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The $\mathbb{R}^3$ exponential x-ray transform inversion in quaternion analysis

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Abstract

In this paper, we present a new derivation of the inverse of the exponential x-ray transform in the three dimensions, based on quaternion analysis. An explicit formula is obtained using a set of three-dimensional x-ray projection data. The result without attenuation is recovered as a special case.

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1. Introduction

The exponential x-ray transform in $\mathbb{R}^3$ is an integral transform on the space of integrable functions in $\mathbb{R}^3$, which serves as mathematical basis for a three-dimensional imaging process and may be viewed as a single-photon emission computed tomography (SPECT) imaging and also in intensity modulated radiation therapy [1]. In three-dimensional SPECT, it provides a way to perform accurate attenuation correction without transmission measurements [2].

To 'see' the inside of an object, it is necessary to probe its hidden three-dimensional structure by a physical agent. One way to achieve this goal is to use an external source of x-ray to illuminate the studied object and measure the transmitted x-ray intensity along all possible directions in space. Given a calibrated x-ray source, this measurement gives the integrated attenuation map $f_0(\mathbf{x})$ of traversing ionizing radiation along straight line paths through this object. The set of such line integrals represents a mapping: $f_0 \mapsto (X_a f_0)(\mathbf{x}, \mathbf{n})$ called the exponential x-ray transform of f_0. Here, $\mathbf{x} \in \mathbb{R}^3$ is the x-ray source position and $\mathbf{n}$ is a unit vector of the direction of the straight line, at the end of which a measurement is performed. Thus, $(X_a f_0)(\mathbf{x}, \mathbf{n})$ depends on five apparent variables but only four variables (two for $\mathbf{n}$ and two for the position of the line in a plane containing the coordinate system origin and orthogonal to $\mathbf{n}$) are independent. Reconstructing f_0 from the data $(X_a f_0)(\mathbf{x}, \mathbf{n})$ is the main problem to be solved.

The solution, without restriction on the set of source points $\mathbf{x}$, has been worked out mathematically in [3–6]. The reconstruction formula involves the average of the x-ray data on the unit sphere of $\mathbb{R}^3$, centered at a site in space. The case of point sources lying on a space curve is given by [7–10]. Finally, among the large amount of indirect inversion procedures, the most well known for efficiency and appeal are those of Smith, who developed a technique that converts divergent beam data into parallel beam data and used its known inversion procedure [11], and of Grangeat, who made a conversion of x-ray data into three-dimensional radon data before using radon inversion [12]. The inversion of exponential radon transform in two dimension is discussed by [13, 14]. Approximated inversion for the exponential x-ray transform in $\mathbb{R}^N$ is obtained by [15].

In this work, we show that the use of quaternion analysis leads to a new inversion formula for the exponential x-ray transform in $\mathbb{R}^3$. Quaternions are higher dimensional generalization of complex numbers. Although not widely used, they provide elegant compact local formulation for electromagnetism [16], solid mechanics and some other fields in engineering [17]. Recently, quaternions have been used in integral transforms, for example in geophysical processes [19] or in signal processing [18]. Recently, in [20] the quaternions are used to obtain an inversion formula for the x-ray transform. In this paper we generalized the method presented in [20] to get an inversion formula for the exponential x-ray transform.

In the next section, we introduce some useful notions on the algebra of the real quaternions $\mathbb{H}$ and collect the main results of quaternion analysis needed for our problem. Section 3 describes the derivation of the inversion formula giving the reconstructed function in terms of the x-ray data. This paper ends with a conclusion and sketches the perspectives for inverting a more general case of x-ray transforms in $\mathbb{R}^3$.

2. Preliminary

Quaternions were invented by Hamilton in the first half of the 19th century, when he looked for a three-dimensional generalization of complex numbers [21]. But, this theory did not generate widespread interest until nearly a century after it was discovered. Subsequently, Fueter introduced the notion of 'regular' quaternionic functions as functions satisfying an analog of the Cauchy–Riemann equations. With this new concept, he is led to Cauchy's theorem, Cauchy's integral formula and Laurent expansion for analytic functions [22]. A comprehensive review of quaternions can be found in [23].

Let $\mathbf{x} = (x_1, x_2, x_3)$ be an element of $\mathbb{R}^3$, expressed in an orthonormal basis formed by three unit vectors $\imath_1, \imath_2$ and $\imath_3$ by $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$. The conventional vector space structure is given by a scalar (inner) product rule for the basis unit vectors, i.e. $(\imath_n \cdot \imath_m) = \delta_{mn}$ and by a vector (cross) product, i.e. $\imath_1 \times \imath_2 = \imath_3$ with its cyclic permutations and the non-commutativity $\imath_m \times \imath_n = -\imath_m \times \imath_n$.

To this structure, one can add a new one

- by promoting the unit vectors to be imaginary units, i.e. ${\imath_1}^2 = {\imath_2}^2 = {\imath_3}^2 = -1$,
- by introducing a non-commutative multiplication rule between them: $\imath_i \imath_j = -\imath_j \imath_i$ for $i \neq j$ and $\imath_i \imath_j = \imath_k$ for all cyclic permutations of (i, j, k).

Then to each $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$, as a three-dimensional vector, corresponds a new object $\underline{\mathbf{x}}$ (also called Vec x by some authors), which has the same formal expression but with $\imath_m$ following the new multiplication rule. Consequently, the identification

$$\mathbf{x} \in \mathbb{R}^3 \quad \mapsto \quad \underline{\mathbf{x}} = \sum_{m=1}^{3} x_m \imath_m \tag{1}$$

is an isomorphism of $\mathbb{R}^3$ onto the set of 'vector parts' {Vec} of more general objects called quaternions by Hamilton.

In fact, a quaternion x has four components, i.e. besides its imaginary vector part, there is also a scalar part $\mathrm{Sc}\, x = x_0\iota_0$, where ι_0 is the real (or non-imaginary) unit part (usually identified with the real unit $1 = \iota_0 \in \mathbb{R}$) and $x_0 \in \mathbb{R}$, such that

$$x = x_0\iota_0 + \sum_{m=1}^{3} x_m\iota_m = \mathrm{Sc}\, x + \mathrm{Vec}\, x = x_0\iota_0 + \underline{\mathbf{x}}, \ (x_0, x_1, x_2, x_3 \in \mathbb{R}). \tag{2}$$

The set of quaternions with real components should be called $\mathbb{H}(\mathbb{R})$,[1] but for simplicity, will be denoted by $\mathbb{H}$.

Following [23], we give some of their properties:

$$\text{conjugate operation:} \quad \overline{x} = x_0\iota_0 - \sum_{m=1}^{3} x_m\iota_m, \tag{3}$$

$$\text{square norm:} \quad |x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{4}$$

$$\text{inverse:} \quad x^{-1} = \frac{\overline{x}}{|x|^2} \quad \text{if and only if} \quad x\overline{x} \neq 0. \tag{5}$$

Finally, the ordered product of the two quaternions $y = y_0\iota_0 + \underline{\mathbf{y}}$ and $x = x_0\iota_0 + \underline{\mathbf{x}}$ is a quaternion $w = yx = (\mathrm{Sc}\, w + \mathrm{Vec}\, w)$, where

$$w_0 = \mathrm{Sc}\, w = y_0x_0 - (\mathbf{y}\cdot\mathbf{x}) \qquad \text{and} \qquad \underline{\mathbf{w}} = \mathrm{Vec}\, w = \underline{\mathbf{y}}x_0 + y_0\underline{\mathbf{x}} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{6}$$

In particular, i.e. the ordered product of $\underline{\mathbf{y}}$ by $\underline{\mathbf{x}}$ is

$$\underline{\mathbf{y}}\,\underline{\mathbf{x}} = -\mathbf{y}\cdot\mathbf{x} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{7}$$

2.1. Dirac operator D

For our purposes, we do not need the full machinery of quaternionic analyticity as developed by Fueter and others [22, 23]. Here we are only concerned by analytic properties useful for imaging processes in $\mathbb{R}^3$ modeled by the x-ray transform. They are essentially extracted from [24, 25]:

$$D = \sum_{m=1}^{3} \iota_m \frac{\partial}{\partial x_i}. \tag{8}$$

The quaternionic operator D has been given different names according to authors: Dirac operator for [25], three-dimensional Cauchy–Riemann operator for [17], Moisil–Teodorescu differential operator for [26], etc[2].

Inspection shows that it is related to the three-dimensional Laplace operator by $\triangle = -D^2$. The solutions of $Df(\mathbf{x}) = 0$, frequently called left monogenic $\mathbb{H}$-valued functions, satisfy many generalizations of classical theorems from complex analysis to higher dimensional context [24]. Given the elementary solution of the Laplace operator $\triangle E(\mathbf{x}) = -D^2E(\mathbf{x}) = \delta(\mathbf{x})$, as

$$E(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|} \tag{9}$$

[1] Quaternions with complex-valued components are called *biquaternions* and denoted by $\mathbb{H}(\mathbb{C})$.

[2] There is another right D_r, which can be defined with the ι_m on the right side of the partial derivatives of $\underline{\mathbf{f}}$. We shall not need it here.

the elementary solution of D can be worked out as [25]

$$\underline{\mathbf{K}}(\mathbf{x}) = \sum_{m=1}^{3} K_m(\mathbf{x})\imath_m = -\frac{\underline{\mathbf{x}}}{4\pi|\mathbf{x}|^3}, \qquad \underline{\mathbf{x}} \neq 0, \tag{10}$$

where

$$K_m(\mathbf{x}) = -\frac{x_m}{4\pi|\mathbf{x}|^3}, \qquad (m = 1, 2, 3). \tag{11}$$

Note that $\underline{\mathbf{K}}(\mathbf{x})$ is a $\mathbb{H}$-valued fundamental solution of D and therefore monogenic in $G\backslash\{0\}$.

2.2. *New operator D_a*

Here we define the operator $D_{\mathbf{a}}$ as

$$D_a := D + \underline{\mathbf{a}} \tag{12}$$

where $\underline{\mathbf{a}}$ is a pure quaternion vector part constant. Its fundamental solution is given by [16, 24]

$$\underline{\mathbf{K}}_a(\mathbf{x}) = -\mathrm{e}^{-\mathbf{a}\cdot\mathbf{x}}\frac{\underline{\mathbf{x}}}{4\pi|\mathbf{x}|^3}, \qquad \mathbf{x} \neq \mathbf{0}. \tag{13}$$

Consequently, there exists a three-dimensional Cauchy integral representation for continuous left monogenic $\mathbb{H}$-valued functions on $\overline{G}$ [24]:

$$(F_a f)(\mathbf{x}) := \int_\Gamma \underline{\mathbf{K}}_a(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\Gamma_{\mathbf{y}}, \qquad \mathbf{x} \in G\backslash\Gamma, \tag{14}$$

where $\underline{\alpha}(\mathbf{y}) = \sum_{m=1}^{3}\alpha_m(\mathbf{y})\imath_m$ is the quaternionic outward pointing unit vector at $\mathbf{y}$ on the boundary $\partial G = \Gamma$, and $\mathrm{d}\Gamma_{\mathbf{y}}$ is the Lebesgue measure on Γ. Moreover one has $D_a(F_a f)(\mathbf{x}) = 0$.

The operator D_a has an inverse, called the Teodorescu transform [27]. It is defined for all $f(\mathbf{x}) \in \mathcal{C}(G, \mathbb{H})$ by

$$(T_a f)(\mathbf{x}) := \int_G \underline{\mathbf{K}}_a(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \mathbf{x} \in G \subset \mathbb{R}^3. \tag{15}$$

Roughly speaking, D_a is a kind of directional derivative and T_a is just the integration, the inverse of this directional derivative.

Conversely, for any $f(\mathbf{x}) \in \mathcal{C}^1(G, \mathbb{H}) \cap \mathcal{C}(\overline{G}, \mathbb{H})$, it can be shown that it satisfies the so-called Borel–Pompeiu formula [25]:

$$(F_a f)(\mathbf{x}) + (T_a D_a) f(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^3\backslash\overline{G}. \end{cases} \tag{16}$$

Notation. Here we review our notation in this paper. Only 'bold' letters are used for vectors or vector functions in $\mathbb{R}^3$, such as $\mathbf{x}$ or $\mathbf{f}(\mathbf{x})$. The index 'zero' indicates the scalar part of a quaternion or quaternion function, e.g. x_0 or $a_0(\mathbf{x})$. Underlined bold letters are used for the vector part of the quaternions or quaternion functions, e.g. $\underline{\mathbf{x}}$ or $\underline{\mathbf{f}}(\mathbf{x})$. Operators with index '$a$' are operators with attenuation, e.g. T_a, X_a.

3. The exponential x-ray transform and its inverse

3.1. *Single-photon emission imaging*

In this imaging modality, one deals with a radiating object characterized by a non-uniform activity distribution density having a compact support in a closed set G with smooth boundary

$\partial G = \Gamma$. Data are collected outside G by a planar-collimated detector, which registers the received photon flux density at a detector site. However emitted photon flux from a site would travel through out the object and get attenuated along the traveling path before being detected. To account for this photon depletion due to scattering and absorption, one introduces a phenomenological constant linear attenuation coefficient a_0 to describe the traversed medium. Let the object activity density be $f_0(\mathbf{x})$, a real value, of compact support and integrable function in $\mathbb{R}^3$. The detected photon flux density in the direction of unit vector $\mathbf{n} \in \mathbb{S}^2$, which is the entering direction of the collimator hole of the detector, is given by the following integral, or exponential x-ray transform of $f_0(\mathbf{x})$:

$$(X_a f_0)(\mathbf{x}, \mathbf{n}) = \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-a_0 t}\, f_0(\mathbf{x} + t\mathbf{n}), \tag{17}$$

where $\mathbf{x} \in \Gamma$ such that $\mathbf{n}$ points to the inside of G. Integral (17) represents the sum of the activity density flux detected from a set of points on a line starting from $\mathbf{x}$ in the direction of an inward $\mathbf{n}$. The exponential factor describes the loss by medium absorption along this line as one moves nearer to the detection site.

We observe that $(X_a f_0)(\mathbf{x}, \mathbf{n})$ satisfies a very simple partial differential equation, namely[3]

$$(\mathbf{n} \cdot \nabla_{\mathbf{x}} + a_0)\, (X_a f_0)(\mathbf{x}, \mathbf{n}) = -f_0(\mathbf{x}). \tag{18}$$

This can be checked if we let the $(\mathbf{n} \cdot \nabla_{\mathbf{x}} + a_0)$ operator act under the integral sign. After adding and subtracting terms, the non-trivial integrand just turns into the differential of $\mathrm{e}^{-a_0 t}\, f_0(\mathbf{x})$ under the integral sign. As the radiating object is of finite extent, we have necessarily $f_0(|\mathbf{x}| \to \infty) = 0$ and equation (18) is straightforwardly obtained. This is in fact a simplified stationary photon transport equation with constant attenuation and without a source or sink term [30]. By construction, the solution of this partial differential equation is subjected to the following boundary condition. For a given direction $\mathbf{n}$, because of the support hypothesis and because of the prescription on the direction of integration, $(X_a f_0)(\mathbf{x}, \mathbf{n}) = 0$, whenever $\mathbf{x}$ is on the boundary $\Gamma = \partial G$ of G and $\mathbf{n}$ points outward of Γ.

As $\mathbf{n}$ does not depend on $\mathbf{x}$, we can rewrite the above equation (18) in the following form:

$$\mathbf{n} \cdot (\nabla + \mathbf{a})\, u_0(\mathbf{x}, \mathbf{n}) = -f_0(\mathbf{x}), \tag{19}$$

where $\mathbf{a} := a_0\mathbf{n}$ and $u_0 = (X_a f_0)$.

In two dimensions, in order to gain more insight into the nature of this equation, it is convenient to go to a complex version of the equation, by analytically continuing some parameter into the complex plane. But in three dimensions, it seems that the quaternion formalism is more appropriate. The question is how one can recast equation (19) in a quaternionic framework.

To this end, we consider the following equation:

$$\underline{\mathbf{n}} D_a u_0(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in G, \tag{20}$$

where $f(\mathbf{x}) = f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x})$ is an $\mathbb{H}$-valued 'source' function on its right-hand side and the unknown function has the form $u_0\mathbf{x})$.

The quaternionic product rule (6) can be applied to the left-hand side of equation (20) and yields

$$\underline{\mathbf{n}} D_a u_0 = -\mathbf{n} \cdot (\nabla_{\mathbf{x}} + \mathbf{a}) u_0 + \underline{\mathbf{n} \times \nabla_{\mathbf{x}}} u_0 = f_0 + \underline{\mathbf{f}}, \tag{21}$$

which leads to a set of two equations for the scalar field $u_0(\mathbf{x})$:

$$\begin{cases} (\mathbf{n} \cdot \nabla_{\mathbf{x}} + a_0) u_0(\mathbf{x}, \mathbf{n}) = -f_0(\mathbf{x}) \\ (\mathbf{n} \times \nabla_{\mathbf{x}}) u_0(\mathbf{x}, \mathbf{n}) = \mathbf{f}(\mathbf{x}). \end{cases} \tag{22}$$

[3] This is not the second-order ultra-hyperbolic partial differential equation of John [29].

Here the the quaternion formalism gives two equations. Similar to the use of complex analysis for the Laplace equation in two dimensions, here we get two equations (potential and equipotential equations) [28]. The first equation in the above equations is precisely the equation satisfied by the exponential x-ray transform (19). The second equation may be viewed as a derivation of u_0 in the perpendicular surface to $\mathbf{n}$. Imposing $\mathbf{f} = 0$ implies a constant value for u_0 on the surface perpendicular to $\mathbf{n}$.

By solving equation (21), we can obtain as by product the solution of equation (22), for given $f_0(\mathbf{x})$, first. Then, $\mathbf{f}(\mathbf{x})$ can be computed from the curl term and the gradient term of the solution.

Now, by multiplying equation (20) by $-\underline{\mathbf{n}}$ from left we have

$$D_a u_0(\mathbf{x}, \mathbf{n}) = -\underline{\mathbf{n}} f(\mathbf{x}) \tag{23}$$

where we used '$\underline{\mathbf{nn}} = -1$'.

From [16], we can give this solution $u_0(\mathbf{x}, \mathbf{n})$ in terms of the elementary solution (or Green's function) of D_a. Then by using an appropriate change of variable, $u_0(\mathbf{x}, \mathbf{n})$ may be represented as a combination of exponential x-ray transforms, which are in fact measured data.

Finally by applying the operator D_a on this form of solution, one can recover $f(\mathbf{x})$ in terms of the measurements, which are the exponential x-ray transforms, and thereby achieve the reconstruction of $f_0(\mathbf{x})$.

The Borel–Pompeiu formula (equation (16)) yields

$$T_a D_a u_0 = D_a T_a u_0 - F_a u_0 = u_0 - F_a u_0. \tag{24}$$

Now, by using equation (23) we have

$$u_0 = -T_a \mathbf{n} f + F_a u_0. \tag{25}$$

To get the explicit form of $u_0(\mathbf{x})$, we must compute the Teodorescu transform of f.

For $G \subset \mathbb{R}^3$ is a dense set in $\mathbb{R}^3$, the defining integral of the Teodorescu transform can be evaluated knowing its kernel (13):

$$\begin{aligned}(T_a \mathbf{n} f)(\mathbf{x}, \mathbf{n}) &= \int_G \mathrm{d}\mathbf{y}\, \underline{\mathbf{K}}_a(\mathbf{x}-\mathbf{y})\mathbf{n} f(\mathbf{y}) = \frac{1}{4\pi}\int_G \mathrm{d}\mathbf{y}\, \mathrm{e}^{-(\mathbf{x}-\mathbf{y})\cdot\mathbf{a}}\, \frac{\underline{\mathbf{x}} - \underline{\mathbf{y}}}{|\mathbf{x}-\mathbf{y}|^3}\mathbf{n} f(\mathbf{y}) \\ &= \frac{1}{4\pi}\sum_{i,j=1}^{3}\sum_{\beta=0}^{3}\int_G \mathrm{d}\mathbf{y}\, \mathrm{e}^{-(\mathbf{x}-\mathbf{y})\cdot\mathbf{a}}\, \frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3}\, \mathbf{n}_j f_\beta(\mathbf{y})\ \iota_i\, \iota_j\, \iota_\beta.\end{aligned} \tag{26}$$

Now let us make the following change of variables, $\mathbf{y} = (\mathbf{x} + \mathbf{m}\,t)$ in $\mathbb{R}^3$, where $t \in \mathbb{R}^+$ and $\mathbf{m}$ is a unit vector. Consequently, the volume element $\mathrm{d}\mathbf{y}$ in spherical coordinates becomes $\mathrm{d}\mathbf{y} = t^2\,\mathrm{d}t\,\mathrm{d}\Omega_{\mathbf{m}}$, where $\mathrm{d}\Omega_{\mathbf{m}}$ is the area element of the unit sphere $\Omega_{\mathbf{m}}$ in $\mathbb{R}^3$.

Thus, we have

$$\begin{aligned}(T_a \mathbf{n} f)(\mathbf{x}, \mathbf{n}) &= \frac{1}{4\pi}\sum_{i,j=1}^{3}\sum_{\beta=0}^{3}\int_{\Omega_{\mathbf{m}}} \mathrm{d}\Omega_{\mathbf{m}} \int_{\mathbb{R}^+} t^2\,\mathrm{d}t\, \mathrm{e}^{-a_0 t}\, \frac{m_i t}{|\mathbf{m}t|^3} n_j f_\beta(\mathbf{x} + \mathbf{m}t)\, \iota_i \iota_j\, \iota_\beta \\ &= \frac{1}{4\pi}\sum_{i,j=1}^{3}\sum_{\beta=0}^{3}\int_{\Omega_{\mathbf{m}}} \mathrm{d}\Omega_{\mathbf{m}}\ m_i n_j \left(\int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-a_0 t}\, f_\beta(\mathbf{x} + \mathbf{m}t)\right) \iota_i \iota_j\, \iota_\beta.\end{aligned} \tag{27}$$

In equation (27), the x-ray transform of the component f_β of f arises in a natural way. Thus for f_β, we may introduce its exponential x-ray transform

$$X_a f_\beta(\mathbf{x}, \mathbf{m}) := \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-a_0 t}\, f_\beta(\mathbf{x} + \mathbf{m}t), \qquad (\beta = 0, \ldots, 3). \tag{28}$$

Then, by summing up over the quaternion imaginary units, we get the exponential x-ray transform of the $\mathbb{H}$-valued function f in $\mathbb{R}^3$, as

$$X_a f(\mathbf{x}, \mathbf{m}) := \sum_{\beta=0}^{3} X_a f_\beta(\mathbf{x}, \mathbf{m})\imath_\beta = \int_{\mathbb{R}^+} \mathrm{d}t\, \mathrm{e}^{-a_0 t} f(\mathbf{x} + \mathbf{m}t). \tag{29}$$

Consequently, the Teorodescu transform of $\underline{\mathbf{n}} f$ appears as

$$(T_a \underline{\mathbf{n}} f)(\mathbf{x}) = \frac{1}{4\pi} \int_{\Omega_\mathbf{m}} \mathrm{d}\Omega_\mathbf{m}\, \underline{\mathbf{mn}}\, (X_a f)(\mathbf{x}, \mathbf{m}). \tag{30}$$

3.2. *The reconstruction formula for* f

Having expressed the solution of equation (20) in terms of x-ray data, on may let D_a act on equation (25), to obtain $-\underline{\mathbf{n}} f$, since $-\underline{\mathbf{n}} f = D_a u_0 = D_a(-T_a \underline{\mathbf{n}} f + F_a u_0)$. Thus, we have

$$-\underline{\mathbf{n}} f(\mathbf{x}) = D_a u(\mathbf{x}) = -\frac{1}{4\pi} D_a \int_{\Omega_\mathbf{m}} \mathrm{d}\Omega_\mathbf{m}\, \underline{\mathbf{mn}}\, (X_a f)(\mathbf{x}, \mathbf{m}). \tag{31}$$

The quaternion reconstruction formula for f is as follows:

$$\begin{aligned} f(\mathbf{x}) = -\underline{\mathbf{n}} D_a u(\mathbf{x}) &= -\frac{1}{4\pi} \underline{\mathbf{n}} D_a \int_{\Omega_\mathbf{m}} \mathrm{d}\Omega_\mathbf{m}\, \underline{\mathbf{mn}}\, (X_a f)(\mathbf{x}, \mathbf{m}) \\ &= -\frac{1}{4\pi} \int_{\Omega_\mathbf{m}} \mathrm{d}\Omega_\mathbf{m}\, \underline{\mathbf{n}} D_a \underline{\mathbf{mn}}\, (X_a f)(\mathbf{x}, \mathbf{m}). \end{aligned} \tag{32}$$

3.2.1. *Vector and scalar parts of the reconstructed* f.

To obtain the vector and scalar parts of f from equation (32), we have to compute the action of $\underline{\mathbf{n}} D_a \underline{\mathbf{mn}}(X_a f)$ on the integrand of equation (32) as follows:

$$\begin{aligned} \underline{\mathbf{n}} D_a &= -\mathbf{n} \cdot (\nabla_\mathbf{x} + \mathbf{a}) + \underline{\mathbf{n} \times (\nabla_\mathbf{x} + \mathbf{a})} = -(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0) + \underline{\mathbf{n} \times \nabla_\mathbf{x}} \\ \underline{\mathbf{mn}} &= -\mathbf{m} \cdot \mathbf{n} + \underline{\mathbf{m} \times \mathbf{n}}, \end{aligned} \tag{33}$$

where $\underline{\mathbf{a}} = \underline{\mathbf{n}} a_0$. Then, we have

$$\begin{aligned} \underline{\mathbf{n}} D_a \underline{\mathbf{mn}} = &+(\mathbf{m} \cdot \mathbf{n})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0) - (\mathbf{n} \times \nabla_\mathbf{x}) \cdot (\mathbf{m} \times \mathbf{n}) \\ &- (\mathbf{m} \cdot \mathbf{n})(\underline{\mathbf{n} \times \nabla_\mathbf{x}}) - (\underline{\mathbf{m} \times \mathbf{n}})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0), \end{aligned} \tag{34}$$

where $\underline{(\mathbf{m} \times \mathbf{n}) \times (\mathbf{n} \times \nabla_\mathbf{x})} = 0$. Now, the scalar and vector parts of $\underline{\mathbf{n}} D_a \underline{\mathbf{mn}}(X_a f)$ are obtained, respectively, as follows:

$$\begin{aligned} \mathrm{Sc}[\underline{\mathbf{n}} D_a \underline{\mathbf{mn}}(X_a f)] = &+[(\mathbf{m} \cdot \mathbf{n})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0) - (\mathbf{n} \times \nabla_\mathbf{x}) \cdot (\mathbf{m} \times \mathbf{n})](X_a f_0) \\ &+ [(\mathbf{m} \cdot \mathbf{n})(\mathbf{n} \times \nabla_\mathbf{x}) + (\mathbf{m} \times \mathbf{n})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0)] \cdot (X_a \mathbf{f}), \end{aligned} \tag{35}$$

and

$$\begin{aligned} \mathrm{Vec}[\underline{\mathbf{n}} D_a \underline{\mathbf{mn}}(X_a f)] = &+[(\mathbf{m} \cdot \mathbf{n})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0) - (\mathbf{n} \times \nabla_\mathbf{x}) \cdot (\mathbf{m} \times \mathbf{n})](X_a \underline{\mathbf{f}}) \\ &+ [(\mathbf{m} \cdot \mathbf{n})(\underline{\mathbf{n} \times \nabla_\mathbf{x}}) + (\underline{\mathbf{m} \times \mathbf{n}})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0)](X_a f_0) \\ &+ \underline{[(\mathbf{m} \cdot \mathbf{n})(\mathbf{n} \times \nabla_\mathbf{x}) + (\mathbf{m} \times \mathbf{n})(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0)] \times (X_a \mathbf{f})}. \end{aligned} \tag{36}$$

Now, we try to simplify the above equations by considering the following relations:

$$\begin{aligned} (\mathbf{n} \cdot \nabla_\mathbf{x} + a_0)(X_a \mathbf{f}) &= (\mathbf{n} \cdot \nabla_\mathbf{x} + a_0)[X_a(\mathbf{n} \times \nabla_\mathbf{x}) u_0] \\ &= (\mathbf{n} \times \nabla_\mathbf{x})[X_a(\mathbf{n} \cdot \nabla_\mathbf{x} + a_0) u_0] = -(\mathbf{n} \times \nabla_\mathbf{x})(X_a f_0) \end{aligned} \tag{37}$$

and

$$\begin{aligned}\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})\times(X_a\mathbf{f})} &= \underline{(\mathbf{n}\times\nabla_{\mathbf{x}})}\times[X_a\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})}u_0]\\ &= [X_a\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})\times(\mathbf{n}\times\nabla_{\mathbf{x}})u_0}] = 0,\end{aligned} \tag{38}$$

where we have used the relations in (22) and note that $u_0 = X_a f_0$. Thus, we use the first equation in (37) for the last term on the right-hand side of equations (35) and (36).

Also, the first term in the third line of equation (36) is zero from the second equation of (38). Finally, equations (35) and (36) are written as

$$\begin{aligned}\mathrm{Sc}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)] = &+[(\mathbf{m}\cdot\mathbf{n})(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0) - 2(\mathbf{n}\times\nabla_{\mathbf{x}})\cdot(\mathbf{m}\times\mathbf{n})](X_a f_0)\\ &+[(\mathbf{m}\cdot\mathbf{n})(\mathbf{n}\times\nabla_{\mathbf{x}})]\cdot(X_a\mathbf{f}),\end{aligned} \tag{39}$$

and

$$\begin{aligned}\mathrm{Vec}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)] = &+[(\mathbf{m}\cdot\mathbf{n})(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0) - (\mathbf{n}\times\nabla_{\mathbf{x}})\cdot(\mathbf{m}\times\mathbf{n})](X_a\underline{\mathbf{f}})\\ &+[(\mathbf{m}\cdot\mathbf{n})\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})} + \underline{(\mathbf{m}\times\mathbf{n})}(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)](X_a f_0)\\ &+\underline{(\mathbf{m}\times\mathbf{n})\times(\mathbf{n}\times\nabla_{\mathbf{x}})}(X_a f_0).\end{aligned} \tag{40}$$

The first and third terms on the right-hand side of the above equation cancel each other because

$$\begin{aligned}(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)(X_a\underline{\mathbf{f}}) &= (\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)[X_a\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})}u_0]\\ &= \underline{(\mathbf{n}\times\nabla_{\mathbf{x}})}[X_a(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)u_0] = -\underline{(\mathbf{n}\times\nabla_{\mathbf{x}})}X_a f_0.\end{aligned} \tag{41}$$

Now, (40) can be rewritten as

$$\begin{aligned}\mathrm{Vec}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)] = &-(\mathbf{n}\times\nabla_{\mathbf{x}})\cdot(\mathbf{m}\times\mathbf{n})(X_a\underline{\mathbf{f}}) + \underline{(\mathbf{m}\times\mathbf{n})}(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)(X_a f_0)\\ &+\underline{(\mathbf{m}\times\mathbf{n})\times(\mathbf{n}\times\nabla_{\mathbf{x}})}(X_a f_0).\end{aligned} \tag{42}$$

Thus, the scalar and vector parts of (32) are obtained as

$$f_0(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathbf{m}}} \mathrm{d}\Omega_{\mathbf{m}}\, \mathrm{Sc}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)](\mathbf{x},\mathbf{m}), \tag{43}$$

and

$$\underline{\mathbf{f}}(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathbf{m}}} \mathrm{d}\Omega_{\mathbf{m}}\, \mathrm{Vec}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)](\mathbf{x},\mathbf{m}), \tag{44}$$

where $\mathrm{Sc}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)]$ and $\mathrm{Vec}[\underline{\mathbf{n}}D_a\underline{\mathbf{mn}}(X_a f)]$ are given by equations (39) and (42), respectively. These are the reconstruction formulas for the scalar and vector parts of f.

3.2.2. Special case of '$\underline{\mathbf{f}}=0$*'.* From (22), the formula $\underline{\mathbf{f}} = (\underline{\mathbf{n}\times\nabla_{\mathbf{x}}})u_0 = 0$ means that the areolar derivative of the u_0 is equal to zero. In other words, the derivative of u_0 on the plane perpendicular to $\underline{\mathbf{n}}$ is equal to zero, or u_0 is constant on the plane perpendicular to $\underline{\mathbf{n}}$.

From the condition $\underline{\mathbf{f}} = 0$, equation (20) becomes

$$\underline{\mathbf{n}}D_a u_0(\mathbf{x}) = f_0(\mathbf{x}), \qquad \mathbf{x}\in G, \tag{45}$$

from which we can easily see that the above equation is a simplified transport equation (18) in the quaternion formalism. Thus, by setting $\underline{\mathbf{f}}$ equal to zero, formula (43) for f_0 gives a reconstruction formula of the transport equation. In this section, we first give a proof of this fact and then we obtain the explicit form of f_0 with $\underline{\mathbf{f}} = 0$. Finally, we compare our result in the special case $a_0 = 0$ with known results.

The condition $\underline{\mathbf{f}} = 0$ requires that equation (42) is equal to zero, which means that $\underline{\mathbf{n}\times\mathbf{m}} = 0$ and consequently $\mathbf{n}\cdot\mathbf{m} = 1$. In this case, from (39) the scalar part (44) has the following simple form:

$$f_0(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}} \mathrm{d}\Omega_{\mathbf{n}}(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0)(X_a f_0)(\mathbf{x},\mathbf{n}). \tag{46}$$

As earlier in this paper we have introduced $(X_a f_0)(\mathbf{x}, \mathbf{n})$, this is a monogenic (analytic) function on $\Omega_\mathbf{n}$; thus, $[a_0(X_a f_0)(\mathbf{x}, \mathbf{n})]_{\Omega_\mathbf{n}} \to 0$ as $t \to 0$. Finally, the reconstruction formula for f_0 is obtained as

$$f_0(\mathbf{x}) = \frac{1}{4\pi} \int_{\Omega_\mathbf{n}} \mathrm{d}\Omega_\mathbf{n} (\mathbf{n} \cdot \nabla_\mathbf{x})(X_a f_0)(\mathbf{x}, \mathbf{n}). \tag{47}$$

3.3. Special case with $a_0 = 0$

When attenuation is neglected, we have $a_0 = 0$. Equation (47) becomes

$$f_0(\mathbf{x}) = \frac{1}{4\pi} \int_{\Omega_\mathbf{n}} \mathrm{d}\Omega_\mathbf{n} (\mathbf{n} \cdot \nabla_\mathbf{x})(X f_0)(\mathbf{x}, \mathbf{n}), \tag{48}$$

where $X f_0 = \int_0^\infty \mathrm{d}t\, f_0(\mathbf{x} + \mathbf{n}t)$ is the x-ray transform without attenuation. Here the result is the one obtained by [20]. A comparison of the above formula with other results given by [20] is presented in the appendix.

4. Conclusion and perspectives

In this paper, by using quaternion analysis we have established successfully an inverse formula for the exponential x-ray transform in three dimensions. The result without attenuation is recovered although it has appeared under a different form. The generalization of our results to the non-uniform attenuation map in the case of transmission imaging is very interesting because it is much closer to reality and has up to now evaded complete resolution except in two dimensions, where an analytic solution has been given in [31, 32]. We shall go into this topic in a future work.

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Appendix. Comparison with known results [20]

In [5] the inverse formula of the x-ray transform in three dimensions is given by

$$f_0(\mathbf{x}) = -\frac{1}{2\pi^2} \triangle_\mathbf{x} \mathfrak{R}^1 \int_{\Omega_\mathbf{n}} (X f_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n} = -\frac{1}{2\pi^2} \triangle_\mathbf{x} \mathfrak{R}^1 \int_{\Omega_\mathbf{n}} u_0(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}, \tag{A.1}$$

where

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2} \int \frac{1}{|\mathbf{x} - \mathbf{y}|^2} f_0(\mathbf{y})\, \mathrm{d}\mathbf{y}. \tag{A.2}$$

Setting $\mathbf{y} = \mathbf{x} + \mathbf{n}t$ in the above equation, we find

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} \int_{\mathbb{R}^+} f_0(\mathbf{x} + \mathbf{n}t)\, \mathrm{d}t\, \mathrm{d}\Omega_\mathbf{n} = \frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} (X f_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}. \tag{A.3}$$

In equation (A.1) we may define $\frac{1}{4\pi} \int_{\Omega_\mathbf{n}} u_0(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n} = \langle u_0 \rangle_\mathbf{n}$ as the average of u_0 over a unit ball. Thus, using the above relation, equation (A.3) can be written as

$$f_0(\mathbf{x}) = -\triangle_\mathbf{x} \mathfrak{R}^1 \langle u_0 \rangle_{\mathbf{n}'} = -\frac{2}{\pi} \triangle_\mathbf{x} (X \langle u_0 \rangle_\mathbf{n})(\mathbf{x}) = -\frac{2}{\pi} \int_{\Omega_\mathbf{n}} \triangle_\mathbf{x} (X u_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}. \tag{A.4}$$

Here, we obtain another form for $\triangle_{\mathbf{x}}(X_a u_0)$. From equation (22), which expresses $\mathbf{f}$ as $\mathbf{f} = \mathbf{n} \times \nabla_{\mathbf{x}} u_0 = -\nabla_{\mathbf{x}} \times \mathbf{n} u_0$, we deduce that $\nabla_{\mathbf{x}} \cdot \mathbf{f} = 0$. Thus

$$\nabla_{\mathbf{x}} \cdot (\nabla_{\mathbf{x}} \times \mathbf{n} u_0) = \mathbf{n} \triangle_{\mathbf{x}} u_0 - \nabla_{\mathbf{x}} (\mathbf{n} \cdot \nabla_{\mathbf{x}}) u_0 = \mathbf{n} \triangle_{\mathbf{x}} u_0 + \nabla_{\mathbf{x}} f_0 = 0, \tag{A.5}$$

which yields

$$\triangle_{\mathbf{x}} u_0 = -(\mathbf{n} \cdot \nabla_{\mathbf{x}}) f_0. \tag{A.6}$$

Substitution of this expression in equation (A.4) gives an alternative form of the reconstructed f_0

$$f_0(\mathbf{x}) = \frac{2}{\pi} \int_{\Omega_{\mathbf{n}}} (\mathbf{n} \cdot \nabla_{\mathbf{x}})(X f_0)(\mathbf{x}, \mathbf{n}) \, \mathrm{d}\Omega_{\mathbf{n}}, \tag{A.7}$$

which, up to a normalization factor, has the same form as our equation (48).

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