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Inversion formula for the non-uniformly attenuated x-ray transform for emission imaging in $\mathbb{R}^3$ using quaternionic analysis

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Abstract
In this paper, we present a new derivation of the inverse of the non-uniformly attenuated x-ray transform in three dimensions, based on quaternion analysis. An explicit formula is obtained using a set of three-dimensional x-ray projection data. The result without attenuation is recovered as a special case.

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1. Introduction

When a radiopharmaceutical emits radiation of photon energy E_0, an ideal SPECT camera records only emitted photons, which arrive perpendicularly to its surface. We are dealing uniquely with photons of energy E_0; thus, we have to solve a simplified photon transport equation, which may be expressed as

$$\mathbf{n} \cdot (\nabla u_0)(\mathbf{r}, \mathbf{n}, E_0) = -a_0(\mathbf{r}, E_0) u_0(\mathbf{r}, \mathbf{n}, E_0) - f_0(\mathbf{r}, \mathbf{n}, E_0). \tag{1}$$

Here $u_0(\mathbf{r}, \mathbf{n}, E_0)$ represents the photon flux density in the direction $\mathbf{n}$ of energy E_0, i.e. number of photons per unit surface perpendicular to $\mathbf{n}$ per second. Recall that $a_0(\mathbf{r}, E_0)$ is the linear attenuation coefficient or rate of depletion per unit length traversed and finally $-f_0(\mathbf{r}, \mathbf{n}, E_0)$ is the number of photons emitted in the direction $\mathbf{n}$ per unit volume matter (of the extended radiation source). For simplicity, the energy label E_0 will be omitted hereafter.

The aim is to solve this partial differential equation with an isotropic source term $f_0(\mathbf{r})$:

$$\mathbf{n} \cdot (\nabla u_0)(\mathbf{r}, \mathbf{n}) = -a_0(\mathbf{r}) u_0(\mathbf{r}, \mathbf{n}) - f_0(\mathbf{r}), \tag{2}$$

where the unknown photon flux density is $u_0(\mathbf{r}, \mathbf{n})$. Reconstructing f_0 from the data $u_0(\mathbf{x}, \mathbf{n})$ is the main problem posed here.

In three dimensions *without attenuation*, the solution is represented by the 'x-ray cone beam', without restriction on the set of source points $\mathbf{x}$. This has been worked out mathematically in [1–4]. The reconstruction formula contains the average of the x-ray data on the unit sphere of $\mathbb{R}^3$. The case of point sources lying on a space curve is given by [5–8]. Finally, among the large amount of indirect inversion procedures, the most well known for efficiency and appeal are those by Smith, who developed a technique that converts divergent beam data into parallel beam data and used its known inversion procedure [9] and by Grangeat, who made a conversion of x-ray data into three-dimensional Radon data before using Radon inversion [10].

Reconstructing f_0 from equation (2) in two dimensions has been worked out by Novikov [11]. In this paper, we show that the use of quaternion analysis leads to a new inversion formula for the non-uniformly attenuated x-ray transform in $\mathbb{R}^3$. Quaternions are higher dimensional generalization of complex numbers. Although not widely used, they provide elegant compact local formulation for electromagnetism, solid mechanics and some other fields in engineering [12]. Recently, quaternions have been used in integral transforms, for example, in geophysical processes [13] or in signal processing [14]. In imaging science, [15] gets an inversion formula for the x-ray transform without attenuation. In another work [16], the inversion of exponential x-ray transform is given. The generalization of these works for the non-uniform attenuation is the subject of this paper. As we see later, this generalization is not trivial, because the fundamental solution of the Dirac operator with the non-uniform function $(D + a(\mathbf{x}))$ in quaternion analysis has been studied only for an approximate vector potential of the form [17]

$$\left\{ \frac{\underline{\mathbf{x}} - \underline{\mathbf{x}}^{(i)}}{|\underline{\mathbf{x}} - \underline{\mathbf{x}}^{(i)}|^3}, i = 1, 2, \ldots \right\}. \tag{3}$$

This is not realistic in practical applications. However, the case of constant '$a =$ constant' has been studied in [18, 19].

In the next section, we introduce some useful notions on the algebra of real quaternions $\mathbb{H}$ and collect the main results of quaternion analysis needed for our problem. Section 3 describes the derivation of the inversion formula giving the reconstructed function in terms of the x-ray data, and we give an interpretation of this new result. This paper ends with a conclusion and some perspectives to invert the x-ray transform in the presence of other effects.

2. Quaternions

Let $\mathbf{x} = (x_1, x_2, x_3)$ be an element of $\mathbb{R}^3$, expressed in an orthonormal basis formed by three unit vectors $\imath_1, \imath_2$ and $\imath_3$ by $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$. The conventional vector space structure is given by a scalar (inner) product rule for the basis unit vectors, i.e. $(\imath_n \cdot \imath_m) = \delta_{mn}$ and by a vector (cross) product, i.e. $\imath_1 \times \imath_2 = \imath_3$ with its cyclic permutations and the non-commutativity $\imath_m \times \imath_n = -\imath_m \times \imath_n$.

To this structure, one can add a new one

- by promoting the unit vectors to be imaginary units, i.e. ${\imath_1}^2 = {\imath_2}^2 = {\imath_3}^2 = -1$ and
- by introducing a non-commutative multiplication rule between them: $\imath_i \imath_j = -\imath_j \imath_i$ for $i \neq j$ and $\imath_i \imath_j = \imath_k$ for all cyclic permutations of (i, j, k).

Then to each $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$, as a three-dimensional vector, corresponds a new object $\underline{\mathbf{x}}$ (also called Vec x by some authors), which has the same formal expression but with $\imath_m$ following the new multiplication rule. Consequently, the identification

$$\mathbf{x} \in \mathbb{R}^3 \quad \mapsto \quad \underline{\mathbf{x}} = \sum_{m=1}^{3} x_m \imath_m \tag{4}$$

is an isomorphism of $\mathbb{R}^3$ onto the set of 'vector parts' {Vec} of more general objects called quaternions by Hamilton [27].

In fact, a quaternion x has four components, i.e. besides its imaginary vector part, there is also a scalar part $\mathrm{Sc}\, x = x_0\imath_0$, where $\imath_0$ is the real (or non-imaginary) unit part (usually identified with the real unit $1 = \imath_0 \in \mathbb{R}$) and $x_0 \in \mathbb{R}$, such that

$$x = x_0\imath_0 + \sum_{m=1}^{3} x_m\imath_m = \mathrm{Sc}\, x + \mathrm{Vec}\, x = x_0\imath_0 + \underline{\mathbf{x}}, \qquad (x_0, x_1, x_2, x_3 \in \mathbb{R}). \tag{5}$$

The set of quaternions with real components should be called $\mathbb{H}(\mathbb{R})$,[1] but for simplicity, will be denoted by $\mathbb{H}$.

Following [20], we give some of their properties:

$$\text{conjugate operation: } \overline{x} = x_0\imath_0 - \sum_{m=1}^{3} x_m\imath_m, \tag{6}$$

$$\text{square norm: } |x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{7}$$

$$\text{inverse: } x^{-1} = \frac{\overline{x}}{|x|^2} \text{ if and only if } x\overline{x} \neq 0. \tag{8}$$

Finally, the ordered product of two quaternions $y = y_0\imath_0 + \underline{\mathbf{y}}$ and $x = x_0\imath_0 + \underline{\mathbf{x}}$ is a quaternion $w = yx = (\mathrm{Sc}\, w + \mathrm{Vec}\, w)$, where

$$w_0 = \mathrm{Sc}\, w = y_0x_0 - (\mathbf{y}\cdot\mathbf{x}) \qquad \text{and} \qquad \underline{\mathbf{w}} = \mathrm{Vec}\, w = \underline{\mathbf{y}}x_0 + y_0\underline{\mathbf{x}} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{9}$$

In particular, i.e. the ordered product of $\underline{\mathbf{y}}$ by $\underline{\mathbf{x}}$ is

$$\underline{\mathbf{y}}\,\underline{\mathbf{x}} = -\mathbf{y}\cdot\mathbf{x} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{10}$$

For our purposes, we do not require the full machinery of quaternionic analyticity as developed by Fueter and others [20, 21]. Here we are only concerned with analytic properties useful for imaging processes in $\mathbb{R}^3$ modeled by the x-ray transform. They are essentially extracted from [18, 22]:

$$D = \sum_{j=1}^{3} \imath_j \frac{\partial}{\partial x_j}. \tag{11}$$

The quaternionic operator D has been given different names according to authors: Dirac operator for [18], three-dimensional Cauchy–Riemann operator for [12], Moisil–Teodorescu differential operator for [23], etc.

Inspection shows that it is related to the three-dimensional Laplace operator by $\triangle = -D^2$. The solutions of $Df(\mathbf{x}) = 0$, called frequently left-monogenic $\mathbb{H}$-valued functions, satisfy many generalizations of classical theorems from complex analysis to higher dimensional context [22]. Given the elementary solution of the Laplace operator, $\triangle E(\mathbf{x}) = -D^2E(\mathbf{x}) = \delta(\mathbf{x})$, as

$$E(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}, \tag{12}$$

the elementary solution of D can be worked out as [18]

$$\underline{\mathbf{K}}(\mathbf{x}) = \sum_{j=1}^{3} K_j(\mathbf{x})\imath_j = -\frac{\underline{\mathbf{x}}}{4\pi|\mathbf{x}|^3}, \qquad \underline{\mathbf{x}} \neq 0, \tag{13}$$

[1] Quaternions with complex-valued components are called *biquaternions* and denoted by $\mathbb{H}(\mathbb{C})$.

where

$$K_j(\mathbf{x}) = -\frac{x_j}{4\pi|\mathbf{x}|^3} \qquad (j = 1, 2, 3). \tag{14}$$

Note that $\underline{\mathbf{K}}(\mathbf{x})$ is a $\mathbb{H}$-valued fundamental solution of D and therefore monogenic in $G\backslash\{0\}$ where $G \subset \mathbb{R}^3$.

Now, we write the generalized Leibniz formula in quaternions [18]:

$$D(uw) = \overline{u}Dw + (Du)w + 2\mathrm{Sc}(uD)w, \qquad u, w \in \mathbb{H}(\mathbb{R}^4), \tag{15}$$

where $\mathbb{H}(\mathbb{R}^4)$ is the set of u and v, which are $\mathbb{H}$-valued functions with the domain in $\mathbb{R}^4$.

Consequently, there exists a three-dimensional Cauchy integral representation for continuous left-monogenic $\mathbb{H}$-valued functions on $\overline{G}$ [22],

$$(Ff)(\mathbf{x}) := \int_\Gamma \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\Gamma_{\mathbf{y}}, \qquad \mathbf{x} \in G\backslash\Gamma, \tag{16}$$

where $\underline{\alpha}(\mathbf{y}) = \sum_{j=1}^{3} \alpha_j(\mathbf{y})\imath_j$ is the quaternionic outward pointing unit vector at $\mathbf{y}$ on the boundary $\partial G = \Gamma$, $\mathrm{d}\Gamma_{\mathbf{y}}$ is the Lebesgue measure on Γ. Moreover one has $D\,(F_\Gamma f)(\mathbf{x}) = 0$.

The operator D has a right inverse, called the Teodorescu transform [24]. It is defined for all $f(\mathbf{x}) \in \mathcal{C}(G, \mathbb{H})$ by

$$(Tf)(\mathbf{x}) := \int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\ f(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \mathbf{x} \in G \subset \mathbb{R}^3. \tag{17}$$

Roughly speaking, D is a kind of directional derivative and T is just the integration, the right inverse of this directional derivative.

Conversely, for any $f(\mathbf{x}) \in \mathcal{C}^1(G, \mathbb{H}) \cap \mathcal{C}(\overline{G}, \mathbb{H})$, it can be shown that it satisfies the so-called Borel–Pompeiu formula [18]

$$(Ff)(\mathbf{x}) + (T\ D)f(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^3\backslash\overline{G}. \end{cases} \tag{18}$$

A generalization of the concept of Cauchy principal value for $(Ff)(\mathbf{x})$ can be introduced when the variable $\mathbf{x}$ is approaching the boundary $\partial G = \Gamma$. For a given f, at each regular point $\mathbf{x}' \in \Gamma$ [18], the non-tangential limit of the Cauchy integral representation can be written as

$$\lim_{\mathbf{x}\to\mathbf{x}'} (Ff)(\mathbf{x}) = \tfrac{1}{2}(\pm f(\mathbf{x}') + (Sf)(\mathbf{x}')), \tag{19}$$

where

$$(Sf)(\mathbf{x}) = 2\int_\Gamma \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\,\underline{\alpha}(\mathbf{y})\ f(\mathbf{y})\,\mathrm{d}\Gamma_{\mathbf{y}} \tag{20}$$

is understood as a 'quaternionic Cauchy principal value' of the integral over the smooth boundary Γ because of the singularity of $\underline{\mathbf{K}}(\mathbf{x})$ in the integrand.

A Plemelj–Sokhotzkij-type formula for f, relative to Γ, [22, 24] can now be given as

$$\text{(i)}\quad \lim_{\substack{\mathbf{x}\to\mathbf{x}'\in\Gamma \\ \mathbf{x}\in G}} (Ff)(\mathbf{x}) = (Pf)(\mathbf{x}'), \qquad \text{(ii)}\quad \lim_{\substack{\mathbf{x}\to\mathbf{x}'\in\Gamma \\ \mathbf{x}\in\mathbb{R}^3\backslash\overline{G}}} (Ff)(\mathbf{x}) = -(Qf)(\mathbf{x}'), \tag{21}$$

where P is the projection operator ($P^2 = P$) onto $\mathbb{H}$-valued functions, which have a left-monogenic extension into the domain G, and Q is the projection operator ($Q^2 = Q$) onto $\mathbb{H}$-valued functions, which have a left-monogenic extension into the domain $\mathbb{R}^3\backslash\overline{G}$ and vanish at infinity.

P and Q can be given, in turn, an alternative form in terms of the quaternionic principal value operator S as

$$P := \tfrac{1}{2}(I + S) \qquad Q := \tfrac{1}{2}(I - S), \tag{22}$$

with the following operator relations

$$SP = P, \qquad SQ = -Q, \qquad S^2 = SS = I. \tag{23}$$

Finally, we define a *trace* operator tr as a restriction map for an $\mathbb{H}$-valued function f on Γ, smooth boundary of $G \in \mathbb{R}^3$, by

$$\mathrm{tr}\, f = f|_\Gamma. \tag{24}$$

Notation. Here we review our notation in this paper. Only 'bold' letters are used for vectors or vector functions in $\mathbb{R}^3$, such as $\mathbf{x}$ or $\mathbf{f}(\mathbf{x})$. The index 'zero' indicates the scalar part of a quaternion or quaternion function, e.g. x_0 or $a_0(\mathbf{x})$. Underlined bold letters are used for the vector part of the quaternions or quaternion functions, e.g. $\underline{\mathbf{x}}$ or $\underline{\mathbf{f}}(\mathbf{x})$. Operators with index '$a$' are the operators with attenuation, e.g. T_a, X_a.

3. The x-ray transform and its inverse

We are now in a position to tackle the inversion problem for the non-uniform attenuated x-ray transform of a physical density $f_0(\mathbf{x})$. By definition, this transform consists of integrating $f_0(\mathbf{x})$, assumed to be an integrable function with compact support in a convex set G, along a straight line from the source point $\mathbf{x}$ to infinity in the direction of the unit vector $\mathbf{n}$, i.e.

$$(X_a f_0)(\mathbf{x}, \mathbf{n}) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-\mathfrak{D}a_0(\mathbf{x})t}\, f_0(\mathbf{x} + t\mathbf{n}), \tag{25}$$

where

$$\mathfrak{D}a_0(\mathbf{x}) = -\frac{1}{4\pi} \int_{\Omega_\mathbf{n}} (Xa_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}, \tag{26}$$

where $\mathrm{d}\Omega_\mathbf{n}$ is the area element of the unit sphere $\Omega_\mathbf{n}$ in $\mathbb{R}^3$ and (Xa_0) is the x-ray transform

$$(Xa_0)(\mathbf{x}, \mathbf{n}) = \int_0^\infty \mathrm{d}t\, a_0(\mathbf{x} + \mathbf{n}t). \tag{27}$$

In transmission modality, f_0 represents the attenuation map of the object under study, whereas in emission modality f_0 is its radiation activity density.

The next point is that if $f_0(\infty) = 0$, it can be verified that $(X_a f_0)(\mathbf{x}, \mathbf{n})$ satisfies a very simple partial differential equation, namely

$$(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0(\mathbf{x}))\, (X_a f_0)(\mathbf{x}, \mathbf{n}) = -f_0(\mathbf{x}). \tag{28}$$

This can be checked if we let the $(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0(\mathbf{x}))$ operator act under the integral sign. After a change of variables, the integrand just turns into the differential of $f_0(\mathbf{x})$ under the integral sign. Equation (28) is in fact a simplified stationary photon transport equation with loss by attenuation function $a_0(\mathbf{x})$ and without source or sink term [25]. Since $(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0(\mathbf{x}))$ is a directional derivative plus the attenuated term, clearly its inverse is an integration[2]. The solution of this partial differential equation is subjected to the following boundary condition. For a given direction $\mathbf{n}$, because of the support hypothesis and because of the prescription on the direction of integration, $(X_a f_0)(\mathbf{x}, \mathbf{n}) = 0$, whenever $\mathbf{x}$ is on the boundary $\Gamma = \partial G$ of G and $\mathbf{n}$ points outward of Γ.

To obtain the solution of the above equation by using real analysis, we write the solution of the homogenous form of equation (28), i.e.

$$(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0(\mathbf{x}))v_0(\mathbf{x}, \mathbf{n}) = 0 \tag{29}$$

[2] This is not the second-order ultra-hyperbolic partial differential equation of John [26].

from which $v_0(\mathbf{x}, \mathbf{n})$ is obtained as

$$v_0(\mathbf{x}, \mathbf{n}) = \mathrm{e}^{-\int_{\mathbb{R}^3} \widetilde{G}_0(\mathbf{x}-\mathbf{y},\mathbf{n})a_0(\mathbf{y})\mathrm{d}\mathbf{y}}, \qquad \mathbf{y} \in \mathbb{R}^3, \tag{30}$$

where $\widetilde{G}_0(\mathbf{x} - \mathbf{y}, \mathbf{n})$ is the Green's function of the $(\mathbf{n} \cdot \nabla_{\mathbf{x}})$ operator.

At this point, we define $u_0(\mathbf{x}, \mathbf{n})$ in (28) as

$$u_0(\mathbf{x}, \mathbf{n}) = C_0(\mathbf{x}) v_0(\mathbf{x}, \mathbf{n}). \tag{31}$$

By substituting $u_0(\mathbf{x}, \mathbf{n})$ into equation (28), we have

$$C_0(\mathbf{x}) = \int_{\mathbb{R}^3} \widetilde{G}_0(\mathbf{x} - \mathbf{y}, \mathbf{n}) v_0^{-1}(\mathbf{y}, \mathbf{n}) f_0(\mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{32}$$

and

$$u_0(\mathbf{x}, \mathbf{n}) = -\int_{\mathbb{R}^3} R_0(\mathbf{x}, \mathbf{y}, \mathbf{n}) f_0(\mathbf{y})\, \mathrm{d}\mathbf{y}, \tag{33}$$

where

$$R_0(\mathbf{x}, \mathbf{y}, \mathbf{n}) = v_0(\mathbf{x}, \mathbf{n}) \widetilde{G}_0(\mathbf{x} - \mathbf{y}, \mathbf{n}) v_0^{-1}(\mathbf{y}, \mathbf{n}). \tag{34}$$

We will use a similar method in the quaternion analysis to obtain an inversion solution for equation (28).

3.1. Quaternion solution

By considering $\mathbf{n}$ independent of $\mathbf{x}$, we can rewrite equation (28) in the following form:

$$\mathbf{n} \cdot (\nabla + \mathbf{a}) u_0(\mathbf{x}) = -f_0(\mathbf{x}), \tag{35}$$

where we define $\mathbf{a} := a_0 \mathbf{n}$.

We would like to use the machinery of quaternion analysis to obtain the inversion of the three-dimensional x-ray transform. The idea is to consider equation (28) as part of an inhomogeneous equation (11), with an $\mathbb{H}$-valued 'source' function $f = f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x})$ on its right-hand side for an unknown scalar function $u_0(\mathbf{x})$. As can be checked, the quaternionic product rule (9) yields

$$\underline{\mathbf{n}} D_a u_0(\mathbf{x}) = f(\mathbf{x}), \tag{36}$$

where D_a is defined as follows:

$$D_a = D + \underline{\mathbf{a}}. \tag{37}$$

Explicitly equation (36) has the following form:

$$\underline{\mathbf{n}} D_a u_0(\mathbf{x}) = -\mathbf{n} \cdot (\nabla_{\mathbf{x}} + \mathbf{a}(\mathbf{x})) u_0(\mathbf{x}) + \underline{\mathbf{n} \times (\nabla_{\mathbf{x}} + \mathbf{a}(\mathbf{x}))} u_0(\mathbf{x}) = f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x}), \tag{38}$$

which leads to a set of two equations for u_0:

$$\begin{aligned} &(\mathbf{n} \cdot \nabla_{\mathbf{x}} + a_0(\mathbf{x})) u_0(\mathbf{x}) = -f_0(\mathbf{x}) \\ &(\mathbf{n} \times \nabla_{\mathbf{x}}) u_0(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \end{aligned} \tag{39}$$

the first one being exactly the one of the x-ray transform. By solving equation (39), we can obtain the solution of equation (39) as a by product, for given $f_0(\mathbf{x})$, first. Then, $\mathbf{f}(\mathbf{x})$ can be computed from the curl term and the gradient term of the solution.

From (39) the case $\mathbf{f} = (\mathbf{n} \times \nabla_{\mathbf{x}}) u_0 = 0$ means that the areolar derivative of u_0 is equal to zero. In the other words, the derivative of u_0 on the plane perpendicular to $\mathbf{n}$ is equal to zero, or u_0 is constant on the plane perpendicular to $\mathbf{n}$.

Considering $\underline{\mathbf{f}} = 0$, equation (36) becomes

$$\underline{\mathbf{n}} D_a u_0(\mathbf{x}) = f_0(\mathbf{x}), \qquad \mathbf{x} \in G, \tag{40}$$

in which we can easily see that the above equation is the transport equation (28) in the quaternion formalism.

Now, by multiplying equation (40) by $-\underline{\mathbf{n}}$ from the left-hand side we have

$$D_a u_0(\mathbf{x}, \mathbf{n}) = -\underline{\mathbf{n}} f_0(\mathbf{x}), \tag{41}$$

where we used '$\underline{\mathbf{nn}} = -1$'.

We choose the Dirichlet boundary condition for equation (41), i.e. the boundary value of u_0 on the smooth boundary surface of Γ is equal to w_0 which is a monogenic function. Hence, we have

$$D_a u_0(\mathbf{x}) = -\underline{\mathbf{n}} f_0(\mathbf{x}), \qquad \mathbf{x} \in G \tag{42}$$

$$u_0(\mathbf{x})|_\Gamma = w_0(\mathbf{x}), \tag{43}$$

where w_0 is a scalar function.

Before obtaining the solution of the above equation, we derive an explicit form of $T\underline{\mathbf{a}}$:

$$\begin{aligned}(T\underline{\mathbf{a}})(\mathbf{x}) &= \int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\underline{\mathbf{a}}(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \frac{1}{4\pi}\int_G \frac{\underline{\mathbf{x}}-\underline{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3}\underline{\mathbf{a}}(\mathbf{y})\,\mathrm{d}\mathbf{y} = \frac{1}{4\pi}\sum_{i,j=1}^{3}\int_G \frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3} a_j(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\iota_i\iota_j.\end{aligned} \tag{44}$$

The photon transport in the $\mathbf{n}$ direction of the x-ray source located at $\mathbf{x}$ required that $\mathbf{y} = \mathbf{x}+\mathbf{n}t$, where $t \in \mathbb{R}^+$. Consequently, the volume element $\mathrm{d}\mathbf{y}$ in spherical coordinates becomes $\mathrm{d}\mathbf{y} = t^2\mathrm{d}t\,\mathrm{d}\Omega_{\mathbf{n}}$, where $\mathrm{d}\Omega_{\mathbf{n}}$ is the area element of the unit sphere Ω_n in $\mathbb{R}^3$. Then, we have

$$\begin{aligned}(T\underline{\mathbf{a}})(\mathbf{x}) &= \frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}}\int_{\mathbb{R}^+} \frac{\underline{\mathbf{n}}t}{|\mathbf{n}t|^3}\underline{\mathbf{n}}a_0(\mathbf{x}+\mathbf{n}t)t^2\,\mathrm{d}\Omega_{\mathbf{n}}\,\mathrm{d}t \\ &= -\frac{1}{4\pi}\int_{\Omega_{\mathbf{m}}}\left(\int_{\mathbb{R}^+} a_0(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t\right)\mathrm{d}\Omega_{\mathbf{n}},\end{aligned} \tag{45}$$

where we have $\underline{\mathbf{a}} = \underline{\mathbf{n}}a_0$ and $\underline{\mathbf{nn}} = -1$. Now, we use the definition of the x-ray transform of component f_β:

$$Xa_0(\mathbf{x}, \mathbf{n}) := \int_{\mathbb{R}^+} a_0(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t. \tag{46}$$

Then, we have

$$\mathfrak{D}a_0(\mathbf{x}) := (T\underline{\mathbf{a}})(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}}(Xa_0)(\mathbf{x}, \mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}. \tag{47}$$

Here we showed that the $(T\underline{\mathbf{a}}) = (T\underline{\mathbf{n}}a_0)$ is a scalar function. Thus, the following theorem gives the solution u_0 of the above equation when $(T\underline{\mathbf{a}})$ is a scalar function. This is sufficient to solve our problem.

Theorem. *Assuming that $(T\underline{\mathbf{a}}) = \mathrm{Sc}(T\underline{\mathbf{a}}) = (T\underline{\mathbf{a}})_0$, and for f_0 and u_0, v_0 differentiable or weakly differentiable functions in a normed space with domain in $G \subset \mathbb{R}^3$, the solution of equation (42) is given by*

$$v_0 = \mathrm{e}^{-T\underline{\mathbf{a}}} \tag{48}$$

$$u_0 = -(T_a\underline{\mathbf{n}}f_0) + F_a w_0, \tag{49}$$

where $T_a\underline{\mathbf{n}}f_0 = v_0 T v_0^{-1}\underline{\mathbf{n}}f_0$ and $F_a w_0 = v_0 F v_0^{-1} w_0$. With the condition

$$Q_a w_0 = -\mathrm{tr}\big(v_0 T v_0^{-1}\underline{\mathbf{n}}f_0\big), \tag{50}$$

where $Q_a w_0 = v_0 Q v_0^{-1}$ the above condition follows from Plemelj–Sokhotzkij's formula. This means that there exists an extension onto the domain $\mathbb{R}^3 \backslash \overline{G}$.

Proof. v_0 is the solution of the following homogenous equation:

$$D_a v_0 = (D + \underline{\mathbf{a}}) v_0 = D v_0 + \underline{\mathbf{a}} v_0 = 0, \tag{51}$$

where the solution of the above equation is equal to

$$v_0 = \mathrm{e}^{-T\underline{\mathbf{a}}}. \tag{52}$$

In appendix A, we solve equation (51) using real analysis. We can verify the above solution by substituting expression (52) into equation (51):

$$D\,\mathrm{e}^{-T\underline{\mathbf{a}}} + \underline{\mathbf{a}} v_0 = -(DT\underline{\mathbf{a}})\,\mathrm{e}^{-T\underline{\mathbf{a}}} + \underline{\mathbf{a}} v_0 = -(DT\underline{\mathbf{a}}) v_0 + \underline{\mathbf{a}} v_0 = -\underline{\mathbf{a}} v_0 + \underline{\mathbf{a}} v_0 = 0, \tag{53}$$

where we used the fact that $DTv_0 = v_0$ in G, which means that T is the right inverse of D [22]. Now, we introduce the general solution of (42) as $u_0^{(g)} := v_0 C_0$, where C_0 is a function with domain in $G \subset \mathbb{R}^3$ and $\operatorname{tr} C_0 = 0$. We replace it in equation (42). Thus, we have

$$D(v_0 C_0) + \underline{\mathbf{a}} v_0 C_0 = -\underline{\mathbf{n}} f_0. \tag{54}$$

Finally,

$$D(v_0 C_0) = v_0 D C_0 + (D v_0) C_0 = v_0 D C_0 - \underline{\mathbf{a}} v_0 C_0 = -\underline{\mathbf{n}} f_0. \tag{55}$$

Consequently, (54) gives

$$v_0 D C_0 = -\underline{\mathbf{n}} f_0. \tag{56}$$

By acting v_0^{-1} on the above equation we obtain

$$D C_0 = -v_0^{-1}\,\underline{\mathbf{n}} f_0. \tag{57}$$

Now, taking into account that '$\operatorname{tr} C_0 = 0$' (which means that '$F C_0 = 0$'), C_0 has the following form:

$$C_0 = -T v_0^{-1}\,\underline{\mathbf{n}} f_0. \tag{58}$$

In a similar way where it was shown before that $(T\underline{\mathbf{a}}) = (T\underline{\mathbf{n}} a_0)$ is a scalar function, one can show that $(T v_0 \underline{\mathbf{n}} f_0)$ is a scalar function. Finally, $u_0^{(g)}$ is equal to

$$u_0^{(g)} = v_0 C_0 = -v_0 T v_0^{-1}\,\underline{\mathbf{n}} f_0 = -T_a\,\underline{\mathbf{n}} f_0. \tag{59}$$

The proper solution $u_0^{(p)}$ of equation (42) which takes the value u_0 on the boundary, i.e. equation (43) is

$$\operatorname{tr} u_0 = \operatorname{tr} u_0^{(g)} + \operatorname{tr} u_0^{(p)} = -\operatorname{tr}\left(v_0 T v_0^{-1} \underline{\mathbf{n}} f_0\right) + \operatorname{tr} u_0^{(p)}. \tag{60}$$

Using condition (50), we obtain

$$\operatorname{tr} u_0 = Q_a w_0 + \operatorname{tr} u_0^{(p)} = w_0. \tag{61}$$

Consequently,

$$\operatorname{tr} u_0^{(p)} = (I - Q_a) w_0 = v_0 (I - Q) v_0^{-1} w_0 = v_0 P v_0^{-1} w_0 = P_a w_0, \tag{62}$$

where $P_a = v_0 P v_0^{-1}$. Thus, from the definition of $P w_0 = \operatorname{tr}(F w_0)$ [18], $u_0^{(p)}$ is equal to

$$u_0^{(p)} = v_0 F v_0^{-1} w_0 = F_a w_0. \tag{63}$$

Finally, by considering $u_0^{(g)}$ and $u_0^{(p)}$, $u_0 = u_0^{(g)} + u_0^{(p)}$ is obtained by equation (49). We verify our solution by acting D_a on equation (42). Then, we have

$$D_a u_0 = -D_a v_0 T v_0^{-1} \underline{\mathbf{n}} f_0 + D_a (F_a w_0) = -v_0 D_a T v_0^{-1} \underline{\mathbf{n}} f_0 + (D_a v_0) T v_0^{-1} \underline{\mathbf{n}} f_0, \tag{64}$$

where we used the generalized Leibniz formula (15) and

$$D(F_a w) = D v_0 F v_0^{-1} w_0 = (D v_0) F v_0^{-1} w_0 + v_0 D F v_0^{-1} w_0 = -\underline{\mathbf{a}} F v_0^{-1} w_0 + 0, \tag{65}$$

where we use $D F v_0^{-1} w_0 = 0$, which means that $(F v_0^{-1} w_0)$ is a monogenic function [22]. Thus, as a result, we can conclude that $D_a F_a w_0 = 0$.

Thus, equation (64) is obtained as

$$D_a u_0 = -v_0 v_0^{-1}\,\underline{\mathbf{n}} f_0 - v_0\,\underline{\mathbf{a}} T v_0^{-1}\,\underline{\mathbf{n}} f_0 + (v_0\,\underline{\mathbf{a}}) T v_0^{-1}\,\underline{\mathbf{n}} f_0 = -\underline{\mathbf{n}} f_0, \tag{66}$$

where in the first term on the right-hand side we used $D T u_0 = u_0$.

Now, we check the solution at the boundary condition (43). Thus, by substituting u_0 from (48) into (43), we obtain

$$\begin{aligned} w_0 &= -\mathrm{tr}(v_0 T v^{-1}\underline{\mathbf{n}} f_0 + F_a w_0) \\ &= -\mathrm{tr}\big(v_0 T v_0^{-1}\underline{\mathbf{n}} f_0\big) + \mathrm{tr}(F_a w_0) = -\mathrm{tr}\big(v_0 T v_0^{-1}\underline{\mathbf{n}} f_0\big) + P_a w_0, \end{aligned} \tag{67}$$

where in the last equation we use: $\mathrm{tr}(F_a w_0) = (F_a w_0)_\Gamma = P_a w_0$. Then, (67) yields

$$-\mathrm{tr}(v_0 T v_0^{-1}\underline{\mathbf{n}} f_0) = (I - P_a) w_0 = Q_a w_0. \tag{68}$$

□

3.2. The x-ray representation

Now, we reconstruct f_0 by using equation (49). As shown in (66), '$D_a u_0 = D_a\big(-v_0 T v_0^{-1}\underline{\mathbf{n}} f_0 + F_a w_0\big) = -\underline{\mathbf{n}} f_0$'; thus,

$$\underline{\mathbf{n}} f_0 = D_a\big(v_0 T v_0^{-1}\underline{\mathbf{n}} f_0\big) \tag{69}$$

which gives $\underline{\mathbf{n}} f_0$. Now, replacing $\underline{\mathbf{n}} f_0$ by f_0, we can obtain f_0:

$$f_0 = D_a\big(v_0 T v_0^{-1} f_0\big), \tag{70}$$

or by using $D_a v_0 = 0$ (equation (51)), we have

$$f_0 = v_0 D_a\big(T v_0^{-1} f_0\big). \tag{71}$$

To get the explicit form of $f_0(\mathbf{x})$ in terms of the imaging data set, we first compute the Teodorescu transform of $v_0^{-1} f_0$. Thus, by using equation (47), v_0 is written as

$$v_0(\mathbf{x}) = \mathrm{e}^{-T\underline{\mathbf{a}}(\mathbf{x})} = \mathrm{e}^{\frac{1}{4\pi}\int_{\Omega_\mathbf{n}}(X a_0)(\mathbf{x},\mathbf{n})\mathrm{d}\Omega_\mathbf{n}} = \mathrm{e}^{\mathfrak{D} a_0(\mathbf{x})}. \tag{72}$$

Then, we obtain $(T v_0^{-1} f_0)(\mathbf{x})$ by using the same method with which we obtained $(T\underline{\mathbf{a}})$ in equation (47):

$$\begin{aligned} \big(T v_0^{-1} f_0\big)(\mathbf{x}) &= \int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\big(v_0^{-1} f_0\big)(\mathbf{y})\,\mathrm{d}\mathbf{y} \\ &= \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} \underline{\mathbf{n}}\big[X\big(v_0^{-1} f_0\big)\big](\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \end{aligned} \tag{73}$$

Now, we define the attenuated x-ray transform as follows:

$$(X_a f)(\mathbf{x},\mathbf{n}) := \int_{\mathbb{R}^+} \mathrm{e}^{-\mathfrak{D} a_0(\mathbf{x}+\mathbf{n}t)} f_0(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t. \tag{74}$$

Thus, equation (73) is rewritten as

$$\big(T v_0^{-1} f_0\big)(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} \underline{\mathbf{n}}\,(X_a f_0)(\mathbf{x},\mathbf{n})\mathrm{d}\Omega_\mathbf{n}. \tag{75}$$

Hence, f_0 is obtained by

$$f_0(\mathbf{x}) = v_0 D_a\big(T v_0^{-1} f_0\big)(\mathbf{x}) = \frac{1}{4\pi}\,\mathrm{e}^{(\mathfrak{D} a_0)(\mathbf{x})}\,D_a\int_{\Omega_\mathbf{n}} \underline{\mathbf{n}}\,(X_a f_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{76}$$

As earlier in this paper we have introduced $(X_a f_0)(\mathbf{x}, \mathbf{n})$, this is a monogenic (analytic) function on $\Omega_\mathbf{n}$; thus, $[a_0(\mathbf{x})(X_a f_0)(\mathbf{x}, \mathbf{n})]_{\Omega_\mathbf{n}} \to 0$ as $t \to 0$. Finally, the reconstruction formula for f_0 is obtained as

$$f_0(\mathbf{x}) = -\frac{1}{4\pi}\, \mathrm{e}^{\mathfrak{D} a_0(\mathbf{x})} \int_{\Omega_\mathbf{n}} (\mathbf{n} \cdot \nabla_\mathbf{x})\, (X_a f_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}. \tag{77}$$

Case $\boldsymbol{a_0 = 0}$. In the special case where $a_0 = 0$, $\mathfrak{D} a_0 = 0$. Equation (77) is given as

$$f_0(\mathbf{x}) = \frac{1}{4\pi} \int_{\Omega_\mathbf{n}} \mathrm{d}\Omega_\mathbf{n}\, (\mathbf{n} \cdot \nabla_\mathbf{x})(X f_0)(\mathbf{x}, \mathbf{n}), \tag{78}$$

where $(X f_0)(\mathbf{x} + \mathbf{n}t) = \int_0^\infty \mathrm{d}t\, f_0(\mathbf{x} + \mathbf{n}t)$ is the x-ray transform without attenuation. Here the result is the one obtained by [15]. A comparison of the above formula with other results given by [15] is presented in appendix B.

4. Conclusion

In this paper, by using quaternion analysis we have obtained a successful inverse formula for the non-uniform x-ray transform in three dimensions. As we have shown in equation (77) for the case without attenuation $a_0 = 0$ has a different form, but it is essentially equivalent to the result obtained many years ago in previous works.

Appendix A.

In this appendix we compute a solution for equation (51) using real analysis. Equation (51) can be written as

$$\nabla_\mathbf{x} v_0 + \mathbf{n} a_0 v_0 = 0. \tag{A.1}$$

Multiplying by $\mathbf{n}$ the left-hand side yields

$$(\mathbf{n} \cdot \nabla_\mathbf{x}) v_0 + a_0 v_0 = 0, \tag{A.2}$$

or

$$(\mathbf{n} \cdot \nabla_\mathbf{x}) \ln v_0 = -a_0, \tag{A.3}$$

where by introducing $\phi_0 := \ln v_0$ and $\rho_0 := -a_0$, we have

$$(\mathbf{n} \cdot \nabla_\mathbf{x}) \phi_0 = \rho_0. \tag{A.4}$$

The above equation is a stationary transport equation with the source term ρ_0 and without attenuation. The solution of this equation is known to be given by a divergent x-ray transform of the data [15, 28], i.e.

$$\phi_0(\mathbf{x}, \mathbf{n}) = (X\rho_0)(\mathbf{x}, \mathbf{n}) := \int_{\mathbb{R}^+} \rho_0(\mathbf{x} + \mathbf{n}t)\, \mathrm{d}t. \tag{A.5}$$

Now, by replacing ρ_0 and ϕ_0 by $\ln v_0$ and $-a_0$, respectively, we obtain

$$v_0(\mathbf{x}, \mathbf{n}) = \mathrm{e}^{-\int_{\mathbb{R}^+} a_0(\mathbf{x} + \mathbf{n}t)\mathrm{d}t}, \tag{A.6}$$

which is the same result as obtained from quaternion analysis. This solution is obtained without restriction on a_0. Thus, equation (51) or (A.1) does not impose any restriction on a_0.

Appendix B. Comparison of formula (78) with known results [15]

In [3], the inverse formula of the x-ray transform in three dimensions is given by

$$f_0(\mathbf{x}) = -\frac{1}{2\pi^2}\triangle_{\mathbf{x}}\mathfrak{R}^1 \int_{\Omega_{\mathbf{n}}} (Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} = -\frac{1}{2\pi^2}\triangle_{\mathbf{x}}\mathfrak{R}^1 \int_{\Omega_{\mathbf{n}}} u_0(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}, \tag{B.1}$$

where

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2}\int \frac{1}{|\mathbf{x}-\mathbf{y}|^2} f_0(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{B.2}$$

Setting $\mathbf{y} = \mathbf{x} + \mathbf{n}t$ in the above equation, we find

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2}\int_{\Omega_{\mathbf{n}}}\int_{\mathbb{R}^+} f_0(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t\,\mathrm{d}\Omega_{\mathbf{n}} = \frac{1}{2\pi^2}\int_{\Omega_{\mathbf{n}}} (Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}. \tag{B.3}$$

In equation (B.1) we may define $\frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}} u_0(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} = \langle u_0\rangle_{\mathbf{n}}$ as the average of u_0 over a unit ball. Thus, by using the above relation, equation (B.3) can be written as

$$f_0(\mathbf{x}) = -\triangle_{\mathbf{x}}\mathfrak{R}^1\langle u_0\rangle_{\mathbf{n}'} = -\frac{2}{\pi}\triangle_{\mathbf{x}}(X\langle u_0\rangle_{\mathbf{n}})(\mathbf{x}) = -\frac{2}{\pi}\int_{\Omega_{\mathbf{n}}} \triangle_{\mathbf{x}}(Xu_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}. \tag{B.4}$$

Here we obtain another form for $\triangle_{\mathbf{x}}(X_a u_0)$. From equation (39), which expresses $\mathbf{f}$ as $\mathbf{f} = \mathbf{n}\times\nabla_{\mathbf{x}}u_0 = -\nabla_{\mathbf{x}}\times\mathbf{n}u_0$, we deduce that $\nabla_{\mathbf{x}}\cdot\mathbf{f} = 0$. Thus,

$$\nabla_{\mathbf{x}}\cdot(\nabla_{\mathbf{x}}\times\mathbf{n}u_0) = \mathbf{n}\triangle_{\mathbf{x}}u_0 - \nabla_{\mathbf{x}}(\mathbf{n}\cdot\nabla_{\mathbf{x}})u_0 = \mathbf{n}\triangle_{\mathbf{x}}u_0 + \nabla_{\mathbf{x}}f_0 = 0, \tag{B.5}$$

which yields

$$\triangle_{\mathbf{x}}u_0 = -(\mathbf{n}\cdot\nabla_{\mathbf{x}})f_0. \tag{B.6}$$

Substitution of this expression into equation (B.4) gives an alternative form of the reconstructed f_0:

$$f_0(\mathbf{x}) = \frac{2}{\pi}\int_{\Omega_{\mathbf{n}}} (\mathbf{n}\cdot\nabla_{\mathbf{x}})(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}, \tag{B.7}$$

which, up to a normalization factor, has the same form as equation (78).

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