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2009 J. Phys. A: Math. Theor. 42 415205

(http://iopscience.iop.org/1751-8121/42/41/415205)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 415205 (14pp) doi:10.1088/1751-8113/42/41/415205

A quaternionic approach to x-ray transform inversion in $\mathbb{R}^3$

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Received 7 July 2009, in final form 25 August 2009
Published 29 September 2009
Online at stacks.iop.org/JPhysA/42/415205

Abstract
A new derivation of the inverse of the x-ray transform is presented based on quaternion analysis. As pointed out by practitioners, a direct inversion formula offers more efficient reconstruction algorithms than tomographic inversion. It is shown that the new inverse formula is equivalent to the existing one. The advantage of this approach is that it paves the way for a potential inversion of the x-ray transform with a non-uniform attenuation map in three dimensions, which models single photon emission imaging in nuclear medicine.

PACS numbers: 02.30.Jr, 02.30.Uu, 02.30.Zz

1. Introduction

The x-ray (or divergent cone-beam) transform in $\mathbb{R}^3$ is an integral transform on the space of integrable functions in $\mathbb{R}^3$, which serves as a mathematical basis for a three-dimensional imaging process and may be viewed as a generalization of conventional radiographic imaging. It plays an analogous role in electron microscopy [1, 2], crystallography [3], biochemistry [4], molecular biology [5], aerodynamics [6], radio astronomy [7], radiography [8] and in industrial non-destructive testing [9]. As a pure mathematical tool, it has helped to bring advances to partial differential equations theory [10] and most importantly to integral geometry as well as to group representation theory [11].

To 'see' the inside of an object, it is necessary to probe its hidden three-dimensional structure by a physical agent. One way to achieve this goal is to use an external source of x-ray to illuminate the studied object and measure the transmitted x-ray intensity along all possible directions in space. Given a calibrated x-ray source, this measurement gives the integrated attenuation map $f(\mathbf{x})$ of traversing ionizing radiation along straight line paths through this object. The set of such line integrals represents a mapping: $f \mapsto (Xf)(\mathbf{x}, \mathbf{n})$ called the x-ray transform of f. Here, $\mathbf{x} \in \mathbb{R}^3$ is the x-ray source position and $\mathbf{n}$ is a unit vector of the direction of the straight line, at the end of which a measurement is performed. Thus, $(Xf)(\mathbf{x}, \mathbf{n})$ depends on five apparent variables but only four variables (two for $\mathbf{n}$ and two for the position of the line in a plane containing the coordinate system origin and orthogonal to $\mathbf{n}$) are independent. Reconstructing f from the data $(Xf)(\mathbf{x}, \mathbf{n})$ is the main problem to be solved.

The solution, without restriction on the set of source points $\mathbf{x}$, has been worked out mathematically in [12–15]. The reconstruction formula involves the average of the x-ray data on the unit sphere of $\mathbb{R}^3$, centered at a site in space. The case of point sources lying on a space curve is given by [16–19]. Finally, among the large amount of indirect inversion procedures, the most well known for efficiency and appeal are those of B D Smith, who developed a technique that converts divergent beam data into parallel beam data and used its known inversion procedure [20], and of P Grangeat, who made a conversion of x-ray data into three-dimensional Radon data before using Radon inversion [21].

In this work, we show that the use of quaternion analysis leads to a new inversion formula for the x-ray transform in $\mathbb{R}^3$. Quaternions are higher dimensional generalization of complex numbers. Although not widely used, they provide elegant compact local formulation for electromagnetism [22], solid mechanics and some other fields in engineering [26]. Recently, quaternions have been used in integral transforms, for example, in geophysical processes [27] or in signal processing [28]. A general approach to Radon and x-ray transforms in higher dimensional Clifford analysis has been given in [29], but the question of their inversion has not been treated.

Our motivation stems from the impressive success of the use of complex analysis for finding the inverse of the attenuated x-ray (or Radon) transform in $\mathbb{R}^2$, as shown in several works [30–32]. It is natural to raise the question whether such remarkable results can also be achieved in $\mathbb{R}^3$ by going to some higher dimensional 'complex' space . It turns out that quaternions are the appropriate objects to be considered. However, although the starting point is a simple partial differential equation satisfied by the x-ray transform, as in $\mathbb{R}^2$, no extension of this equation in the complex domain of its parameters, as done in [30], is performed, and only the existing quaternionic structure is directly used.

In the following section, we introduce some useful notions on the algebra of real quaternions $\mathbb{H}$ and collect the main results of quaternion analysis required for our problem. Section 3 describes the derivation of the inversion formula giving the reconstructed function in terms of the x-ray data. Finally, we give an alternate derivation of this new result in section 4. This paper ends with a conclusion and sketches the perspectives for inverting a more general case of x-ray transforms in $\mathbb{R}^3$.

2. Quaternions

Quaternions were invented by H W Hamilton in the first half of the 19th century, when he looked for a three-dimensional generalization of complex numbers [23]. But, this theory did not generate widespread interest until nearly a century after it was discovered. Subsequently, R Fueter introduced the notion of 'regular' quaternionic functions as functions satisfying an analog of the Cauchy–Riemann equations. With this new concept, he is led to Cauchy's theorem, Cauchy'integral formula and Laurent expansion for analytic functions [24]. A comprehensive review of quaternions can be found in [25].

2.1. Algebra

As known in elementary mathematics, the two-dimensional Euclidean vector space $\mathbb{R}^2$, may be equipped with a new algebraic structure with the introduction of an imaginary number i, with

$i^2 = -1$, affected the Oy-axis of a Cartesian reference system. To each point $\mathbf{x} = (x, y) \in \mathbb{R}^2$ corresponds a complex number $z = (x + \mathrm{i}y) \in \mathbb{C}$, i.e., $\mathbb{R}^2$ is isomorphically identified to $\mathbb{C}$. As we shall see in this section, a similar idea was introduced by Hamilton in $\mathbb{R}^3$.

Let $\mathbf{x} = (x_1, x_2, x_3)$ be an element of $\mathbb{R}^3$, expressed in an orthonormal basis formed by three unit vectors $\imath_1, \imath_2$ and $\imath_3$ by $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$. The conventional vector space structure is defined by a scalar (inner) product rule for the basis unit vectors, i.e. $(\imath_n \cdot \imath_m) = \delta_{mn}$ and by a vector (cross) product, i.e. $\imath_1 \times \imath_2 = \imath_3$ with its cyclic permutations and the non-commutativity $\imath_m \times \imath_n = -\imath_m \times \imath_n$.

To this structure, one can add a new one

- by promoting the unit vectors to be imaginary units, i.e. ${\imath_1}^2 = {\imath_2}^2 = {\imath_3}^2 = -1$,
- by introducing a non-commutative multiplication rule between them: $\imath_i \imath_j = -\imath_j \imath_i$ for $i \neq j$ and $\imath_i \imath_j = \imath_k$ for all cyclic permutations of (i, j, k).

Then to each $\mathbf{x} = \sum_{m=1}^{3} x_m \imath_m$, as a three-dimensional vector, corresponds a new object $\underline{\mathbf{x}}$ (also called $\mathrm{Vec}\, x$ by some authors), which has the same formal expression but with $\imath_m$ following the new multiplication rule. Consequently, the identification

$$\mathbf{x} \in \mathbb{R}^3 \quad \mapsto \quad \underline{\mathbf{x}} = \sum_{m=1}^{3} x_m \imath_m \tag{1}$$

is an isomorphism of $\mathbb{R}^3$ onto the set of 'vector parts' {Vec}. In fact, $\mathrm{Vec}\, x$ is the imaginary part of more general objects, called quaternions by H W Hamilton.

An arbitrary quaternion x has four components, i.e. besides its imaginary (or $\mathrm{Vec}\, x$ vector part), there is also a scalar part $\mathrm{Sc}\, x = x_0 \imath_0$, where $\imath_0$ is the real (or non-imaginary) unit part (usually identified with the real unit $1 = \imath_0 \in \mathbb{R}$) and $x_0 \in \mathbb{R}$, such that

$$x = x_0 \imath_0 + \sum_{m=1}^{3} x_m \imath_m = \mathrm{Sc}\, x + \mathrm{Vec}\, x = x_0 \imath_0 + \underline{\mathbf{x}}, \qquad (x_0, x_1, x_2, x_3 \in \mathbb{R}). \tag{2}$$

The set of quaternions with real components should be called $\mathbb{H}(\mathbb{R})$[3], but for simplicity, will be denoted by $\mathbb{H}$.

Following [25], we give some of their properties:

$$\text{conjugate operation:} \quad \overline{x} = x_0 \imath_0 - \sum_{m=1}^{3} x_m \imath_m, \tag{3}$$

$$\text{square norm:} \quad |x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{4}$$

$$\text{inverse:} \quad x^{-1} = \frac{\overline{x}}{|x|^2} \text{ if and only if } x\overline{x} \neq 0. \tag{5}$$

The other useful property of a quaternion is that, in contrast to complex numbers, each of its coordinates x_0, x_1, x_2 and x_3 can themselves be written as quaternionic polynomials, i.e.

$$\begin{cases} x_0 = \dfrac{1}{4}(\imath_0 x \imath_0 - \imath_1 x \imath_1 - \imath_2 x \imath_2 - \imath_3 x \imath_3), \\ x_1 = \dfrac{-\imath_1}{4}(\imath_0 x \imath_0 - \imath_1 x \imath_1 + \imath_2 x \imath_2 + \imath_3 x \imath_3) \\ x_2 = \dfrac{-\imath_2}{4}(\imath_0 x \imath_0 + \imath_1 x \imath_1 - \imath_2 x \imath_2 + \imath_3 x \imath_3) \\ x_3 = \dfrac{-\imath_3}{4}(\imath_0 x \imath_0 + \imath_1 x \imath_1 + \imath_2 x \imath_2 - \imath_3 x \imath_3), \end{cases} \tag{6}$$

[3] Quaternions with complex-valued components are called *biquaternions* and denoted by $\mathbb{H}(\mathbb{C})$.

so every real polynomial in x_0, x_1, x_2 and x_3 can be expressed as a quaternionic polynomial in x. Thus, the theory of quaternionic power series is the same as the theory of the real analytic function on $\mathbb{R}^4$ [25].

Finally, the ordered product of two quaternions $y = y_0\imath_0 + \underline{\mathbf{y}}$ and $x = x_0\imath_0 + \underline{\mathbf{x}}$ is a quaternion $w = yx = (\mathrm{Sc}\, w + \mathrm{Vec}\, w)$, where

$$\mathrm{Sc}\, w = y_0x_0 - (\mathbf{y}\cdot\mathbf{x}) \qquad \text{and} \qquad \mathrm{Vec}\, w = \underline{\mathbf{y}}x_0 + y_0\underline{\mathbf{x}} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{7}$$

In particular, i.e., the ordered product of two 'pure imaginary' quaternions $\underline{\mathbf{y}}$ by $\underline{\mathbf{x}}$ is a full quaternion

$$\underline{\mathbf{y}}\underline{\mathbf{x}} = -\mathbf{y}\cdot\mathbf{x} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{8}$$

2.2. Analysis

Let $G \subset \mathbb{R}^3$ be an open set with a smooth boundary $\partial G = \Gamma$. Let f be an $\mathbb{H}$-valued function in $\mathbb{R}^4$. Its differential is the linear mapping $\mathrm{d}f : \mathbb{H} \to \mathbb{H}$, such that, by identifying the tangent space at each point of $\mathbb{H}$ with $\mathbb{H}$ itself [25], it has the expression

$$\mathrm{d}f = \frac{\partial f}{\partial x_0}\,\mathrm{d}x_0 + \frac{\partial f}{\partial x_1}\,\mathrm{d}x_1 + \frac{\partial f}{\partial x_2}\,\mathrm{d}x_2 + \frac{\partial f}{\partial x_3}\,\mathrm{d}x_3. \tag{9}$$

In particular, the differential of the identity function is

$$\mathrm{d}x = \imath_0\,\mathrm{d}x_0 + \imath_1\,\mathrm{d}x_1 + \imath_2\,\mathrm{d}x_2 + \imath_3\,\mathrm{d}x_3, \tag{10}$$

the exterior product with itself is

$$\mathrm{d}x \wedge \mathrm{d}x = \imath_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \imath_2\,\mathrm{d}x_3 \wedge \mathrm{d}x_1 + \imath_3\,\mathrm{d}x_1 \wedge \mathrm{d}x_2, \tag{11}$$

where $\wedge$ is the exterior product. The exterior product of different fundamental differential 1-form is called 2-form [34]. The 3-form $\mathbb{D}x$ is defined, as in [25], by

$$\mathbb{D}x = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \imath_1\,\mathrm{d}x_0 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \imath_2\,\mathrm{d}x_0 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 - \imath_3\,\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2. \tag{12}$$

Finally, the essentially unique 4-form is

$$\mathrm{d}V_{(4)} = \mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3. \tag{13}$$

In the special case of a $\mathrm{Vec}\, x$ (a quaternion $\underline{\mathbf{x}}$ with $(x_0 = 0)$), the unique constant 3-form $\mathbb{D}\underline{\mathbf{x}}$ is, using (12),

$$\mathrm{d}V_{(3)} = \mathbb{D}\underline{\mathbf{x}} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3. \tag{14}$$

An $\mathbb{H}$-valued function defined on $G \subset \mathbb{R}^3$ has the expression

$$f(\mathbf{x}) = \imath_0 f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x}) = \sum_{m=0}^{3} f_m(\mathbf{x})\imath_m \qquad \text{where} \quad f_m \in \mathbb{R}, \quad (\mathbf{x} \in G). \tag{15}$$

Continuity, differentiability, integrability and growth at infinity of f are properties contained in its components f_m, $(m = 0, 1, 2, 3)$; see e.g. [33, 34].

2.3. Analytic properties

For our purposes, we do not need the full machinery of quaternionic analyticity as developed by Fueter and others [24, 25]. Here we are only concerned with analytic properties useful for imaging processes in $\mathbb{R}^3$ modeled by the x-ray transform. They are essentially extracted from [33, 34].

A classical $\mathcal{C}^1$-vector field $\mathbf{w}(\mathbf{x})$ in $\mathbb{R}^3$, which fulfills

$$\begin{aligned} \text{div } \mathbf{w}(\mathbf{x}) = 0 \quad &\Leftrightarrow \quad \nabla \cdot \mathbf{w}(\mathbf{x}) = 0, \\ \text{rot } \mathbf{w}(\mathbf{x}) = 0 \quad &\Leftrightarrow \quad \nabla \times \mathbf{w}(\mathbf{x}) = 0, \end{aligned} \tag{16}$$

can be conveniently represented by a three-dimensional Cauchy-type equation,

$$D\underline{\mathbf{w}}(\mathbf{x}) = \left(\sum_{m=1}^{3} \imath_m \frac{\partial}{\partial x_i}\right) \underline{\mathbf{w}}(\mathbf{x}) = -(\nabla \cdot \mathbf{w}(\mathbf{x})) + \underline{\nabla \times \mathbf{w}}(\mathbf{x}) = 0, \tag{17}$$

where, according to our isomorphic embedding, $\underline{\mathbf{w}}(\mathbf{x})$ is the vector part of a quaternionic field on $\mathbb{R}^3$ corresponding to $\mathbf{w}(\mathbf{x})$. Equations (16), which imply that the vector field $\mathbf{w}(\mathbf{x})$ derives from a harmonic scalar potential function, result from a simple application of formula (8). In two dimensions, this fact can be understood nicely in the context of the complex-valued analytic function in $\mathbb{C}$, which are characterized by the Cauchy–Riemann equations. Here the quaternionic operator D generalizes the two-dimensional $\overline{\partial} = \partial/\partial\overline{z}$ operator of complex analysis, has been given different names according to authors: the Dirac operator for [29, 34], the three-dimensional Cauchy–Riemann operator for [26], the Moisil–Teodorescu differential operator for [37], etc[4].

In the more general setting of quaternion analysis, D is the vector part of the quaternionic Cauchy–Riemann operator $\mathcal{D}$, defined in [33] as

$$\mathcal{D} = \sum_{\beta=0}^{3} \imath_\beta \frac{\partial}{\partial x_\beta} = \imath_0 \frac{\partial}{\partial x_0} + D. \tag{18}$$

But here, we shall restrict ourselves to D. Inspection shows that it is related to the three-dimensional Laplace operator by $\triangle = -D^2$. The solutions of $Df(\mathbf{x}) = 0$, called frequently left monogenic $\mathbb{H}$-valued functions, satisfy many generalizations of classical theorems from complex analysis to higher dimensional context [33]. Since the elementary solution of the Laplace operator is known, the elementary solution of D can be worked out as, see e.g. [34],

$$K_i(\mathbf{x}) = -\frac{x_i}{4\pi|\mathbf{x}|^3}, \qquad (i = 1, 2, 3) \tag{19}$$

$$\underline{\mathbf{K}}(\mathbf{x}) = \sum_{m=1}^{3} K_m(\mathbf{x})\imath_m = -\frac{\underline{\mathbf{x}}}{4\pi|\mathbf{x}|^3}, \qquad \underline{\mathbf{x}} \neq 0. \tag{20}$$

Note that $\underline{\mathbf{K}}(\mathbf{x})$ is a $\mathbb{H}$-valued fundamental solution of D and therefore monogenic in $G\backslash\{0\}$, or $D\underline{\mathbf{K}}(\mathbf{x}) = \delta(\mathbf{x})$.

Consequently, there exists a three-dimensional Cauchy integral representation for continuous left monogenic $\mathbb{H}$-valued functions on $\overline{G}$ [33],

$$(F_\Gamma f)(\mathbf{x}) := \int_\Gamma \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y}) f(\mathbf{y})\, \mathrm{d}\Gamma_{\mathbf{y}}, \qquad \mathbf{x} \in G\backslash\Gamma, \tag{21}$$

where $\underline{\alpha}(\mathbf{y}) = \sum_{m=1}^{3} \alpha_m(\mathbf{y})\imath_m$ is the quaternionic outward pointing unit vector at $\mathbf{y}$ on the boundary $\partial G = \Gamma, \mathrm{d}\Gamma_{\mathbf{y}}$ is the Lebesgue measure on Γ. Moreover one can see that $D(F_\Gamma f)(\mathbf{x}) = 0$.

[4] There is another right D_r, which can be defined with $\imath_m$ on the right side of the partial derivatives of $\underline{\mathbf{w}}$. We shall not need it here.

The $\mathbb{H}$-valued fundamental solution of D is also used to define the so-called Teodorescu transform [36]. For all $f(\mathbf{x}) \in \mathcal{C}(G, \mathbb{H})$ we have

$$(T_G f)(\mathbf{x}) := \int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\mathbf{y} \quad \mathbf{x} \in G \subset \mathbb{R}^3. \tag{22}$$

We can also see that it is the right inverse of D, since $D(T_G f)(\mathbf{x}) = f(\mathbf{x})$.

Conversely, for all $\underline{\mathbf{f}}(\mathbf{x}) \in \mathcal{C}^2(G, \mathbb{H}) \cap \mathcal{C}(\overline{G}, \mathbb{H})$ and $\mathbf{x} \in G$, the equation

$$D\,\underline{\mathbf{u}}(\mathbf{x}) = f(\mathbf{x}) \tag{23}$$

has the general solution $\underline{\mathbf{u}}(\mathbf{x}) = (T_G f)(\mathbf{x}) + (F_\Gamma \underline{\mathbf{v}})(\mathbf{x})$, where $\underline{\mathbf{v}}(\mathbf{x})$ is an arbitrary function chosen to satisfy a given boundary condition of the form

$$\underline{\mathbf{u}}(\mathbf{x})|_\Gamma = \underline{\mathbf{v}}(\mathbf{x}). \tag{24}$$

Thus, for any $f \in \mathcal{C}^1(G, \mathbb{H}) \cap \mathcal{C}(\overline{G}, \mathbb{H})$, it can be shown that it satisfies the so-called Borel–Pompeiu formula [34]:

$$(F_\Gamma f)(\mathbf{x}) + (T_G D) f(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^3 \backslash \overline{G}. \end{cases} \tag{25}$$

A generalization of the concept of the Cauchy principal value for $(F_\Gamma f(\mathbf{x})$ can be introduced, when the variable $\mathbf{x}$ is approaching the boundary $\partial G = \Gamma$. For a given f, at each regular point $\mathbf{x}' \in \Gamma$ [34], the non-tangential limit of the Cauchy integral representation can be written as

$$\lim_{\mathbf{x}\to\mathbf{x}'} (F_\Gamma f)(\mathbf{x}) = \tfrac{1}{2}(\pm f(\mathbf{x}') + (S_\Gamma f)(\mathbf{x}')), \tag{26}$$

where

$$(S_\Gamma f)(\mathbf{x}) = 2 \int_\Gamma f(\mathbf{x}-\mathbf{y}) \underline{\alpha}(\mathbf{y}) f(\mathbf{y})\,\mathrm{d}\Gamma_{\mathbf{y}} \tag{27}$$

is understood as a 'quaternionic Cauchy principal value' of the integral over the smooth boundary Γ, because of the singularity of $\underline{\mathbf{K}}(\underline{\mathbf{x}})$ in the integrand.

A Plemelj–Sokhotzkij'-type formula for f, relative to Γ, [33, 36] can now be given as

$$\begin{aligned} &\text{(i)} \quad \lim_{\mathbf{x} \xrightarrow[\mathbf{x}\in G]{} \mathbf{x}' \in \Gamma} (F_\Gamma f)(\mathbf{x}) = (P_\Gamma f)(\mathbf{x}'), \\ &\text{(ii)} \quad \lim_{\mathbf{x} \xrightarrow[\mathbf{x}\in \mathbb{R}^3\backslash\overline{G}]{} \mathbf{x}' \in \Gamma} (F_\Gamma f)(\mathbf{x}) = -(Q_\Gamma f)(\mathbf{x}'), \end{aligned} \tag{28}$$

where P_Γ is the projection operator $\left(P_\Gamma^2 = P_\Gamma\right)$ onto $\mathbb{H}$-valued functions, which have a left-monogenic extension into the domain G, and Q_Γ is the projection operator $\left(Q_\Gamma^2 = Q_\Gamma\right)$ onto $\mathbb{H}$-valued functions, which have a left-monogenic extension into the domain $\mathbb{R}^3\backslash\overline{G}$ and vanish at infinity.

P_Γ and Q_Γ can be given, in turns, an alternative form in terms of the quaternionic principal value operator S_Γ, as

$$P_\Gamma := \tfrac{1}{2}(I + S_\Gamma) \qquad Q_\Gamma := \tfrac{1}{2}(I - S_\Gamma), \tag{29}$$

with the following operator relations:

$$S_\Gamma P_\Gamma = P_\Gamma, \qquad S_\Gamma Q_\Gamma = -Q_\Gamma, \qquad S_\Gamma^2 = S_\Gamma S_\Gamma = I. \tag{30}$$

Finally, we define a *trace* operator tr as a restriction map for an $\mathbb{H}$-valued function $\underline{\mathbf{u}}$ on Γ, smooth boundary of $G \in \mathbb{R}^3$, by

$$\mathrm{tr}\,\underline{\mathbf{u}} = \underline{\mathbf{u}}|_\Gamma. \tag{31}$$

3. The x-ray transform and its inverse

3.1. The setting of the problem

We are now in a position to tackle the inversion problem for the x-ray transform of a physical density $f_0(\mathbf{x})$. By definition, this transform consists of integrating $f_0(\mathbf{x})$, assumed to be an integrable function with compact support in a convex set G, along a straight line from the source point $\mathbf{x}$ to infinity in the direction of the unit vector $\mathbf{n}$, i.e.

$$(Xf_0)(\mathbf{x}, \mathbf{n}) = \int_0^\infty \mathrm{d}t\ f_0(\mathbf{x} + t\mathbf{n}). \tag{32}$$

Concretely, this transform describes two working modalities in medical imaging. In transmission modality, f_0 represents the attenuation map of the object under study, whereas in emission modality f_0 is its radiation activity density.

- The next point is that if $f_0(\infty) = 0$ as assumed (f_0 is compactly supported), it can be verified that $(Xf_0)(\mathbf{x}, \mathbf{n})$ satisfies a very simple partial differential equation, namely
$$\mathbf{n} \cdot \nabla (Xf_0)(\mathbf{x}, \mathbf{n}) = -f_0(\mathbf{x}). \tag{33}$$
This can be checked if we let the $\mathbf{n} \cdot \nabla$ operator act under the integral sign. After a change of variables, the integrand just turns into the differential of $f_0(\mathbf{x})$ under the integral sign. Equation (33) is in fact a simplified stationary photon transport equation without loss by attenuation and without source or sink term [38]. Since $\mathbf{n} \cdot \nabla$ is a directional derivative, its inverse is a directional integration[5]. Physically $(Xf_0)(\mathbf{x}, \mathbf{n})$ is subjected to the following boundary condition. For a given direction $\mathbf{n}$, because of the support hypothesis and because of the prescription on the direction of integration, $(Xf_0)(\mathbf{x}, \mathbf{n}) = 0$, whenever $\mathbf{x}$ is on the boundary $\Gamma = \partial G$ of G and $\mathbf{n}$ points outward of Γ.
- As $\mathbf{n}$ is fixed unit vector in $\mathbb{R}^3$, equation (33) can also be written as a divergence equation for a vector field $\mathbf{u}(\mathbf{x})$ having a constant direction $\mathbf{n}$, namely $\mathbf{u}(\mathbf{x}) = \mathbf{n}(Xf_0)(\mathbf{x}, \mathbf{n})$
$$\mathbf{n} \cdot \nabla (Xf_0)(\mathbf{x}, \mathbf{n}) = \nabla \cdot \mathbf{u}(\mathbf{x}) = -f_0(\mathbf{x}). \tag{34}$$

3.2. Formulation in quaternionic analysis

The problem of inversion of the three-dimensional x-ray transform will now be set in the framework of quaternion analysis. The idea is to consider equation (34) as part of an inhomogeneous equation (17), with an $\mathbb{H}$-valued 'source' function $f = f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x})$ on its right-hand side for an unknown function $\underline{\mathbf{u}}(\mathbf{x}) = \sum_{m=1}^{3} u_m(\mathbf{x})\imath_m$, i.e.

$$D\underline{\mathbf{u}}(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in G. \tag{35}$$

As can be checked, the quaternionic product rule (9) yields

$$D\underline{\mathbf{u}}(\mathbf{x}) = -(\nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x})) + \underline{\nabla_{\mathbf{x}} \times \mathbf{u}}(\mathbf{x}) = f_0(\mathbf{x}) + \underline{\mathbf{f}}(\mathbf{x}), \tag{36}$$

which implies the following two equations for the vector field $\mathbf{u}(\mathbf{x})$:

$$\nabla_{\mathbf{x}} \cdot \mathbf{u}(\mathbf{x}) = -f_0(\mathbf{x}), \qquad \nabla_{\mathbf{x}} \times \mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}), \tag{37}$$

the first one, being exactly equation (33).

- So following the general approach of [30], equation (35) shall be worked out with appropriate boundary conditions and its solution $\mathbf{u}$ will be reexpressed in terms of x-ray transform data. Then equation (35) is used to invert the three-dimensional x-ray transform, by expressing $f_0(\mathbf{x})$ in terms of the measurements.

[5] This is not the second-order ultra-hyperbolic partial differential equation of John [10].

Theorem [34]. *For all $f(\mathbf{x}) \in \mathcal{C}^2(G,\mathbb{H}) \cap \mathcal{C}(\overline{G},\mathbb{H})$ and $\mathbf{x} \in G$, the inhomogeneous equation (35)*

$$D\underline{\mathbf{u}}(\mathbf{x}) = f(\mathbf{x}), \tag{38}$$

with the boundary condition $\underline{\mathbf{u}}(\mathbf{x})|_\Gamma = \underline{\mathbf{v}}(\mathbf{x})$, has a unique solution, where $\underline{\mathbf{v}}(\mathbf{x})$ is given on Γ.

Proof. In [34], for f in $\mathcal{C}^2(G,\mathbb{H}) \cap \mathcal{C}(\overline{G},\mathbb{H})$ the solution of equation (23) is given by

$$\underline{\mathbf{u}} = T_G f + F_\Gamma \underline{\mathbf{v}} \tag{39}$$

with the condition

$$Q_\Gamma \underline{\mathbf{v}} = \mathrm{tr}\, T_G f, \tag{40}$$

which follows from the Plemelj–Sokhotzkij's formula, as an extension onto $\mathbb{R}^3\backslash\overline{G}$.

By letting D act on equation (39) we have

$$D\,\underline{\mathbf{u}} = DT_G f + DF_\Gamma \underline{\mathbf{v}}. \tag{41}$$

Now, as T_G is the right inverse of D, for $f \in \mathcal{C}^1(G)\bigcap\mathcal{C}(\overline{G})$ from [33, 36] we have the following statements:

$$(DT_G f)(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^3\backslash\overline{G} \end{cases} \quad (T_G \text{ is 'right inverse' of } D) \tag{42}$$

$$(DF_\Gamma \underline{\mathbf{v}})(\mathbf{x}) = 0, \qquad \mathbf{x} \in \mathbb{R}^3\backslash\Gamma \quad (F_\Gamma \text{ is a monogenic function}). \tag{43}$$

We deduce that $D\underline{\mathbf{u}}(\mathbf{x}) = f(\mathbf{x})$. We now show that it satisfies the boundary condition (40). Thus, by substituting $\underline{\mathbf{u}}$ from (39) in (40), we have

$$\underline{\mathbf{v}} = \mathrm{tr}\,\underline{\mathbf{u}} = \mathrm{tr}(T_G f + F_\Gamma \underline{\mathbf{v}}) = \mathrm{tr}(T_G f) + \mathrm{tr} F_\Gamma \underline{\mathbf{v}} = \mathrm{tr}(T_G f) + P_\Gamma \underline{\mathbf{v}}, \tag{44}$$

where we have used: $\mathrm{tr}(F_\Gamma \underline{\mathbf{v}}) = (F_\Gamma \underline{\mathbf{v}})_\Gamma = P_\Gamma \underline{\mathbf{v}}$. Hence, equation (44) becomes

$$\mathrm{tr}(T_G f) = (I - P_\Gamma)\underline{\mathbf{v}} = Q_\Gamma \underline{\mathbf{v}}. \tag{45}$$

□

3.3. Computation of the Teodorescu transform of f

For $f \in \mathcal{C}^{1,\beta}_{\mathbb{H}}(G)$, $0 < \beta \leqslant 1$ and if $G \subset \mathbb{R}^3$ is a dense set in $\mathbb{R}^3$, we show that $(T_G f)(\mathbf{x})$ can be expressed in terms of x-ray measurements by explicit computation

$$(T_G f)(\mathbf{x}) = \frac{1}{4\pi}\int_G \frac{\underline{\mathbf{x}} - \underline{\mathbf{y}}}{|\mathbf{x} - \mathbf{y}|^3} f(\mathbf{y})\,\mathrm{d}\mathbf{y} = \frac{1}{4\pi}\sum_{i=1}^{3}\sum_{\beta=0}^{3}\int_G \frac{x_i - y_i}{|\mathbf{x} - \mathbf{y}|^3} f_\beta(\mathbf{y})\,\mathrm{d}\mathbf{y}\imath_i\imath_\beta. \tag{46}$$

Let $\mathbf{y} = \mathbf{x} + \mathbf{n}t$ in $\mathbb{R}^3$, where $t \in \mathbb{R}^+$ and $\mathbf{n}$ is a unit vector. Consequently, under the above change of variables, we may express the volume element $\mathrm{d}\mathbf{y}$ in spherical coordinates $\mathrm{d}\mathbf{y} = t^2\,\mathrm{d}t\,\mathrm{d}\Omega_{\mathbf{n}}$, where $\mathrm{d}\Omega_{\underline{\mathbf{n}}}$ is the area element of the unit sphere $\Omega_{\mathbf{n}}$ in $\mathbb{R}^3$.

Thus, we have

$$\begin{aligned}(T_G f)(\mathbf{x}) &= \frac{1}{4\pi}\sum_{i=1}^{3}\sum_{\beta=0}^{3}\int_{\Omega_\mathbf{n}}\int_{\mathbb{R}^+}\frac{n_i t}{|\mathbf{n}t|^3} f_\beta(\mathbf{x} + \mathbf{n}t)t^2\,\mathrm{d}\Omega_\mathbf{n}\,\mathrm{d}t\,\imath_i\imath_\beta \\ &= \frac{1}{4\pi}\sum_{i=1}^{3}\sum_{\beta=0}^{3}\int_{\Omega_\mathbf{n}} n_i\left(\int_{\mathbb{R}^+} f_\beta(\mathbf{x} + \mathbf{n}t)\,\mathrm{d}t\right)\mathrm{d}\Omega_\mathbf{n}\imath_i\imath_\beta.\end{aligned} \tag{47}$$

We can now see that the x-ray transform arises in a natural way in the Teodorescu transform. For each quaternionic component f_β of f, we have an integration on a half-line along the direction of $\mathbf{n}$,

$$(Xf_\beta)(\mathbf{x},\mathbf{n}) := \int_{\mathbb{R}^+} f_\beta(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t, \qquad (\beta = 0,\ldots,3), \tag{48}$$

after summing up over the quaternionic imaginary unit, we obtain the x-ray transform of an $\mathbb{H}$-valued function f defined in $\mathbb{R}^3$, as

$$(Xf)(\mathbf{x},\mathbf{n}) = \sum_{\beta=0}^{3} Xf_\beta(\mathbf{x},\mathbf{n})\imath_\beta = \int_{\mathbb{R}^+} f(\mathbf{x}+\mathbf{n}t)\,\mathrm{d}t. \tag{49}$$

Consequently, the Teodorescu transform of f is a weighted vector spherical mean value of $(Xf)(\mathbf{x},\mathbf{n})$ in the form

$$(T_G f)(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} \underline{\mathbf{n}}(Xf)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{50}$$

3.4. f-reconstruction's formula

Now by letting D act on $(T_G f)(\mathbf{x})$ (or equation (38)), we get f through $f = D\underline{\mathbf{u}} = D(T_G f + DF_\Gamma \underline{\mathbf{v}})$.

This result may be called the quaternionic reconstruction formula:

$$f(\mathbf{x}) = D\underline{\mathbf{u}}(\mathbf{x}) = \frac{1}{4\pi} D\int_{\Omega_\mathbf{n}} \underline{\mathbf{n}}(Xf)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{51}$$

It remains to sort out the reconstruction formulae for the scalar part $f_0(\mathbf{x})$ and the vector part $\mathbf{f}(\mathbf{x})$. This computation is done in the following section.

4. Explicit calculation of the vector and scalar parts of the reconstructed f

4.1. Generalities

To obtain the vector and scalar parts of f, we use equation (36) to rewrite equation (51) explicitly as follows:

$$f(\mathbf{x}) = -\frac{1}{4\pi}\sum_{i=1}^{3}\partial_i\int_{\Omega_\mathbf{n}} n_i(Xf)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n} + \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_\mathbf{n}} n_j(Xf)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}\imath_k. \tag{52}$$

The last term on the right-hand side, called I, is the sum of two terms involving successively f_0 and $\mathbf{f}$ (or its components f_l),

$$I = \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_\mathbf{n}} n_j(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}\imath_k + \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\sum_{l=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_\mathbf{n}} n_j(Xf_l)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}\imath_k\imath_l. \tag{53}$$

The second term of the above equation, called J, can be expressed successively as

$$\begin{aligned} J &= \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\sum_{l=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_\mathbf{n}} n_j(Xf_l)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}\imath_k\imath_l \\ &= \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\sum_{l,m=1}^{3}\varepsilon_{ijk}\varepsilon_{klm}\partial_i\int_{\Omega_\mathbf{n}} n_j(Xf_l)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}\imath_m. \end{aligned} \tag{54}$$

By using the identity

$$\sum_{k=1}^{3}\varepsilon_{ijk}\varepsilon_{lmk} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}), \tag{55}$$

we reduce J to

$$J = \frac{1}{4\pi}\sum_{i,j=1}^{3}\partial_i\int_{\Omega_{\mathbf{n}}} n_j((Xf_i)(\mathbf{x},\mathbf{n})\imath_j - (Xf_j)(\mathbf{x},\mathbf{n})\imath_i)\,\mathrm{d}\Omega_{\mathbf{n}}, \tag{56}$$

such that finally, I has the expression

$$I = \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_{\mathbf{n}}} n_j(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}\imath_k + \frac{1}{4\pi}\sum_{i,j=1}^{3}\partial_i\int_{\Omega_{\mathbf{n}}} n_j((Xf_i)(\mathbf{x},\mathbf{n})\imath_j - (Xf_j)(\mathbf{x},\mathbf{n})\imath_i)\,\mathrm{d}\Omega_{\mathbf{n}}, \tag{57}$$

or after rearrangement of all terms according to their k-component, I reads

$$I = \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_{\mathbf{n}}} n_j(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}\imath_k + \frac{1}{4\pi}\sum_{i,k=1}^{3}\left(\partial_i\int_{\Omega_{\mathbf{n}}} n_k(Xf_i)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} - \partial_k\int_{\Omega_{\mathbf{n}}} n_i(Xf_i)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}\right)\imath_k. \tag{58}$$

4.2. Reconstruction of vector part of f

From the previous expression of I, we can derive the components f_k with $k = 1, 2, 3$ of $\mathbf{f}$ as

$$f_k(\mathbf{x}) = -\frac{1}{4\pi}\sum_{i=1}^{3}\partial_i\int_{\Omega_{\mathbf{n}}} n_i(Xf_k)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} + \frac{1}{4\pi}\sum_{i,j=1}^{3}\varepsilon_{ijk}\partial_i\int_{\Omega_{\mathbf{n}}} n_j(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} + \frac{1}{4\pi}\sum_{i=1}^{3}\left(\partial_i\int_{\Omega_{\mathbf{n}}} n_k(Xf_i)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} - \partial_k\int_{\Omega_{\mathbf{n}}} n_i(Xf_i)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}\right). \tag{59}$$

Hence the reconstructed $\mathbf{f}$ has the following compact vector form:

$$\mathbf{f}(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}}[-\nabla_{\mathbf{x}}\cdot(\mathbf{n}X\mathbf{f}) + \nabla_{\mathbf{x}}\times(\mathbf{n}Xf_0) + \mathbf{n}(\nabla_{\mathbf{x}}\cdot X\mathbf{f}) - \nabla_{\mathbf{x}}(\mathbf{n}\cdot X\mathbf{f})]\,(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}. \tag{60}$$

Now for each given $\mathbf{n}$, a constant vector field, we have $\nabla_{\mathbf{x}}\times\mathbf{n} = \nabla_{\mathbf{x}}\cdot\mathbf{n} = 0$. In equation (60) and from the definitions of $\mathbf{f}$ and f_0, one can show that the last two terms in equation (60) are zero and that the two first terms are equal. This is so because

$$\begin{aligned}-\nabla_{\mathbf{x}}\cdot(\mathbf{n}X\mathbf{f}) &= -(\mathbf{n}\cdot\nabla_{\mathbf{x}})(X\mathbf{f}) = -(\mathbf{n}\cdot\nabla_{\mathbf{x}})(\nabla_{\mathbf{x}}\times\mathbf{n})(Xu_0)\\ &= -(\nabla_{\mathbf{x}}\times\mathbf{n})(\mathbf{n}\cdot\nabla_{\mathbf{x}})(Xu_0) = \nabla_{\mathbf{x}}\times(\mathbf{n}Xf_0),\end{aligned} \tag{61}$$

where in the above derivation we have made use of the commutativity between the x-ray transform and the differential operators with respect to $\mathbf{x}$. Hence, we obtain the reconstruction of the vector part of f in terms of the x-ray data $(Xf_0)(\mathbf{x},\mathbf{n})$ as

$$\mathbf{f}(\mathbf{x}) = \frac{1}{2\pi}\int_{\Omega_{\mathbf{n}}}(\mathbf{n}\cdot\nabla_{\mathbf{x}})(X\mathbf{f})(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}} = \frac{1}{2\pi}\int_{\Omega_{\mathbf{n}}}(\mathbf{n}\times\nabla_{\mathbf{x}})(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_{\mathbf{n}}. \tag{62}$$

4.3. Reconstruction of the scalar part of f

f_0 is given by

$$f_0(\mathbf{x}) = -\frac{1}{4\pi}\sum_{i=1}^{3}\partial_i \int_{\Omega_\mathbf{n}} n_i (Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n} - \frac{1}{4\pi}\sum_{i,j,k=1}^{3}\varepsilon_{ijk}\partial_i \int_{\Omega_\mathbf{n}} n_j (Xf_k)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}$$
$$= -\frac{1}{4\pi}\int_{\Omega_\mathbf{n}} [\nabla_\mathbf{x}\cdot(\mathbf{n}Xf_0)(\mathbf{x},\mathbf{n}) + (\nabla_\mathbf{x}\times\mathbf{n})\cdot(X\mathbf{f})(\mathbf{x},\mathbf{n})]\,\mathrm{d}\Omega_\mathbf{n}. \tag{63}$$

Putting the expressions of $f_0 = -(\mathbf{n}\cdot\nabla_\mathbf{x})u_0$ and of $\mathbf{f} = -(\mathbf{n}\times\nabla_\mathbf{x})u_0$ in equation (63), we get f_0 as

$$f_0(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} [(\mathbf{n}\cdot\nabla_\mathbf{x})(X(\mathbf{n}\cdot\nabla_\mathbf{x})u_0)(\mathbf{x},\mathbf{n}) + (\nabla_\mathbf{x}\times\mathbf{n})\cdot(X(\nabla_\mathbf{x}\times\mathbf{n})u_0)(\mathbf{x},\mathbf{n})]\,\mathrm{d}\Omega_\mathbf{n}$$
$$= \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} [(\mathbf{n}\cdot\nabla_\mathbf{x})^2 + |\nabla_\mathbf{x}\times\mathbf{n}|^2](Xu_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{64}$$

Finally, after simplifications, we arrive at the final form of the reconstructed f_0:

$$f_0(\mathbf{x}) = \frac{1}{4\pi}\int_{\Omega_\mathbf{n}} \triangle_\mathbf{x}(Xu_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}, \tag{65}$$

where $\triangle_\mathbf{x}$ is the Laplace operator in $\mathbf{x}$.

4.4. Alternative form of the result

From equation (37), which expresses $\mathbf{f}$ as the curl of $\mathbf{u}$: $\mathbf{f} = \nabla_\mathbf{x}\times\mathbf{u}$, we deduce that $\nabla_\mathbf{x}\cdot\mathbf{f} = 0$. Thus

$$\nabla_\mathbf{x}\cdot(\nabla_\mathbf{x}\times\mathbf{u}) = \mathbf{n}\triangle_\mathbf{x}u_0 - \nabla_\mathbf{x}(\mathbf{n}\cdot\nabla_\mathbf{x})u_0 = \mathbf{n}\triangle_\mathbf{x}u_0 + \nabla_\mathbf{x}f_0 = 0, \tag{66}$$

which yields

$$\triangle_\mathbf{x}u_0 = -(\mathbf{n}\cdot\nabla_\mathbf{x})f_0. \tag{67}$$

Substitution of this expression in equation (65) gives an alternative form of the reconstructed f_0:

$$f_0(\mathbf{x}) = -\frac{1}{4\pi}\int_{\Omega_\mathbf{n}} (\mathbf{n}\cdot\nabla_\mathbf{x})(Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{68}$$

4.5. Comparison with known results

In [14], an inverse formula for the x-ray transform in the three dimensions has been given in terms of the so-called $f_0(\mathbf{x})$ Riesz transform $\mathfrak{R}^1$ of a function

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2}\int \frac{1}{|\mathbf{x}-\mathbf{y}|^2} f_0(\mathbf{y})\,\mathrm{d}\mathbf{y}. \tag{69}$$

The reconstructed function f_0 is given by

$$f_0(\mathbf{x}) = -\frac{1}{2\pi^2}\triangle_\mathbf{x}\mathfrak{R}^1\int_{\Omega_\mathbf{n}} (Xf_0)(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n} = -\frac{1}{2\pi^2}\triangle_\mathbf{x}\mathfrak{R}^1\int_{\Omega_\mathbf{n}} u_0(\mathbf{x},\mathbf{n})\,\mathrm{d}\Omega_\mathbf{n}. \tag{70}$$

Now setting $\mathbf{y} = \mathbf{x} + \mathbf{n}t$ in the above equation yields

$$\mathfrak{R}^1 f_0(\mathbf{x}) = \frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} \int_{\mathbb{R}^+} f_0(\mathbf{x} + \mathbf{n}t)\, \mathrm{d}t\, \mathrm{d}\Omega_\mathbf{n} = \frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} (Xf_0)(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n}. \tag{71}$$

In equation (70) we put $\frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} u_0(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n} = \langle u_0 \rangle_\mathbf{n}$, which is the average of u_0 over the unit ball in $\mathbb{R}^3$. Thus, by using the above relation, equation (71) can be written as

$$\begin{aligned} f_0(\mathbf{x}) &= -\triangle_\mathbf{x} \mathfrak{R}^1 \langle u_0 \rangle_{\mathbf{n}'} = -\frac{1}{2\pi^2} \int_{\Omega_\mathbf{n}} (X \langle u_0 \rangle_{\mathbf{n}'})(\mathbf{x}, \mathbf{n})\, \mathrm{d}\Omega_\mathbf{n} = -\triangle_\mathbf{x} \langle X \langle u_0 \rangle_{\mathbf{n}'} \rangle_\mathbf{n} \\ &= -\triangle_\mathbf{x} \langle X u_0 \rangle_\mathbf{n} \end{aligned} \tag{72}$$

which, up to a normalization factor, has the same form as our equation (65).

4.6. Alternate derivation of the inversion formula

We saw that $\mathbf{f}$ is defined by following relation $\mathbf{f} = \nabla_\mathbf{x} \times \mathbf{n} u_0$, and its inversion formula is obtained by equation (62). If we compute the integral of $\mathbf{f}$ on the smooth surface S, we obtain

$$\int_S \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \int_S \nabla_\mathbf{x} \times \mathbf{n} u_0 \cdot \mathrm{d}\mathbf{S} = \frac{1}{2\pi} \int_S \int_{\Omega_\mathbf{n}} (\nabla_\mathbf{x} \times \mathbf{n})(Xf_0) \cdot \mathrm{d}\mathbf{S}\, \mathrm{d}\Omega_\mathbf{n}. \tag{73}$$

Thus, by using the Stokes's theorem we have

$$\oint_C u_0 \mathbf{n} \cdot \mathrm{d}\mathbf{l} = \frac{1}{2\pi} \oint_C \int_{\Omega_\mathbf{n}} (Xf_0)\mathbf{n} \cdot \mathrm{d}\mathbf{l}\, \mathrm{d}\Omega_\mathbf{n} \tag{74}$$

or, by introducing $\mathrm{d}l_\mathbf{n} := \mathbf{n} \cdot \mathrm{d}\mathbf{l}$,

$$\oint_C \left[(Xf_0) - \frac{1}{2\pi} \int_{\Omega_\mathbf{n}} (Xf_0) \mathrm{d}\Omega_\mathbf{n} \right] \mathrm{d}l_\mathbf{n} = 0. \tag{75}$$

We conclude that the integrand is equal to zero. Thus,

$$(Xf_0) = \frac{1}{2\pi} \int_{\Omega_\mathbf{n}} (Xf_0)\, \mathrm{d}\Omega_\mathbf{n}. \tag{76}$$

By acting $(\mathbf{n} \cdot \nabla_\mathbf{x})$ on the above equation, we obtain

$$f_0 = \frac{-1}{2\pi} (\mathbf{n} \cdot \nabla_\mathbf{x}) \int_{\Omega_\mathbf{n}} (Xf_0)\, \mathrm{d}\Omega_\mathbf{n}, \tag{77}$$

which is the same result as that in equation (68).

5. Conclusion and perspectives

In this paper, we have shown that the use of quaternion analysis has been successful in obtaining the inverse formula for the x-ray transform in three dimensions. Although appearing under a different form, this result is essentially equivalent to the result obtained many years ago. The reason we have started this study is that quaternion formalism lends itself to a generalization whereby the operator D is replaced by $D + \underline{\mathbf{a}}(\mathbf{x})$, $\underline{\mathbf{a}}(\mathbf{x})$ is a smooth compactly supported $\mathbb{H}$-valued function. Such a function represents, for example, the non-uniform attenuation map in the case of transmission imaging. This is of high interest because it is much more close to reality and has up to now evaded complete resolution except in two dimensions, where an analytic solution has been given in [30, 31]. We shall go into this topic in future work.

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