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Green function of the inhomogeneous Helmholtz equation with nonuniform refraction index, using quaternion analysis

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In this paper we first show in the framework of quaternion analysis how the fundamental solutions of the Dirac operators with vector potential can be obtained. Then, we use the obtained results to present a derivation of the exact analytic Green function for the Helmholtz equation, i.e., $(\triangle + |\mathbf{a}(\mathbf{x})|^2)G_0(\mathbf{x}) = \delta(\mathbf{x})$, for the case $\mathbf{a}(\mathbf{x})$ is a monogenic (analytic) vector potential. © *2010 American Institute of Physics*. [doi:10.1063/1.3524507]

I. INTRODUCTION

The Helmholtz equation with nonhomogeneous refraction index is of interest for different subjects in science and engineering. In acoustics as well as in propagation in non-homogeneous matter some authors have used a transformation to treat the Helmholtz equation with constant refraction index.[1,2] There are many other works devoted to numerical techniques, which are specifically based on (i) one-dimensional direct and inverse scattering theory, (ii) parabolic approximation, (iii) geometrical approximation, and (iv) direct Fourier synthesis of the wave field.[3,4] In this paper, we obtain an exact analytic solution for the Helmholtz equation with nonuniform refraction index for some cases, which shall be explained later at the end of the Sec. I, by using quaternion analysis. This paper is concerned with wave propagation in 3 dimensions.

The Helmholtz equation for a wave $\psi_0(\mathbf{x})$ propagating in a medium of nonuniform index of refraction $|\mathbf{a}(\mathbf{x})|^2$ reads

$$\triangle\psi_0(\mathbf{x}) + |\mathbf{a}(\mathbf{x})|^2\psi_0(\mathbf{x}) = f_0(\mathbf{x}). \tag{1}$$

Its general solution is obtainable from its Green function, which satisfies the following equation:

$$(\triangle + |\mathbf{a}(\mathbf{x})|^2)\mathcal{G}_0(\mathbf{x}) = \delta(\mathbf{x}). \tag{2}$$

Knowing the Green function, $\psi_0(\mathbf{x})$ in (1) can be obtained as

$$\psi_0(\mathbf{x}) = \int \mathcal{G}_0(\mathbf{x}-\mathbf{y}) f_0(\mathbf{y}) d\mathbf{y}. \tag{3}$$

To construct the Green function $\mathcal{G}_0$, we need to know the fundamental solution of Dirac operator $D + \underline{\mathbf{a}}(\mathbf{x})$ in quaternion analysis. The solution of this operator when $\underline{\mathbf{a}} = \frac{1}{v_0}\nabla v_0$ is obtained by Ref. 5. Recently, quaternion approach is used to obtain an inversion formula for nonattenuated and attenuated x-ray transform in three dimensions by Refs. 6–8. To obtain an inversion formula for nonuniformly attenuated x-ray emission in Ref. 8, the author has studied the fundamental solution of the above Dirac operator with nonuniform vector potential in the special x-ray emission case. In this paper, we obtain the solution of the Dirac operator with varying vector potential in general and without restriction. As we see later, this paper also gives the general solution for the Dirac operator with an analytic vector potential, for which one can recover the results of Refs. 5 and 8–10.

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In this paper, we will construct the Green function of the Helmholtz equation under the following condition:

- "$\mathbf{a}(\mathbf{x})$" is an analytic vector potential, in quaternion analysis a monogenic pure quaternion function, i.e., $D\underline{\mathbf{a}}(\mathbf{x}) = 0$.

In quaternion analysis, the condition $\underline{\mathbf{a}}$ specifies a monogenic function, i.e., $D\underline{\mathbf{a}} = 0$ gives two following conditions in vector analysis:

$$\nabla \cdot \mathbf{a} = 0, \tag{4}$$

$$(\nabla \times \mathbf{a}) = 0. \tag{5}$$

From the above conditions, $\mathbf{a}$ is a gradient of the following harmonics and nonharmonics functions:

$$\mathbf{a} = \nabla h_0 \quad \text{and} \quad \triangle h_0 = 0, \tag{6}$$

$$\mathbf{a} = \frac{\nabla v_0}{v_0} \quad \text{and} \quad v_0 = e^{w_0}. \tag{7}$$

Equation (6) yields h_0 which is a harmonic function, i.e., $\triangle h_0 = 0$. Equation (7) introduces all nonharmonic scalar potential, v_0. But, w_0 is a harmonic function.

This article is organized as follows. In Sec. II we give a short introduction to quaternion analysis. In Sec. III, we obtain the fundamental solution for the Dirac operator with varying vector potential. Section IV represents a reformulation of the Helmholtz equation in quaternionic form, then we obtain its Green function. In Sec. V, the comparison with known results is given. Finally, in conclusion we discuss about the results and perspectives.

II. QUATERNION REVIEW

A comprehensive review of quaternions can be found in Ref. 11. A quaternion x has four components, expressed as follows:

$$x = x_0\iota_0 + \sum_{m=1}^{3} x_m\iota_m = \mathrm{Sc}\,x + \mathrm{Vec}\,x = x_0\iota_0 + \underline{\mathbf{x}}, \quad (x_0, x_1, x_2, x_3 \in \mathbb{R}), \tag{8}$$

where $\iota_0 = 1$ and $\iota_m = \sqrt{-1}$, $(m = 1, 2, 3)$ and we have the following relations between "ι"s:

- by promoting the unit vectors to be imaginary units, i.e., $\iota_1{}^2 = \iota_2{}^2 = \iota_3{}^2 = -1$,
- by introducing a noncommutative multiplication rule between them: $\iota_i\iota_j = -\iota_j\iota_i$ for $i \neq j$ and $\iota_i\iota_j = \iota_k$ for all cyclic permutations of (i, j, k).

The set of quaternions with real components should be called $\mathbb{H}(\mathbb{R})$, but for simplicity, will be denoted by $\mathbb{H}$.

Following Ref. 11, we give some of their properties,

$$\text{conjugate operation :} \quad \overline{x} = x_0\iota_0 - \sum_{m=1}^{3} x_m\iota_m, \tag{9}$$

$$\text{square norm :} \quad |x|^2 = x\overline{x} = \overline{x}x = x_0^2 + x_1^2 + x_2^2 + x_3^2, \tag{10}$$

$$\text{inverse :} \quad x^{-1} = \frac{\overline{x}}{|x|^2} \quad \text{if and only if} \quad x\overline{x} \neq 0. \tag{11}$$

Quaternions with complex valued components are called *biquaternions* and denoted by $\mathbb{H}(\mathbb{C})$. For $x = p + \mathrm{i}q$ in $\mathbb{H}(\mathbb{C})$, where $p, q \in \mathbb{H}(\mathbb{R})$ and $\mathrm{i} = \sqrt{-1}$, we have[12]

$$\overline{x} = \overline{p} + \mathrm{i}\overline{q}, \qquad x^* = p - \mathrm{i}q. \tag{12}$$

Finally, the ordered product of two quaternions, $y = y_0\imath_0 + \underline{\mathbf{y}}$ and $x = x_0\imath_0 + \underline{\mathbf{x}}$, is a quaternion $w = yx = (\mathrm{Sc}\, w + \mathrm{Vec}\, w)$, where

$$\mathrm{Sc}\, w = y_0x_0 - (\mathbf{y}\cdot\mathbf{x}) \quad \text{and} \quad \mathrm{Vec}\, w = \underline{\mathbf{y}}x_0 + y_0\underline{\mathbf{x}} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{13}$$

In particular, i.e., the ordered product of two "pure quaternions," $\underline{\mathbf{y}}$ by $\underline{\mathbf{x}}$ is

$$\underline{\mathbf{y}}\,\underline{\mathbf{x}} = -\mathbf{y}\cdot\mathbf{x} + \underline{\mathbf{y}\times\mathbf{x}}. \tag{14}$$

The operator D is introduced by[13]

$$D = \sum_{m=1}^{3} \imath_m \frac{\partial}{\partial x_i}. \tag{15}$$

The quaternion operator D has been given different names according to different authors: Dirac operator for Ref. 13, three-dimensional Cauchy–Riemann operator for Ref. 14, *etc*.

Inspection shows that it is related to the three-dimensional Laplace operator by $\triangle = -D^2$. The solutions of $Du(\mathbf{x}) = 0$, called frequently left monogenic $\mathbb{H}$-valued functions, satisfy many generalizations of classical theorems from complex analysis to a higher dimensional context.[12] Given the elementary solution of the Laplace operator $\triangle E_0(\mathbf{x}) = -D^2E_0(\mathbf{x}) = \delta(\mathbf{x})$, as

$$E_0(\mathbf{x}) = \frac{1}{4\pi\,|\mathbf{x}|}, \tag{16}$$

the elementary solution of D can be worked out as[13]

$$\underline{\mathbf{K}}(\mathbf{x}) = DE_0(\mathbf{x}) = \sum_{m=1}^{3} K_m(\mathbf{x})\imath_m = -\frac{\underline{\mathbf{x}}}{4\pi\,|\mathbf{x}|^3}, \quad \underline{\mathbf{x}} \neq 0, \tag{17}$$

where

$$K_m(\mathbf{x}) = -\frac{x_m}{4\pi\,|\mathbf{x}|^3}, \qquad (m = 1, 2, 3). \tag{18}$$

Note that $\underline{\mathbf{K}}(\mathbf{x})$ is a $\mathbb{H}$-valued fundamental solution of D and therefore monogenic in $G\backslash\{0\}$.

Consequently, there exists a three-dimensional Cauchy integral representation for continuous left monogenic $\mathbb{H}$-valued functions on $\overline{G}$,[13]

$$(Ff)(\mathbf{x}) := \int_\Gamma \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y})f(\mathbf{y})d\Gamma_{\mathbf{y}}, \quad \mathbf{x}\in G\backslash\Gamma, \tag{19}$$

where $\underline{\alpha}(\mathbf{y}) = \sum_{m=1}^{3}\alpha_m(\mathbf{y})\imath_m$ is the quaternionic outward pointing unit vector at $\mathbf{y}$ on the boundary $\partial G = \Gamma$, $d\Gamma_{\mathbf{y}}$ the Lebesgue measure on Γ. Moreover one has $D\,(Ff)(\mathbf{x}) = 0$.

The operator D has an inverse, called the Teodorescu transform.[15] It is defined for all $f(\mathbf{x}) \in \mathcal{C}(G, \mathbb{H})$ by

$$(Tf)(\mathbf{x}) := \int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\;f(\mathbf{y})\,d\mathbf{y} \quad \mathbf{x}\in G\subset\mathbb{R}^3. \tag{20}$$

Roughly speaking, D is a kind of directional derivative and T is just the integration, the inverse of this directional derivative.

Conversely, for any $f(\mathbf{x}) \in \mathcal{C}^1(G,\mathbb{H})\cap\mathcal{C}(\overline{G},\mathbb{H})$, it can be shown that it satisfies the so-called Borel–Pompeiu formula[12]

$$(Ff)(\mathbf{x}) + (T\,D)f(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x}\in G \\ 0, & \mathbf{x}\in\mathbb{R}^3\backslash\overline{G} \end{cases}. \tag{21}$$

A generalization of the concept of Cauchy principal value for $(F_\Gamma f)(\mathbf{x})$ can be introduced, when the variable $\mathbf{x}$ is approaching the boundary $\partial G = \Gamma$. For a given f, at each regular point $\mathbf{x}' \in \Gamma$,[12] the nontangential limit of the Cauchy integral representation can be written as

$$\lim_{\mathbf{x}\to\mathbf{x}'}(Ff)(\mathbf{x}) = \frac{1}{2}(\pm f(\mathbf{x}') + (Sf)(\mathbf{x}')), \tag{22}$$

where

$$(Sf)(\mathbf{x}) = 2\int_\Gamma \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\,\underline{\alpha}(\mathbf{y})\,f(\mathbf{y})\,d\Gamma_{\mathbf{y}} \tag{23}$$

is understood as a "quaternionic Cauchy principal value" of the integral over the smooth boundary Γ because of the singularity of $\underline{\mathbf{K}}(\mathbf{x})$ in the integrand.

The Gauss' formula in quaternion analysis is introduced as follows:[12]

$$\int_G (Du)(\mathbf{y})d\mathbf{y} = \int_\Gamma \underline{\alpha}(\mathbf{y})u(\mathbf{y})d\Gamma_y. \tag{24}$$

Finally, the generalized Leibniz formula for quaternion is as follows:

$$D(uv) = (Du)v + \overline{u}(Dv) + 2\mathrm{Sc}(uD)v \tag{25}$$

and for $D(G_0u)$ where G_0 is a scalar potential, the Leibniz formula has the following simple form

$$D(G_0u) = (DG_0)u + (Du)G_0. \tag{26}$$

A. Generalization to the higher dimensions

The generalization of the above definitions is given in Ref. 12. Here we introduce

$$\underline{\mathbf{K}}(\mathbf{x}) = \frac{1}{\sigma_n}\frac{\underline{\mathbf{x}}}{|\mathbf{x}|^n}, \quad \left(\sigma_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n+1}{2})}\right) \tag{27}$$

as a fundamental solution of the Dirac operator to n-dimensions which is defined as

$$D = \sum_{m=1}^{n} \iota_m \frac{\partial}{\partial x_m}. \tag{28}$$

So that $D\underline{\mathbf{K}}(\mathbf{x}) = \delta(\mathbf{x})$. Now, the Teodorescu operator has the following form:

$$(T\underline{\mathbf{a}})(\mathbf{x}) = \frac{1}{\sigma_n}\int_G \underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\underline{\mathbf{a}}(\mathbf{y})d\mathbf{y}, \quad G\subset\mathbb{R}^n. \tag{29}$$

B. Notation

Here we review our notation in this paper.

- The index "zero" indicates the scalar part of a quaternion or quaternion function, e.g., x_0 or $f_0(\mathbf{x})$.
- Only "bold" letters are used for vectors or vector functions in $\mathbb{R}^3$, such as $\mathbf{x}$ or $\mathbf{f}(\mathbf{x})$.
- "Underlined bold" letters are used for the vector part of the quaternions or quaternion functions, e.g., $\underline{\mathbf{x}}$ or $\underline{\mathbf{f}}(\mathbf{x})$.

III. THE FUNDAMENTAL SOLUTION OF THE DIRAC OPERATOR WITH NONUNIFORM VECTOR POTENTIAL

As mentioned in Sec. I, our results on fundamental solutions of $D_{\pm\underline{\mathbf{a}}} = D \pm \underline{\mathbf{a}}(\mathbf{x})$ and of $D_{\pm i\underline{\mathbf{a}}} = D \pm i\underline{\mathbf{a}}(\mathbf{x})$ are obtained in general. These results can be used in a large number of problems in physics and engineering. An inversion formula for nonuniformly attenuated x-ray emission and the Green's function of the Helmholtz equation with nonhomogeneous refraction index is useful in some restricted special cases. This work is the first step to study the general case of the fundamental solution of the $(D + a(\mathbf{x}))$ operator in which $a(\mathbf{x})$ is a varying $\mathbb{H}$-value function. In the next theorem, we will obtain the action of the Teodorescu operator on $\underline{\mathbf{a}}$.

Theorem 1: *The solution of the following Dirichlet problem:*

$$\begin{aligned} Du_0 &= \underline{\mathbf{a}} \qquad \text{in } G,\\ u_0 &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{30}$$

is

$$u_0 = (T\underline{\mathbf{a}}). \tag{31}$$

Proof: By using the Borel–Pompeiu's formula (21), the solution of the above equations is obtained as

$$u_0 = (T\underline{\mathbf{a}}) + (Fu_0), \tag{32}$$

where $Fu_0 = 0$ on G because of $u_0 = 0$ on Γ. Thus, (32) gives (31).

Remark 2: *The Dirichlet condition for bounded function with* $u_0(\mathbf{x}) = 0$ *on* Γ *means* $\lim_{|\mathbf{x}|\to\infty} u_0(\mathbf{x}) = 0$.[16]

The above theorem is general and it can be used for all vector potentials, which can be written as $\underline{\mathbf{a}} = Du_0$ or in vector analysis $\mathbf{a} = \nabla u_0$. To obtain the fundamental solution of the $D_{\underline{\mathbf{a}}}$, $\underline{\mathbf{a}}$ does not necessary need to be a monogenic function. But, as we see later in the Sec. IV the monogenic condition is necessary to obtain the Green's function for the Helmholtz equation. The next theorem gives the fundamental solution, $\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y})$, of the Dirac operator $D_{\mathrm{i}\underline{\mathbf{a}}} := (D + \mathrm{i}\underline{\mathbf{a}})$.

Theorem 3: *Considering* $\underline{\mathbf{a}} = Du_0$ *is a vector potential. Then, the fundamental solution of the* $D_{\mathrm{i}\underline{\mathbf{a}}}$*-operator is* $\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}$. *Its expression is*

$$\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y}) = e^{\mathrm{i}u_0(\mathbf{x})}\,\underline{\mathbf{K}}(\mathbf{x} - \mathbf{y})e^{-\mathrm{i}u_0(\mathbf{y})}. \tag{33}$$

Proof: To prove this theorem we have to show that $D_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}} = \delta$. First, we consider $C_0 = e^{-\mathrm{i}(T\underline{\mathbf{a}})(\mathbf{x})}$ and D derives relative to $\underline{\mathbf{y}}$. By acting $D_{\mathrm{i}\underline{\mathbf{a}}}$ on $\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}$ we have

$$D_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}} = C_0 D(e^{-\mathrm{i}u_0}\,\underline{\mathbf{K}}) + \mathrm{i}\,\underline{\mathbf{a}}C_0 e^{-\mathrm{i}u_0}\,\underline{\mathbf{K}}, \tag{34}$$

where $D(e^{-\mathrm{i}u_0}\,\underline{\mathbf{K}})$ is obtained, with the use of Leibniz formula (25), as follows:

$$D(e^{-\mathrm{i}u_0}\,\underline{\mathbf{K}}) = (De^{-\mathrm{i}u_0})\,\underline{\mathbf{K}} + e^{-\mathrm{i}u_0}\,(D\underline{\mathbf{K}}). \tag{35}$$

Thus, we have

$$D_{\mathrm{i}\underline{\mathbf{a}}}(e^{-\mathrm{i}u_0}\,\underline{\mathbf{K}}) = -\mathrm{i}\,\underline{\mathbf{a}}e^{-\mathrm{i}u_0} + e^{-\mathrm{i}u_0}D\underline{\mathbf{K}} + \mathrm{i}\,\underline{\mathbf{a}}e^{-\mathrm{i}u_0} = e^{-\mathrm{i}u_0}D\underline{\mathbf{K}}. \tag{36}$$

Now, integrating over G with respect to $\mathbf{y}$, the equation $D\underline{\mathbf{K}}(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$, we get

$$\begin{aligned} \int_G D_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y})d\mathbf{y} &= \int_G C_0 e^{-\mathrm{i}u_0(\mathbf{y})}D\underline{\mathbf{K}}(\mathbf{x} - \mathbf{y})d\mathbf{y}\\ &= \int_G C_0 e^{-\mathrm{i}u_0(\mathbf{y})}\delta(\mathbf{x} - \mathbf{y})d\mathbf{y}\\ &= C_0 e^{-\mathrm{i}u_0(\mathbf{x})} = \int_G \delta(\mathbf{x} - \mathbf{y})d\mathbf{y} = 1. \end{aligned} \tag{37}$$

This yields $C_0 = e^{\mathrm{i}u_0(\mathbf{x})}$.

Now, in analogy with the operators T, F, and S, we introduce three integral transforms $T_{\mathrm{i}\underline{\mathbf{a}}}$, $F_{\mathrm{i}\underline{\mathbf{a}}}$, and $S_{\mathrm{i}\underline{\mathbf{a}}}$ as follows:

$$(T_{\mathrm{i}\underline{\mathbf{a}}}u)(\mathbf{x}) := \int_G \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})\, u(\mathbf{y})\, d\mathbf{y} \quad \mathbf{x} \in G \subset \mathbb{R}^n, \tag{38}$$

$$(F_{\mathrm{i}\underline{\mathbf{a}}}u)(\mathbf{x}) := \int_\Gamma \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y})u(\mathbf{y})d\Gamma_{\mathbf{y}}, \quad \mathbf{x} \in G\backslash\Gamma, \tag{39}$$

$$(S_{\mathrm{i}\underline{\mathbf{a}}}u)(\mathbf{x}) := 2\int_\Gamma \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})\, \underline{\alpha}(\mathbf{y})\, u(\mathbf{y})\, d\Gamma_{\mathbf{y}}, \tag{40}$$

and we have the following relations: $T_{\mathrm{i}\underline{\mathbf{a}}}$ is "right inverse" operator of $D_{\mathrm{i}\underline{\mathbf{a}}}$,

$$(D_{\mathrm{i}\underline{\mathbf{a}}}T_{\mathrm{i}\underline{\mathbf{a}}}u)(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^n\backslash\overline{G} \end{cases}. \tag{41}$$

The Borel–Pompeiu's formula,

$$(F_{\mathrm{i}\underline{\mathbf{a}}}u)(\mathbf{x}) + (T_{\mathrm{i}\underline{\mathbf{a}}}D_{\mathrm{i}\underline{\mathbf{a}}})u(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^n\backslash\overline{G} \end{cases}, \tag{42}$$

and $F_{\mathrm{i}\underline{\mathbf{a}}}$ represent a "monogenic" function,

$$(D_{\mathrm{i}\underline{\mathbf{a}}}F_{\mathrm{i}\underline{\mathbf{a}}})u(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbb{R}^n\backslash\Gamma. \tag{43}$$

Proofs of these relations are given in Appendix. The following lemma gives the fundamental solution of the Dirac operators with other types of vector potentials.

Lemma 4: *The fundamental solutions for the following operators are given as:*

$$\begin{array}{lll} (i) & D_{-\mathrm{i}\underline{\mathbf{a}}} = (D - \mathrm{i}\underline{\mathbf{a}}(\mathbf{y})), & \underline{\mathbf{K}}_{-\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y}) := e^{-\mathrm{i}u_0(\mathbf{x})}\,\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})\,e^{\mathrm{i}u_0(\mathbf{y})}, \\ (ii) & D_{\underline{\mathbf{a}}} = (D + \underline{\mathbf{a}}(\mathbf{y})), & \underline{\mathbf{K}}_{\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y}) := e^{u_0(\mathbf{x})}\,\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})e^{-u_0(\mathbf{y})}, \\ (iii) & D_{-\underline{\mathbf{a}}} = (D - \underline{\mathbf{a}}(\mathbf{y})), & \underline{\mathbf{K}}_{-\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y}) := e^{-u_0(\mathbf{x})}\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})e^{u_0(\mathbf{y})}. \end{array} \tag{44}$$

Proof: By replacing "$\underline{\mathbf{a}}$" by "$-\underline{\mathbf{a}}$" for (i), by "$-\mathrm{i}\underline{\mathbf{a}}$" for (ii), and by "$\mathrm{i}\underline{\mathbf{a}}$" for (iii) in Theorem 3 we obtain the expected results.

Proposition 5: *The Borel–Pompeiu's formula in the case $D_{\pm\mathbf{a}}$ has the following form:*

$$(F_{\pm\underline{\mathbf{a}}}u)(\mathbf{x}) + (T_{\pm\underline{\mathbf{a}}}D_{\pm\underline{\mathbf{a}}}u)(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in G \\ 0, & \mathbf{x} \in \mathbb{R}^n\backslash\overline{G} \end{cases}. \tag{45}$$

Proof is given in Appendix.

IV. THE GREEN FUNCTION OF THE HELMHOLTZ EQUATION

In this section, we will reconstruct the Helmholtz equation in the quaternion formalism. Then, we use the results of the Sec. III to obtain the Green function for the Helmholtz equation with nonuniform refraction index. The following lemma shows the quaternion representation of the Helmholtz equation.

Lemma 6: *The Helmholtz equation can be written in the quaternion form,*

$$\frac{1}{2}\left[D_{\pm\mathrm{i}\underline{\mathbf{a}}}\overline{D_{\mp\mathrm{i}\underline{\mathbf{a}}}} + \overline{D_{\mp\mathrm{i}\underline{\mathbf{a}}}}D_{\pm\mathrm{i}\underline{\mathbf{a}}}\right]\mathcal{G}_0(\mathbf{x}-\mathbf{y}) = (\triangle + |\mathbf{a}(\mathbf{y})|^2)\mathcal{G}_0(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \tag{46}$$

where $\mathcal{G}_0(\mathbf{x}-\mathbf{y})$ is the Green function, $\delta(\mathbf{x}-\mathbf{y})$ is the Dirac delta function and we used the first relation in (12) for biquaternion.

Proof: First, we have

$$D_{\pm\mathrm{i}\underline{\mathbf{a}}}\overline{D_{\mp\mathrm{i}\underline{\mathbf{a}}}}\mathcal{G}_0 := -(D \pm \mathrm{i}\underline{\mathbf{a}})(D \pm \mathrm{i}\underline{\mathbf{a}})\mathcal{G}_0 = [\triangle + |\mathbf{a}|^2 \pm \mathrm{i}(D\underline{\mathbf{a}} - \underline{\mathbf{a}}D)]\mathcal{G}_0 \tag{47}$$

and

$$\overline{D_{\mp\mathrm{i}\underline{\mathbf{a}}}}D_{\pm\mathrm{i}\underline{\mathbf{a}}}\mathcal{G}_0 := -(D \mp \mathrm{i}\underline{\mathbf{a}})(D \pm \mathrm{i}\underline{\mathbf{a}})\mathcal{G}_0 = [\triangle + |\mathbf{a}|^2 \mp \mathrm{i}(D\underline{\mathbf{a}} - \underline{\mathbf{a}}D)]\mathcal{G}_0. \tag{48}$$

The sum of above two equations yields Eq. (46).

Hence, the fundamental solution of the $D_{\mathrm{i}\underline{\mathbf{a}}}$ can be obtained as

$$D_{\mathrm{i}\underline{\mathbf{a}}}G_0(\mathbf{x} - \mathbf{y}) = \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y}), \tag{49}$$

so that

$$D_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}). \tag{50}$$

The following theorem gives the construction of its Green function.

Theorem 7: *The solution of the following equation with Dirichlet Boundary conditions:*

$$\begin{aligned} &D_{\mathrm{i}\underline{\mathbf{a}}}G_0 = \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}, && \text{in } G \subset \mathbb{R}^n, \\ &G_0\,|_\Gamma = 0, && \text{on } \Gamma, \end{aligned} \tag{51}$$

is

$$G_0 = (T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}). \tag{52}$$

Proof: By using the Borel–Pompeiu's formula (42) we have

$$G_0(\mathbf{x}) = (T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}})(\mathbf{y}) + (F_{\mathrm{i}\underline{\mathbf{a}}}G_0)(\mathbf{y}) = (T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}})(\mathbf{y}), \tag{53}$$

where we used $(F_{\mathrm{i}\underline{\mathbf{a}}}G_0)(\mathbf{y}) = 0$ because $G_0\,|_\Gamma\ = 0$. The verification can be done by letting $D_{\mathrm{i}\underline{\mathbf{a}}}$ act on the two sides of Eq. (51). Then, using formula (41) we have $D_{\mathrm{i}\underline{\mathbf{a}}}G_0 = D_{\mathrm{i}\underline{\mathbf{a}}}T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}} = \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}$, which yields Eq. (44).

The computation of $(T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}})$ in Appendix yields

$$G_0(\mathbf{x} - \mathbf{y}) := e^{\mathrm{i}u_0(\mathbf{x})}E_0(\mathbf{x} - \mathbf{y})e^{-\mathrm{i}u_0(\mathbf{y})}. \tag{54}$$

To verify we first use the following relation:

$$D\,e^{-\mathrm{i}u_0} = D\,e^{-\mathrm{i}u_0} = -\mathrm{i}(D\,u_0)\,e^{-\mathrm{i}u_0} = -\mathrm{i}\underline{\mathbf{a}}\,e^{-\mathrm{i}u_0}, \tag{55}$$

where D acts only on $\mathbf{y}$ and $Du_0 = \underline{\mathbf{a}}$. Thus, the following equation holds:

$$D_{\mathrm{i}\underline{\mathbf{a}}}G_0(\mathbf{x} - \mathbf{y}) = e^{\mathrm{i}u_0(\mathbf{x})}\underline{\mathbf{K}}(\mathbf{x} - \mathbf{y})e^{-\mathrm{i}u_0(\mathbf{y})}. \tag{56}$$

The following proposition states the above result for G_0.

Proposition 8: *The solution of the following equation with Dirichlet boundary conditions:*

$$\begin{aligned} &D_{\mathrm{i}\underline{\mathbf{a}}}G_0(\mathbf{x}) = \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}), && \mathbf{x} \in G \subset \mathbb{R}^N, \\ &G_0(\mathbf{x})\,|_\Gamma = 0, && \mathbf{x} \in \Gamma, \end{aligned} \tag{57}$$

is obtained by Eq. (54).

Now, we would like to obtain the Green function for the Helmholtz equation. We first show that G_0 is a solution for the Helmholtz equation. Here we use real analysis because the Helmholtz equation is a scalar equation. To prove this we have to show that

$$\Im = \frac{1}{\sigma_n}\int_G (\triangle + |\mathbf{a}(\mathbf{y})|^2)G_0(\mathbf{x} - \mathbf{y})d\mathbf{y} = 1. \tag{58}$$

First, we substitute G_0 into the above equation by (54). The gradient of G_0 yields

$$\begin{aligned} \nabla G_0(\mathbf{x} - \mathbf{y}) &= \nabla\left(e^{\mathrm{i}u_0(\mathbf{x})}E_0(\mathbf{x} - \mathbf{y})e^{-\mathrm{i}u_0(\mathbf{y})}\right) \\ &= e^{\mathrm{i}u_0(\mathbf{x})}(\nabla E_0)e^{-\mathrm{i}u_0(\mathbf{y})} - \mathrm{i}\nabla u_0 e^{\mathrm{i}u_0(\mathbf{x})}E_0 e^{-\mathrm{i}u_0(\mathbf{y})}, \end{aligned} \tag{59}$$

where ∇ act on $\mathbf{y}$. The divergence acting on the above equation yields

$$\triangle G_0 = e^{\mathrm{i}u_0(\mathbf{x})}(\triangle E_0)e^{-\mathrm{i}u_0(\mathbf{y})} - 2\mathrm{i}e^{\mathrm{i}u_0(\mathbf{x})}(\nabla u_0 \cdot \nabla E_0)e^{-\mathrm{i}u_0(\mathbf{y})} - |\nabla u_0|^2 G_0, \tag{60}$$

where we considered in the quaternion language $\underline{\mathbf{a}} = Du_0 = \underline{\nabla u_0}$ as a *monogenic function* or analytic function. Thus, we used $\nabla \cdot \mathbf{a} = 0$ from (4). Now, (58) is written as

$$\mathfrak{I} = \int_G e^{\mathrm{i}u_0(\mathbf{x})}(\triangle E_0)e^{-\mathrm{i}u_0(\mathbf{y})}d\mathbf{y} - 2\mathrm{i}\int_G e^{\mathrm{i}u_0(\mathbf{x})}(\nabla u_0 \cdot \nabla E_0)e^{-\mathrm{i}u_0(\mathbf{y})}d\mathbf{y}. \tag{61}$$

In the above equation the first integral is equal to 1 because of "$\triangle E_0(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y})$." Using the first Green's identity, i.e.,

$$\int_G \phi_0 \triangle \psi_0 d\mathbf{y} + \int_G \nabla\psi_0 \cdot \nabla\phi_0 d\mathbf{y} = \int_\Gamma \phi_0(\nabla\psi_0 \cdot \alpha(\mathbf{y}))d\Gamma_{\mathbf{y}}, \tag{62}$$

where $\alpha(\mathbf{y})$ is outward normal unit vector on $\Gamma_{\mathbf{y}}$. The second integral in (61) from the first Green's identity is equal to

$$\begin{aligned}\int_G e^{\mathrm{i}u_0(\mathbf{x})}(\nabla u_0 \cdot \nabla E_0)e^{-\mathrm{i}u_0(\mathbf{y})}d\mathbf{y} &= \int_G e^{\mathrm{i}u_0(\mathbf{x})}(\triangle E_0)e^{-\mathrm{i}u_0(\mathbf{y})}d\mathbf{y} \\ &\quad - \int_{\Gamma_{\mathbf{y}}} e^{\mathrm{i}u_0(\mathbf{x})}(\nabla E_0 \cdot \alpha(\mathbf{y}))e^{-\mathrm{i}u_0(\mathbf{y})}d\Gamma_{\mathbf{y}}.\end{aligned} \tag{63}$$

The second integral in the right-hand-side of above equation is the Gauss's law integral on $\Gamma_{\mathbf{y}}$ and it is obtained as follows:

$$\begin{aligned}&\int_{\Gamma_{\mathbf{y}}} e^{\mathrm{i}u_0(\mathbf{x})}(\nabla E_0 \cdot \alpha(\mathbf{y}))e^{-\mathrm{i}u_0(\mathbf{y})}d\Gamma_{\mathbf{y}} \\ &= \int_{\Gamma_\epsilon} e^{\mathrm{i}u_0(\mathbf{x})}(\nabla E_0 \cdot \alpha(\mathbf{y}))e^{-\mathrm{i}u_0(\mathbf{y})}d\Gamma_\epsilon + \lim_{\epsilon\to 0}\int_{\partial B_\epsilon} e^{\mathrm{i}u_0(\mathbf{x})}\left(\frac{1}{\sigma_n \epsilon^{n-1}}\right)e^{-\mathrm{i}u_0(\mathbf{x}-\epsilon)}\epsilon^{n-1}d\Omega_\epsilon \\ &= 0 + 1,\end{aligned} \tag{64}$$

where $G_\epsilon \equiv G \backslash B_\epsilon$ and $\partial G_\epsilon = \Gamma_\epsilon$ with radius $\epsilon \to 0$. Thus, from the two last equations the second integral in (61) is equal to zero and (58) is proved. The similar computation shows that the complex conjugate of G_0, i.e., $\overline{G_0} = e^{-\mathrm{i}u_0(\mathbf{x})}E_0 e^{\mathrm{i}u_0(\mathbf{y})}$, is another solution for the Helmholtz equation. Thus, the superposition of the G_0 and $\overline{G_0}$ is the solution of the Green function. Now, following theorem gives the Green function of the Helmholtz equation (2).

Theorem 9: *If* $\underline{\mathbf{a}}$ *is a monogenic function. The Green function of the Helmholtz equation* (2) *is given as follows:*

$$\mathcal{G}_0 = c_0^{(1)}G_0 + c_0^{(2)}\overline{G_0}, \tag{65}$$

where $c_0^{(1)}$ *and* $c_0^{(2)}$ *are two scalar constants.*

A. The Green function for the $(\nabla - |\mathbf{a}|^2)$-operator

The following theorems give the Green function of the $(\nabla - |\mathbf{a}|^2)$-operator.

Theorem 10: *The* $(\nabla - |\mathbf{a}|^2)$*-operator has the following form in the quaternion analysis:*

$$-\frac{1}{2}\left[D_{-\underline{\mathbf{a}}}D_{\underline{\mathbf{a}}} + D_{\underline{\mathbf{a}}}D_{-\underline{\mathbf{a}}}\right]\widetilde{\mathcal{G}}_0(\mathbf{x}-\mathbf{y}) = (\triangle - |\mathbf{a}(\mathbf{y})|^2)\widetilde{\mathcal{G}}_0(\mathbf{x}-\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}), \tag{66}$$

where $\widetilde{\mathcal{G}}_0$ is the Green function.

Theorem 11: *If* $\underline{\mathbf{a}}$ *is a monogenic function. The above Green function of the* $(\nabla - |\mathbf{a}|^2)$*-operator obtains as*

$$\widetilde{\mathcal{G}}_0(\mathbf{x}-\mathbf{y}) = e^{u_0(\mathbf{x})}E_0(\mathbf{x}-\mathbf{y})e^{-u_0(\mathbf{y})}. \tag{67}$$

The proofs of the above theorems are similar to the case of Helmhotz's equation.

In Sec. V we try to compare our results to results of other authors which have studied this problem up to now.

V. COMPARISON WITH OTHER RESULTS

In this section, we will show that our results recover the known results for both the Green's function of the Helmholtz equation and the fundamental solution of the Dirac operator with the nonuniform vector potential. The forth example is not usable for the Helmholtz equation because the vector potential is not monogenic. We note to that the Helmholtz operator and $(\triangle - |\mathbf{a}|^2)$-operator are exchangeable when we define the vector potential as "i $\underline{\mathbf{a}}$" or "$\underline{\mathbf{a}}$".

A. Case with constant refraction index

The Dirac operator with a constant function "$(D + \mathrm{i}\,\underline{\mathbf{a}})$, $\underline{\mathbf{a}} = \text{Const.}$" has been studied by Refs. 9 and 10. In this step, we will see that our result contains the result $\underline{\mathbf{a}}$ constant. Now, we have

$$u_0(\mathbf{x}) = (T\,\underline{\mathbf{a}})(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a}). \tag{68}$$

It is easy to verify the above equation by acting D on it,

$$D(T\,\underline{\mathbf{a}})(\mathbf{x}) = Du_0(\mathbf{x}) = D(\mathbf{x} \cdot \mathbf{a}) = \underline{\nabla(x_1 a_1 + x_2 a_2 + x_3 a_3)} = \underline{\mathbf{a}}. \tag{69}$$

Thus, we have

$$\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x} - \mathbf{y}) = \frac{1}{\sigma_n} e^{\mathrm{i}(\mathbf{x}\cdot\mathbf{a})} \left(\frac{\underline{\mathbf{x}} - \underline{\mathbf{y}}}{|\mathbf{x} - \mathbf{y}|^n} \right) e^{-\mathrm{i}(\mathbf{y}\cdot\mathbf{a})}, \tag{70}$$

which replacing i$\mathbf{a}$ by $\mathbf{a}$ gives the same result with a constant difference in the exponent as which given by Ref. 10.

"$\underline{\mathbf{a}} = \text{Const.}$" is a monogenic function, i.e. $D\underline{\mathbf{a}} = 0$. Thus, the Green function of the Helmholtz equation $(\triangle + |\mathbf{a}|^2)\mathcal{G}_0(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ for $|\mathbf{a}|^2 = \text{Const.}$ is obtained by Theorem 9, where for three dimensions, $n = 3$, we have

$$G_0(\mathbf{x} - \mathbf{y}) = -\frac{1}{4\pi} e^{\mathrm{i}\mathbf{x}\cdot\mathbf{a}} \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) e^{-\mathrm{i}\mathbf{y}\cdot\mathbf{a}}. \tag{71}$$

The above solution is the result obtained by Ref. 17.

B. Case with $|\mathbf{x}|^{-4}$ refraction index

In Ref. 5, the author considered a vector potential as a regular function in $L_p(G, H)$, which can be approximated by a right-linear combination of functions of the type

$$\left\{ \frac{\underline{\mathbf{x}} - \underline{\mathbf{x}}^{(i)}}{|\mathbf{x} - \mathbf{x}^{(i)}|^3}, \qquad i = 1, 2, \ldots \right\}, \tag{72}$$

where $\{\underline{\mathbf{x}}^i, i = 1, 2, \cdots\}$ is the dense on the closed surface Γ' outside of G and parallel to $\partial G = \Gamma$. Then, Dirac operators disturbed by regular potentials may be reduced to considering of operators of the type

$$D + \frac{\underline{\mathbf{x}} - \underline{\mathbf{x}}^{(i)}}{|\mathbf{x} - \mathbf{x}^{(i)}|^3}. \tag{73}$$

Thus, if we consider the above vector potential then we have

$$u_0(\mathbf{x}) = (T\,\underline{\mathbf{a}})(\underline{\mathbf{x}}) = -\frac{1}{|\mathbf{x} - \mathbf{x}^{(i)}|}. \tag{74}$$

Finally, the fundamental solution of the operator in (73) is given by

$$e^{-\frac{1}{|\mathbf{x}-\mathbf{x}^{(i)}|}}\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})e^{\frac{1}{|\mathbf{y}-\mathbf{y}^{(i)}|}}, \tag{75}$$

this is the same results which obtained by Ref. 5.

In addition, $D\left(\frac{\underline{\mathbf{x}-\mathbf{x}}^{(i)}}{|\mathbf{x}-\mathbf{x}^{(i)}|^3}\right)=0$. Then, the Green function for the $(\nabla-|\mathbf{x}|^4)$-operator, $\widetilde{\mathcal{G}}_0$, is given by applying the Theorem 11, for the $n=3$, we have

$$\widetilde{\mathcal{G}}_0(\mathbf{x})=e^{-\frac{1}{|\mathbf{x}-\mathbf{x}^{(i)}|}}\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}e^{-\frac{1}{|\mathbf{y}-\mathbf{y}^{(i)}|}}. \tag{76}$$

C. X-ray emission with nonuniform attenuation

By defining $\mathbf{a}:=a_0\mathbf{n}$, the following quaternion equation:

$$\underline{\mathbf{n}}D_{\underline{\mathbf{a}}}u_0(\mathbf{x})=\underline{\mathbf{n}}(D+\underline{\mathbf{a}})u_0(\mathbf{x})=f_0(\mathbf{x}) \tag{77}$$

leads us to a set of two equations for the u_0 in real analysis

$$\begin{aligned}&(\mathbf{n}\cdot\nabla_{\mathbf{x}}+a_0(\mathbf{x}))u_0(\mathbf{x})=-f_0(\mathbf{x}),\\&(\mathbf{n}\times\nabla_{\mathbf{x}})u_0(\mathbf{x})=0.\end{aligned} \tag{78}$$

The first one being exactly the stationary transport equation for photons in a nonhomogeneous area. The photon transport in the $\mathbf{n}$-direction of the x-ray source located at $\mathbf{x}$ required that $\mathbf{y}=\mathbf{x}+\mathbf{n}t$, where $t\in\mathbb{R}^+$. Consequently, the volume element $d\mathbf{y}$, in spherical coordinates becomes $d\mathbf{y}=t^2dtd\Omega_{\mathbf{y}}$, where $d\Omega_{\mathbf{y}}$ is the area element of the unit sphere $\Omega_{\mathbf{y}}$ in $\mathbb{R}^3$. Then, we have

$$\begin{aligned}u_0(\mathbf{x})=(T\underline{\mathbf{a}})(\mathbf{x})&=\frac{1}{4\pi}\int_{\Omega_{\mathbf{n}}}\int_{\mathbb{R}^+}\frac{\underline{\mathbf{n}}t}{|\mathbf{n}t|^3}\underline{\mathbf{n}}a_0(\mathbf{x}+\mathbf{n}t)t^2d\Omega_{\mathbf{y}}dt\\&=-\frac{1}{4\pi}\int_{\Omega_{\mathbf{y}}}\left(\int_{\mathbb{R}^+}a_0(\mathbf{x}+\mathbf{n}t)dt\right)d\Omega_{\mathbf{y}}=(\mathfrak{D}a_0)(\mathbf{x}),\end{aligned} \tag{79}$$

where we have $\underline{\mathbf{a}}=\underline{\mathbf{n}}a_0$ and $\underline{\mathbf{nn}}=-1$. Then, fundamental solution of the $D_{\underline{\mathbf{n}}a_0}$-operator as

$$\underline{\mathbf{K}}_{\underline{\mathbf{n}}a_0}(\mathbf{x}-\mathbf{y})=D_{\underline{\mathbf{n}}a_0}G_0(\mathbf{x}-\mathbf{y})=e^{(\mathfrak{D}a_0)(\mathbf{x})}\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})e^{-(\mathfrak{D}a_0)(\mathbf{y})}. \tag{80}$$

This is the same result as in Ref. 8. Then, G_0 is equal to

$$G_0(\mathbf{x}-\mathbf{y})=e^{(\mathfrak{D}a_0)(\mathbf{x})}E_0(\mathbf{x}-\mathbf{y})e^{-(\mathfrak{D}a_0)(\mathbf{y})}. \tag{81}$$

Note that, in general, $D\mathbf{a}=\underline{\mathbf{n}}Da_0\neq 0$. Thus, G_0 is not generally the Green function of the $(\triangle-|a_0|^2)$-operator (note that $\underline{\mathbf{nn}}=-1$). For the case, $D\mathbf{a}=0$ the above G_0 is the Green function.

D. Case with $|\mathbf{x}|^2$ refraction index

The Teodorescu transform of $\underline{\mathbf{x}}$, by using integral in (A10) for $m=1$, and $k=2$ is given as follows:

$$u_0(\mathbf{x})=(T\underline{\mathbf{x}})(\mathbf{x})=\frac{1}{2}|\mathbf{x}|^2. \tag{82}$$

Then, fundamental solution of the $D_{\underline{\mathbf{x}}}$-operator as

$$\underline{\mathbf{K}}_{\underline{\mathbf{x}}}(\mathbf{x}-\mathbf{y})=D_{\underline{\mathbf{x}}}G_0(\mathbf{x}-\mathbf{y})=e^{\frac{1}{2}|\mathbf{x}|^2}\underline{\mathbf{K}}(\mathbf{x}-\mathbf{y})e^{-\frac{1}{2}|\mathbf{y}|^2} \tag{83}$$

and G_0 is given by

$$G_0(\mathbf{x}-\mathbf{y})=e^{\frac{1}{2}|\mathbf{x}|^2}E_0(\mathbf{x}-\mathbf{y})e^{-\frac{1}{2}|\mathbf{y}|^2}. \tag{84}$$

Because of $D\underline{\mathbf{x}}=-3\neq 0$, i.e., $\underline{\mathbf{x}}$ is not a monogenic function. Thus, G_0 is not the Green function of the $(\nabla-|\mathbf{x}|^2)$-operator.

VI. CONCLUSION

The Helmholtz equation with nonuniform reflection index has a vast application domain in physics and engineering. Up to now, in the author understanding there is no exact solution for it. In this paper, using quaternion analysis, we have presented an analytical solution for the inhomogeneous Helmholtz equation with nonhomogeneous refraction index for a large number of cases. Then, we have given some examples of application along with some other results. The results obtained contains the case with the constant refraction index.

We have also obtained the exact fundamental solutions for the Dirac operators ($D_{\pm\underline{\mathbf{a}}} = D \pm \underline{\mathbf{a}}(\mathbf{x})$) and ($D_{\pm i\underline{\mathbf{a}}} = D \pm i\underline{\mathbf{a}}(\mathbf{x})$), without any approximation and restriction conditions. We use these results to construct the solution of the Helmholtz in the case that $\underline{\mathbf{a}}$ is a monogenic function. Our results in quaternion recover the constant refraction index case and show that it is the same as in Refs. 9 and 10. They also gave the same results as in Ref. 5 when $\underline{\mathbf{a}}$ is monogenic function and gradient of nonharmonic function and the results obtained by Ref. 8 for the x-ray emission.

The fundamental solution for the following Dirac operator $D - \lambda + \underline{\mathbf{a}}$, where $\lambda \in \mathbb{H}(\mathbb{R}^4)$ is a *constant*, is easily given by $e^{u_0(\mathbf{x})}\underline{\mathbf{K}}^{\lambda}(\mathbf{x}-\mathbf{y})e^{-u_0(\mathbf{y})}$ where $\underline{\mathbf{K}}^{\lambda}$ is introduced by Ref. 9. The fundamental solution of the Dirac operator with a quaternion index of refraction $a(\mathbf{x}) = a_0(\mathbf{x}) + \underline{\mathbf{a}}(\mathbf{x})$ can be the subject of the future research. That can lead to the general analytical solution for the Helmholtz equation without monogenicity condition.

APPENDIX

In this appendix, we give the proofs of Eqs. (41)–(43), (45), and (54). Here we consider $n \geq 3$.

- We first start by Eq. (41). Before starting, we change our notation in this proof to separate between the differentiation on $\mathbf{x}$ and $\mathbf{y}$. Hence, we use $D^{\mathbf{x}}_{i\underline{\mathbf{a}}}$ for acting on $\mathbf{x}$ and $D^{\mathbf{y}}_{i\underline{\mathbf{a}}}$ for acting on $\mathbf{y}$. Thus, the relation between these two differentiations when acting on $G_0(\mathbf{x}-\mathbf{y})$ is given as follows:

$$-D^{\mathbf{x}}_{-i\underline{\mathbf{a}}}\left(e^{iu_0(\mathbf{x})}E_0(\mathbf{x}-\mathbf{y})e^{-iu_0(\mathbf{y})}\right) = D^{\mathbf{y}}_{i\underline{\mathbf{a}}}\left(e^{iu_0(\mathbf{x})}E_0(\mathbf{x}-\mathbf{y})e^{-iu_0(\mathbf{y})}\right) = \underline{\mathbf{K}}_{i\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y}). \tag{A1}$$

  Let, $\mathbf{x} \in G$ we have

$$\begin{aligned}(D^{\mathbf{x}}_{i\underline{\mathbf{a}}}T_{i\underline{\mathbf{a}}}f)(\mathbf{x}) &= D^{\mathbf{x}}_{i\underline{\mathbf{a}}}\int_G \underline{\mathbf{K}}_{i\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})f(\mathbf{y})d\mathbf{y} \\ &= -D^{\mathbf{x}}_{i\underline{\mathbf{a}}}\int_G \left[D^{\mathbf{x}}_{-i\underline{\mathbf{a}}}G_0(\mathbf{x}-\mathbf{y})\right]f(\mathbf{y})d\mathbf{y} \\ &= (D^{\mathbf{x}}_{i\underline{\mathbf{a}}}D^{\mathbf{x}}_{-i\underline{\mathbf{a}}})\left(-\int_G e^{iu_0(\mathbf{x})}\frac{1}{\sigma_n(2-n)}\frac{1}{|\mathbf{x}-\mathbf{y}|^{n-2}}e^{-iu_0(\mathbf{y})}f(\mathbf{y})d\mathbf{y}\right).\end{aligned} \tag{A2}$$

  The last integral is just the volume potential which solves the $(D^{\mathbf{x}}_{i\underline{\mathbf{a}}}D^{\mathbf{x}}_{-i\underline{\mathbf{a}}})\phi(\mathbf{x}) = f(\mathbf{x})$ equation. Hence, $(D^{\mathbf{x}}_{i\underline{\mathbf{a}}}T_{i\underline{\mathbf{a}}}f)(\mathbf{x}) = f(\mathbf{x})$.

  For $(D^{\mathbf{x}}_{i\underline{\mathbf{a}}}D^{\mathbf{x}}_{i\underline{\mathbf{a}}})G_0(\mathbf{x}) = 0$ in $\mathbf{x} \in \mathbb{R}^n\backslash\overline{G}$, it follows $(D^{\mathbf{x}}_{i\underline{\mathbf{a}}}T_{i\underline{\mathbf{a}}}f)(\mathbf{x}) = 0$.

  Another method to prove is giving by using the differentiation rule for weak singular integrals. The method used in Ref. 12 to drive $DTf = f$.
- The derivation of Borel–Pompeiu's formula (42) is given as follows: we have

$$(T_{i\underline{\mathbf{a}}}D_{i\underline{\mathbf{a}}})f(\mathbf{x}) = \int_G \underline{\mathbf{K}}_{i\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{i\underline{\mathbf{a}}}f(\mathbf{y})d\mathbf{y} = \lim_{\epsilon\to 0}\int_{G_\epsilon}\underline{\mathbf{K}}_{i\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{i\underline{\mathbf{a}}}f(\mathbf{y})d\mathbf{y} \tag{A3}$$

  with $G_\epsilon = G\backslash\{\mathbf{y} : |\mathbf{x}-\mathbf{y}| \leq \epsilon\} = G\backslash B_\epsilon(\mathbf{x})$. Let $S_\epsilon = \partial B_\epsilon$ and $\Gamma = \partial G$. Before, we use the following relation:

$$\int_G [(uD_{,r})v + u(Dv)]d\mathbf{y} = \int_\Gamma u\underline{\alpha}v\,d\Gamma_{\mathbf{y}}, \tag{A4}$$

where $D_{,r}$ means that the operator is acting from right-hand-side, i.e., $(vD_{,r}) = \sum_{m=1}^{n}\left(\frac{\partial u}{\partial y_i}\right)\iota_m$ and $(Dv) = \sum_{m=1}^{n}\iota_m\left(\frac{\partial v}{\partial y_i}\right)$. The proof is[12]

$$\int_G [uDv + (uD_{,r})v]d\mathbf{y} = \int_G \sum_{m=1}^{n}\left(\frac{\partial u}{\partial y_m}\iota_m v + u\iota_m\frac{\partial v}{\partial y_m}\right)d\mathbf{y} = \sum_{m=1}^{n}\int_G \frac{\partial}{\partial y_m}(u\iota_m v)d\mathbf{y} \tag{A5}$$

using Gauss's theorem in the last equation gives right-hand-side (A4). Now, replacing u by $\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}$ gives $(uD_{,r})$ as

$$(\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}D_{,r}) = e^{\mathrm{i}u_0(\mathbf{x})}\underline{\mathbf{K}}(e^{-\mathrm{i}u_0(\mathbf{y})}D_{,r}) + e^{\mathrm{i}u_0(\mathbf{x})}(\underline{\mathbf{K}}D_{,r})e^{-\mathrm{i}u_0(\mathbf{y})} = \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(-\mathrm{i}\underline{\mathbf{a}}) + e^{\mathrm{i}u_0(\mathbf{x})}(\underline{\mathbf{K}}D_{,r})e^{-\mathrm{i}u_0(\mathbf{y})}. \tag{A6}$$

It follows

$$(\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}D_{\mathrm{i}\underline{\mathbf{a}},r}) = [\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(D_{,r} + \mathrm{i}\underline{\mathbf{a}})] = e^{\mathrm{i}u_0(\mathbf{x})}(\underline{\mathbf{K}}D_{,r})e^{-\mathrm{i}u_0(\mathbf{y})}. \tag{A7}$$

In this step, we use above results to prove the Borel–Pompeiu's formula (42). We have

$$(T_{\mathrm{i}\underline{\mathbf{a}}}D_{\mathrm{i}\underline{\mathbf{a}}})f(\mathbf{x}) = \int_G \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{\mathrm{i}\underline{\mathbf{a}}}f(\mathbf{y})d\mathbf{y} = \lim_{\epsilon\to 0}\int_{G_\epsilon}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{\mathrm{i}\underline{\mathbf{a}}}f(\mathbf{y})d\mathbf{y} \tag{A8}$$

with $G_\epsilon = G\backslash\{\mathbf{y} : |\mathbf{x}-\mathbf{y}| \le \epsilon\} = G\backslash B_\epsilon(\mathbf{x})$. Let $S_\epsilon = \partial B_\epsilon$ and $\Gamma = \partial G$. Thus, Eq. (A8) is written as

$$\begin{aligned}\int_{G_\epsilon}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{\mathrm{i}\underline{\mathbf{a}}}f(\mathbf{y})d\mathbf{y} = &-\int_{G_\epsilon}\left[\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{\mathrm{i}\underline{\mathbf{a}},r}\right]f(\mathbf{y})d\mathbf{y}\\ &+\int_\Gamma \underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})\underline{\alpha}(\mathbf{y})f(\mathbf{y})d\Gamma_{\mathbf{y}},\end{aligned}$$

when $\mathbf{y}\in G_\epsilon, \epsilon\to 0$ then, $[\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}}(\mathbf{x}-\mathbf{y})D_{\mathrm{i}\underline{\mathbf{a}},r}] = e^{\mathrm{i}u_0(\mathbf{x})}\delta(\mathbf{x}-\mathbf{y})e^{-\mathrm{i}u_0(\mathbf{y})}$. Hence the first integral in the right-hand-side is equal to $f(\mathbf{x})$. When $\mathbf{y}$ is not in G_ϵ then, the first integral in the right-hand-side is equal to zero. The second integral for $\epsilon\to 0$ is the definition of $F_{\mathrm{i}\underline{\mathbf{a}}}f$. Thus, Eq. (45) is proved.

- Acting the operator $D_{\mathrm{i}\underline{\mathbf{a}}}$ on the Eq. (42) and using equation (41), then, we obtain Eq. (43).
- Here we show that the Eq. (52) gives (54). G_0 gives as follows:

$$\begin{aligned}G_0(\mathbf{x}) &= (T_{\mathrm{i}\underline{\mathbf{a}}}\underline{\mathbf{K}}_{\mathrm{i}\underline{\mathbf{a}}})(\mathbf{x})\\ &= \frac{1}{\sigma_n}\int_{G\subset\mathbb{R}^n} e^{\mathrm{i}u_0(\mathbf{x})}\ \frac{(\underline{\mathbf{x}}-\underline{\mathbf{y}})}{|\mathbf{x}-\mathbf{y}|^n}\ e^{-\mathrm{i}u_0(\mathbf{y})}e^{\mathrm{i}u_0(\mathbf{y})}\ \frac{\underline{\mathbf{y}}}{|\mathbf{y}|^n}\ e^{-\mathrm{i}u_0(0)}\,d\mathbf{y}\\ &= e^{\mathrm{i}u_0(\mathbf{x})}\left(\frac{1}{\sigma_n}\int_{G\subset\mathbb{R}^n}\frac{(\underline{\mathbf{x}}-\underline{\mathbf{y}})}{|\mathbf{x}-\mathbf{y}|^n}\ \frac{\underline{\mathbf{y}}}{|\mathbf{y}|^n}\,d\mathbf{y}\right)e^{-\mathrm{i}u_0(0)}\\ &= e^{\mathrm{i}u_0(\mathbf{x})}\left(T\frac{\underline{\mathbf{y}}}{|\mathbf{y}|^n}\right)e^{-\mathrm{i}u_0(0)}.\end{aligned} \tag{A9}$$

By using the following integral formula in Ref. 12:

$$T(\underline{\mathbf{x}}|\mathbf{x}|^{(\frac{k}{m}-2)}) = \begin{cases}\frac{m}{k}|\mathbf{x}|^{\frac{k}{m}}, & m,k\neq 0\\ \ln|\mathbf{x}|, & m\neq 0, k=0\end{cases}. \tag{A10}$$

It is easy to verify the above formula by action Dirac operator, D on the results which must give the $(\underline{\mathbf{x}}|\mathbf{x}|^{\frac{k}{m}})$ from this fact that T is a right inverse operator of the D, i.e., $DTu = u$.[13] Finally, $G_0(\mathbf{x}-\mathbf{y})$, for $n\neq 2$, is obtained as (54).

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