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ON A CONJECTURE OF A BOUND FOR THE EXPONENT OF THE SCHUR MULTIPLIER OF A FINITE p-GROUP

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Abstract. Let G be a p-group of nilpotency class k with finite exponent $\exp(G)$ and let $m = \lfloor \log_p k \rfloor$. We show that $\exp(M^{(c)}(G))$ divides $\exp(G)p^{m(k-1)}$, for all $c \geq 1$, where $M^{(c)}(G)$ denotes the c-nilpotent multiplier of G. This implies that $\exp(M(G))$ divides $\exp(G)$, for all finite p-groups of class at most $p-1$. Moreover, we show that our result is an improvement of some previous bounds for the exponent of $M^{(c)}(G)$ given by M. R. Jones, G. Ellis and P. Moravec in some cases.

1. Introduction and motivation

Let a group G be presented as a quotient of a free group F by a normal subgroup R. Then, the c-nilpotent multiplier of G (the Baer invariant of G with respect to the variety of nilpotent group of class at most c) is defined to be

$$M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R,\ _cF]},$$

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where $[R,\ {}_cF]$ denotes the commutator subgroup $[R, \underbrace{F, ..., F}_{c-times}]$ and $c \geq 1$. The case $c = 1$ which has been much studied is the Schur multiplier of G, denoted by $M(G)$. When G is finite, $M(G)$ is isomorphic to the second cohomology group $H^2(G, \mathbb{C}^*)$ (see G. Karpilovsky [6] and C. R. Leedham-Green and S. McKay [8] for further details).

It has been interested to find a relation between the exponent of $M^{(c)}(G)$ and the exponent of G. Let G be a finite p-group of nilpotency class $k \geq 2$ with exponent $\exp(G)$. M. R. Jones [5] proved that $\exp(M(G))$ divides $\exp(G)^{k-1}$. This has been improved by G. Ellis [3] who showed that $\exp(M^{(c)}(G))$ divides $\exp(G)^{\lceil k/2 \rceil}$, where $\lceil k/2 \rceil$ denotes the smallest integer n such that $n \geq k/2$. For $c = 1$, P. Moravec [11] showed that $\lceil k/2 \rceil$ can be replaced by $2\lfloor \log_2 k \rfloor$ which is an improvement, if $k \geq 11$.

In this paper, we will show that if G is a finite exponent p-group of class $k \geq 1$, then $\exp(M^{(c)}(G))$ divides $\exp(G)p^{m(k-1)}$, for all $c \geq 1$, where $m = \lfloor \log_p k \rfloor$. Note that this result is an improvement of the results of Jones, Ellis and Moravec, if $\lfloor \log_p k \rfloor (k-1)/k < e$, $\lfloor \log_p k \rfloor (k-1)/\lceil k/2 \rceil - 1 < e$, $\lfloor \log_p k \rfloor (k-1)/2\lfloor \log_2 k \rfloor - 1 < e$, respectively, where $\exp(G) = p^e$.

It was a longstanding open problem as to whether $\exp(M(G))$ divides $\exp(G)$, for every finite group G. In fact, it was conjectured that the exponent of the Schur multiplier of a finite p-group is a divisor of the exponent of the group itself. I. D. Macdonld and J. W. Wamsley [1] constructed an example of a group of order 2^{21} which has exponent 4, whereas its Schur multiplier has exponent 8, hence the conjecture is not true in general. Also, Moravec [12] gave an example of a group of order 2048 and nilpotency class 6 which has exponent 4 and multiplier of exponent 8. He also proved that if G is a group of exponent 4, then $exp(M(G))$ divides 8. Nevertheless, Jones [5] has shown that the conjecture is true for p-groups of class 2 and emphasized that it is true for some p-groups of class 3. S. Kayvanfar and M. A. Sanati [7] have proved the conjecture for p-groups of class 4 and 5, with some conditions. A. Lubotzky and A. Mann [9] showed that the conjecture is true for powerful p-groups. The first and the third authors [10] showed that the conjecture is true for nilpotent multipliers of powerful p-groups. Finally, Moravec [11, 12] showed that the conjecture is true for metabelian groups of exponent p, p-groups with potent filtration and p-groups of maximal class. Note that a consequence of our result shows that the conjecture is true for all finite p-groups of class at most $p-1$.

2. Preliminaries

In this section, we are going to recall some notions we will use in the next section.

Definition 2.1. *(M. Hall [4]). Let X be an independent subset of a free group, and select an arbitrary total order for X. We define the basic commutators on X, their weight wt, and the ordering among them as follows:*

(1) *The elements of X are basic commutators of weight one, ordered according to the total order previously chosen.*
(2) *Having defined the basic commutators of weight less than n, the basic commutators of weight n are the $c_k = [c_i, c_j]$, where:*
   (a) *c_i and c_j are basic commutators and $wt(c_i) + wt(c_j) = n$, and*
   (b) *$c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.*
(3) *The basic commutators of weight n follow those of weight less than n. The basic commutators of weight n are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight n, then $[b_1, a_1] \leq [b_2, a_2]$ if and only if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 < a_2$.*

Lemma 2.2. *(R. R. Struik [13]). Let $x_1, x_2, ..., x_r$ be any elements of a group and let $\upsilon_1, \upsilon_2, ...$ be the sequence of basic commutators of weight at least two in the x_i's, in ascending order. Then,*

$$(x_1x_2...x_r)^\alpha = x_{i_1}^\alpha x_{i_2}^\alpha ... x_{i_r}^\alpha \upsilon_1^{f_1(\alpha)} \upsilon_2^{f_2(\alpha)} ... \upsilon_i^{f_i(\alpha)} ... ,$$

where $\{i_1, i_2, ..., i_r\} = \{1, 2, ..., r\}$, α is a nonnegative integer and

$$f_i(\alpha) = a_1\binom{\alpha}{1} + a_2\binom{\alpha}{2} + ... + a_{w_i}\binom{\alpha}{w_i}, \qquad (I)$$

with $a_1, ..., a_{wi} \in \mathbf{Z}$ and w_i is the weight of υ_i in the x_i's.

Lemma 2.3. *(Struik [13]). Let α be a fixed integer and G be a nilpotent group of class at most k. If $b_1, \ldots, b_r \in G$ and $r < k$, then*

$$[b_1, ..., b_{i-1}, b_i^{\alpha}, b_{i+1}, ..., b_r] = [b_1, ..., b_r]^{\alpha} \upsilon_1^{f_1(\alpha)} \upsilon_2^{f_2(\alpha)} ...,$$

where υ_i's are commutators in $b_1, ..., b_r$ of weight strictly greater than r, and every b_j, $1 \leq j \leq r$, appears in each commutator υ_i, the υ_i's listed in ascending order. The $f_i(\alpha)$'s are of the form (I), with $a_1, ..., a_{w_i} \in \mathbf{Z}$ and w_i is the weight of υ_i (in the b_j's) minus $(r-1)$.

Remark 2.4. *Outer commutators on the letters $x_1, x_2, \ldots, x_n, \ldots$ are defined inductively as follows:*

The letter x_i is an outer commutator word of weight one. If $u = u(x_1, \ldots, x_s)$ and $v = v(x_{s+1}, \ldots, x_{s+t})$ are outer commutator words of weights s and t, then $w(x_1, \ldots, x_{s+t}) = [u(x_1, \ldots, x_s), v(x_{s+1}, \ldots, x_{s+t})]$ is an outer commutator word of weight $s+t$ and will be written $w = [u, v]$.

It is noted by Struik [13] that Lemma 2.3 can be proved by a similar method, if $[b_1, .., b_{i-1}, b_i^{\alpha}, b_{i+1}, ..., b_r]$ and $[b_1, ..., b_r]$ are replaced with outer commutators.

By a routine calculation we have the following useful fact.

Lemma 2.5. *Let p be a prime number and k be a nonnegative integer. If $m = \lfloor \log_p k \rfloor$, then p^t divides $\binom{p^{m+t}}{k}$, for all integers $t \geq 1$.*

3. Main results

In order to prove the main result we need the following lemma.

Lemma 3.1. *Let G be a p-group of class k and exponent p^e with a free presentation F/R. Then, for any $c \geq 1$, every outer commutator of weight $w > c$ in $F/[R, {}_cF]$ has an order dividing $p^{e+m(c+k-w)}$, where $m = \lfloor \log_p k \rfloor$.*

Proof. Since $\gamma_{k+1}(F) \subseteq R$, we have $\gamma_{c+k+1}(F) \subseteq [R, {}_cF]$. Also, for all x in F and $t \geq 0$ we have $x^{p^{e+t}} \in R$ and hence every outer commutator of weight $w > c$ in F, in which $x^{p^{e+t}}$ appears, belongs to $[R, {}_cF]$. Now, we use inverse induction on w to prove the lemma. For the first step, $w = c + k$, the result follows by the above argument and Lemma 2.3.

Now, assume that the result is true, for all $l > w$. Put $\alpha = p^{e+m(c+k-w)}$ and let $u = [x_1, \ldots, x_w]$ be an outer commutator of weight w. Then, by Lemma 2.3 and Remark 2.4 we have

$$[x_1^{\alpha}, \ldots, x_w] = [x_1, \ldots, x_w]^{\alpha} \upsilon_1^{f_1(\alpha)} \upsilon_2^{f_2(\alpha)} \ldots,$$

where the $\upsilon_i^{f_i(\alpha)}$ are as in Lemma 2.3. Note that $w < w_i = wt(\upsilon_i) \leq c+k$ modulo $[R,\ _cF]$ and hence $f_i(\alpha) = a_1\binom{\alpha}{1} + a_2\binom{\alpha}{2} + ... + a_{w_i}\binom{\alpha}{k_i}$, where $k_i = w_i - w + 1 \leq c + k - w + 1 \leq k$, for all $i \geq 1$. Thus, Lemma 2.5 implies that $p^{e+m(c+k-w-1)}$ divides the $f_i(\alpha)$'s. Now, by induction hypothesis $\upsilon_i^{f_i(\alpha)} \in [R,\ _cF]$, for all $i \geq 1$. On the other hand, since $x_1^{\alpha} \in R$ and $w > c$, $[x_1^{\alpha}, \ldots, x_w] \in [R,\ _cF]$. Therefore, $u^{\alpha} \in [R,\ _cF]$ and this completes the proof.

Theorem 3.2. *Let G be a p-group of class k and exponent p^e. Let $G = F/R$ be any free presentation of G. Then, the exponent of $\gamma_{c+1}(F)/[R,\ _cF]$ divides $p^{e+m(k-1)}$, where $m = \lfloor \log_p k \rfloor$, for all $c \geq 1$.*

Proof. It is easy to see that every element g of $\gamma_{c+1}(F)$ can be expressed as $g = y_1 y_2 \ldots y_n$, where y_i's are commutators of weight at least $c+1$. Put $\alpha = p^{e+m(k-1)}$. Now, Lemma 2.2 implies the identity

$$g^{\alpha} = y_{i_1}^{\alpha} y_{i_2}^{\alpha} \ldots y_{i_n}^{\alpha} \upsilon_1^{f_1(\alpha)} \upsilon_2^{f_2(\alpha)} \ldots,$$

where $\{i_1, i_2, \ldots, i_n\} = \{1, 2, \ldots, n\}$ and $\upsilon_i^{f_i(\alpha)}$'s are as in Lemma 2.2. Then, the υ_i's are basic commutators of weight at least two and at most k in the y_i's modulo $[R,\ _cF]$ (note that $\gamma_{c+k+1}(F) \subseteq [R,\ _cF]$). Thus, Lemma 2.5 yields that $p^{e+m(k-2)}$ divides the $f_i(\alpha)$'s. Hence, $\upsilon_i^{f_i(\alpha)} \in [R,\ _cF]$, for all $i \geq 1$ and $y_j^{\alpha} \in [R,\ _cF]$, for all $1 \leq j \leq n$, by Lemma 3.1. Therefore, we have $g^{\alpha} \in [R,\ _cF]$ and the desired result now follows.

Now, we are in a position to state and prove the main result of the paper.

Theorem 3.3. *Let G be a p-group of class k and exponent p^e. Then,* $\exp(M^{(c)}(G))$ *divides* $\exp(G)p^{m(k-1)}$*, where $m = \lfloor \log_p k \rfloor$, for all $c \geq 1$.*

Proof. Let $G = F/R$ be any free presentation of G. Then, $M^{(c)}(G) \leq \gamma_{c+1}(F)/[R,\ _cF]$. Therefore, $\exp(M^{(c)}(G))$ divides $\exp(\gamma_{c+1}(F)/[R,\ _cF])$. Now, the result follows by Theorem 2.3.

Note that the above result improves some previous bounds for the exponent of $M(G)$ and $M^{(c)}(G)$ as follows.
Let G be a p-group of class k and exponent p^e, then we have the following improvements.

(i) If $\lfloor \log_p k \rfloor (k-1)/k < e$, then $\exp(G) p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{k-1}$. Hence, in this case our result is an improvement of Jones's result [5]. In particular, our result improves the Jones's one for every p-group of exponent p^e and of class at most $p^e - 1$.

(ii) If $\lfloor \log_p k \rfloor (k-1)/\lceil k/2 \rceil - 1 < e$, then $\exp(G) p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{\lceil k/2 \rceil}$ which shows that in this case our result is an improvement of Ellis's result [3]. In particular, our result improves the Ellis's one for every p-group of exponent p^e and of class $k < p^{e/3}$, for all $k \geq 3$, or of class $k < p^{e/4}$, for all $k \geq 4$.

(iii) If $\lfloor \log_p k \rfloor (k-1)/2\lfloor \log_2 k \rfloor - 1 < e$, then $\exp(G) p^{\lfloor \log_p k \rfloor (k-1)} < \exp(G)^{2\lfloor \log_2 k \rfloor}$. Thus, in this case our result is an improvement of Moravec's result [11]. In particular, our result improves the Moravec's one for every p-group of exponent p^e and of class $k < e$, for all $k \geq 2$.

Corollary 3.4. *Let G be a finite p-group of class at most $p-1$, then $\exp(M^{(c)}(G))$ divides $\exp(G)$, for all $c \geq 1$. In particular, $\exp(M(G))$ divides $\exp(G)$.*

Note that the above corollary shows that the mentioned conjecture on the exponent of the Schur multiplier of a finite p-group holds for all finite p-group of class at most $p-1$.

Remark 3.5. *Let G be a finite nilpotent group of class k. Then, G is the direct product of its Sylow subgroups, $G = S_{p_1} \times \cdots \times S_{p_n}$. Clearly,*

$$\exp(G) = \prod_{i=1}^{n} \exp(S_{p_i}).$$

By a result of G. Ellis [2, Theorem 5] we have

$$M^{(c)}(G) = M^{(c)}(S_{p_1}) \times \cdots \times M^{(c)}(S_{p_n}).$$

For all $1 \leq i \leq n$, put $m_i = \lfloor \log_{p_i} k \rfloor$. Then, by Theorem 3.3 we have

$$\exp(M^{(c)}(G)) \mid \exp(G) \prod_{i=1}^{n} p_i^{m_i(k-1)}.$$

Hence, the conjecture on the exponent of the Schur multiplier holds for all finite nilpotent group G of class at most $Max\{p_1 - 1, ..., p_n - 1\}$, where $p_1, ..., p_n$ are all the distinct prime divisors of the order of G.

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