamental frequencies of the structure are needed. The customized diagonal mass matrix may then be a competitor for a well-behaved CM matrix, and performs even better than the RCM matrix for several cases. In fact, the proposed method requires less computational time, in comparison to the other discussed diagonal mass matrices, and offers better approximations for natural frequencies. As illustrated by the numerical examples, a good estimation of the free parameters significantly improves the accuracy of the structural natural frequencies. It should be added that the efficiency of the results strongly depends on the customization procedure; since the vibration analysis seems to be very sensitive to the values of the free parameters in most cases.

2. The mode shape is found to be the most effective factor to select appropriate customized mass and stiffness matrices. In other words, good judgment for estimating the general mode shape of the structural members would be helpful for an efficient customization. The suggested mode shapes may lead to the powerful customized matrices, and consequently, acceptable results of the vibration analysis. The phrase is duplicated from below, under note numbered 3. It is also notable that when different modes are to be considered, the CM matrix and RCM are suitable for use in the dynamic analysis.
3. It should be mentioned that the customized mass and stiffness matrices may not necessarily allow for a precise recovery of the deformed shapes corresponding to the evaluated eigen-frequencies. For good customizations, however, the results are duly within acceptable ranges.

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APPENDIX

Notation

A	section area
$\boldsymbol{B}$	strain interpolating matrix
$\boldsymbol{D}$	nodal displacement matrix
$\boldsymbol{D}_m$	constitutive matrix
E	elastic modulus
$\boldsymbol{e}$	standard vector
f	frequency
G	shear modulus
$\boldsymbol{G}$	transformation matrix between canonical and SG bases
$\boldsymbol{G}_q$	transformation matrix between polynomial and SG bases
$\boldsymbol{H}$	inverse of $\boldsymbol{G}$
I	moment of inertia
$\boldsymbol{K}$	stiffness matrix
L	length of beam
m	mass of beam per unit length
$\boldsymbol{M}$	mass matrix
n	number of SGs/DOF for FE
N	base function
$\boldsymbol{N}$	base matrix
$\boldsymbol{P}$	element force vector
q	strain gradient
$\boldsymbol{q}$	SG vector
u	displacement field function
V	element volume
x	coordinate along beam element
α	free parameter in beam stiffness template
$\boldsymbol{\alpha}$	polynomial coefficient vector
β	free parameter in beam stiffness template
Φ	factor in stiffness matrix for Timoshenko-beam element
η	factor in stiffness matrix for Timoshenko-beam element
κ	shear factor
μ	free parameter in beam mass template
ρ	density
ω	radian frequency
w_0	SG for beam deflection
θ_0	SG for beam rotation
κ_{x0}	SG for beam constant curvature
κ_{xx0}	SG for beam linear curvature
i	index for number of SG
$(.)_q$	subscript for SG
$(.)^T$	transpose of matrix
$(.)^{-1}$	inverse of matrix
$L^{(i)}$	differential operator for SG i
Δ	differential operator matrix
$\ddot{}$	second derivative with respect to time
$'$	first derivative with respect to coordinate x
$''$	second derivative with respect to coordinate x
$'''$	third derivative with respect to coordinate x