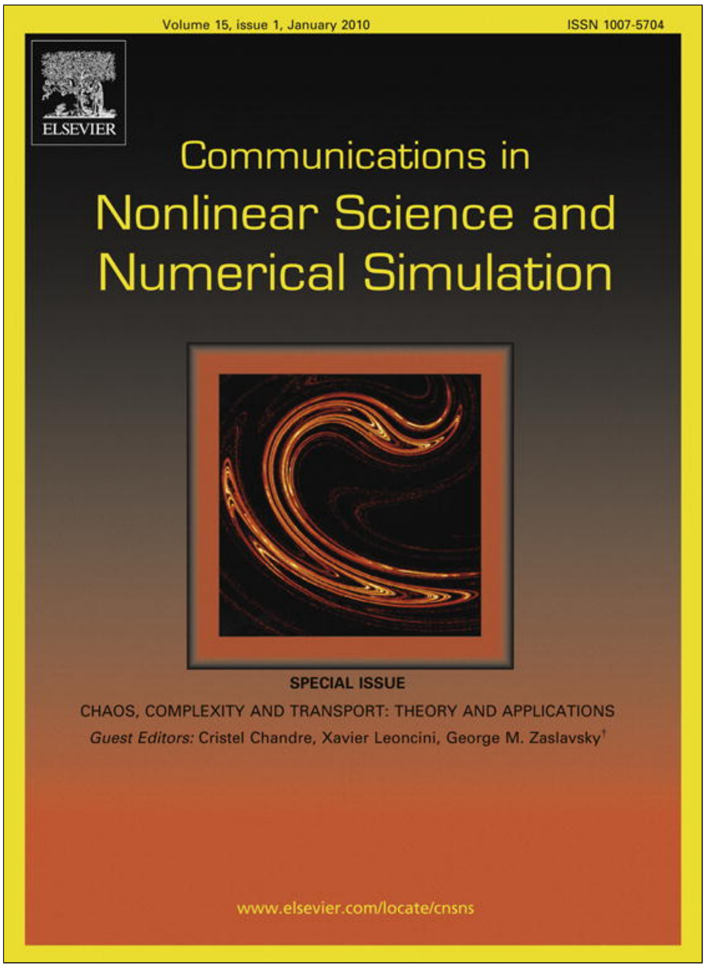
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# Dissipative dynamics of the kaon decay process

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The quantum description of the unstable systems is investigated starting from the Schrödinger equation and using Hamiltonian describing discrete levels interacting with a continuum. This approach is applied to kaons decay processes by using a simple Hamiltonian model. Then, *CP*-violation and decoherence properties are displayed and studied.



## 1. Introduction

The description of the decay processes of unstable quantum systems is a long standing problem [1]. It is generally admitted that the decay probability of radioactive nuclei follows an exponential law which defines the lifetime of the nucleus. Most elementary particles are unstable so a quantum-mechanical description of the decay process was the subject of many investigations. After the general treatment given by Weisskopf and Wigner [2] in order to obtain the exponential law, Khalfin [3] has pointed that, for a quantum system with energy spectrum bounded from below, the decay could not be exponential for large times. It was also observed [4] that short-time behavior of decaying systems could not be exponential and this led to the so-called Zeno effect [5,6]. The departure from the exponential type behavior has been experimentally observed (see [7]).

In this paper, we consider a model of decay which is applied to the case of neutral kaon decay. It is well known [8] that kaons appears in pair $K^0$ and $\overline{K}^0$ each one being conjugated of the other. The decay processes of $K^0$ and $\overline{K}^0$ communicate so that they correspond to combinations of two orthogonal decaying modes $K_1$ and $K_2$, that are distinguished by their lifetime. The discovery of the small *CP*-violation effect, led to non-orthonormality of the short- and long-lived decay modes, now denoted $K_S$ and $K_L$, slightly different from $K_1$ and $K_2$ and depending on a *CP*-violation parameter $\epsilon$. Lee, Oehme and Yang (LOY) [9] proposed a generalization of the Weisskopf and Wigner theory in order to account the empirical data. But, Khalfin again using a Weisskopf and Wigner type theory with lower bounded Hamiltonian, corrected the parameter $\epsilon$. This estimation has been presented and reexamined in [7,10].

Our approach to these problems considers a different model expressing the interaction between $|K_1\rangle$ and $|K_2\rangle$ modes and the continuum. Our approach is distinct from the Weisskopf–Wigner approach in which the approximation consists in

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extending the energy spectrum from $-\infty$ to $+\infty$. It is based on an exactly solvable Hamiltonian model of weak interaction with a positive energy and where the parameter $\alpha$ of the cutoff plays the role of the control of the physical energy band, which could not be arbitrarily extended. The interaction plays also a fundamental role, through the coupling constants and the parameter $\alpha$, in the expression of the effective Hamiltonian (2.34), as well as in the derivation of the *CP*-violation parameter as shown in the formulae (2.35), (2.36) and (2.38). This provides a better estimate of the $\epsilon$ parameter. We also consider decoherence problem in the light of this model in which the continuum is interpreted as an environment.

To be more precise, let us present the fundamental ideas of the theory of spontaneous emission an atom interacting with the electromagnetic field, given by Weisskopf and Wigner. This treatment aims to obtain an exponential time dependence for states by integrating over continuum energy. It is to be noted that the interval of integration over continuum energy is changed to $]-\infty,\infty[$. Now, in the case of two-level atom the Hamiltonian is

$$H = H_0 + H_I, \tag{1.1}$$

$$H_0 = \omega_1|1\rangle\langle 1| + \omega_2|2\rangle\langle 2| + \sum_k \omega_k|k\rangle\langle k|, \tag{1.2}$$

$$H_I = \sum_{i=1}^{2}\sum_k [V_i^*(\omega_k)|i\rangle\langle k| + V_i(\omega_k)|k\rangle\langle i|], \tag{1.3}$$

where we consider $\hbar = c = 1$ and $V_i(\omega_k)$ a factor form. Therefore, the Schrödinger equation is

$$\mathrm{i}\frac{\partial\psi(t)}{\partial t} = H\psi(t), \tag{1.4}$$

where $\psi(t) \equiv \{a_1(t)|1\rangle, a_2(t)|2\rangle, b(\omega_k,t)|k\rangle\}$. Thus, we have

$$\mathrm{i}\frac{\partial a_1(t)}{\partial t} = \omega_1 a_1(t) + \sum_k V_1^*(\omega_k)b(\omega_k,t), \tag{1.5}$$

$$\mathrm{i}\frac{\partial a_2(t)}{\partial t} = \omega_2 a_2(t) + \sum_k V_2^*(\omega_k)b(\omega_k,t) \tag{1.6}$$

and

$$\mathrm{i}\frac{\partial b(\omega_k,t)}{\partial t} = \omega_k b(\omega_k,t) + [V_1(\omega_k)a_1(t) + V_2(\omega_k)a_2(t)]. \tag{1.7}$$

Usually one makes some approximations [11,12]. First, replacing summation with integration as

$$\sum_k V_i(\omega_k) \to \int_0^\infty d\omega\, v_i(\omega), \tag{1.8}$$

i.e. we assume that the modes of fields are closely spaced. Then, we have to assume the variations of $v_1(\omega)$ and $v_2(\omega)$ over $\omega$ are negligible with $|\omega| \lesssim$ "uncertainty of the original state energy", i.e. $v_i(\omega) \approx v_i$ $(i = 1,2)$. Also, another assumption is: the lower limit of integration over $\omega$ is replaced by $-\infty$. Finally, one obtains the following Markovian form of the reduced Schrödinger equation, see e.g. [13]

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = (M - \mathrm{i}\Gamma)\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}, \tag{1.9}$$

where

$$M - \mathrm{i}\Gamma = \begin{pmatrix} \omega_1 - \mathrm{i}2\pi|v_1|^2 & -\mathrm{i}2\pi v_1^* v_2 \\ -\mathrm{i}2\pi v_1 v_2^* & \omega_2 - \mathrm{i}2\pi|v_2|^2 \end{pmatrix}. \tag{1.10}$$

Weisskopf–Wigner treatment also assumes that the unstable state is of the form [11,12]

$$\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} = \psi e^{-\nu t}, \tag{1.11}$$

then subtituting into (1.9) it gives

$$(M - \mathrm{i}\Gamma)\psi = \nu\psi. \tag{1.12}$$

In order to apply this theory to the decay of the neutral kaons, LOY formulated a generalization of the Weisskopf–Wigner theory. Before, to introduce the LOY theory, it is useful to recall some properties of kaons.

Kaons are bosons that were discovered in the 1940s during the study of cosmic rays. They are produced by collision processes in nuclear reactions during which the strong interactions dominate. They appear in pairs $K^0, \overline{K}^0$ [8,14].

The $K$ mesons are eigenstates of the parity operator $P$: $P|K^0\rangle = -|K^0\rangle$ and $P|\overline{K}^0\rangle = -|\overline{K}^0\rangle$. $K^0$ and $\overline{K}^0$ are charge conjugate to each other $C|K^0\rangle = |\overline{K}^0\rangle$ and $C|\overline{K}^0\rangle = |K^0\rangle$. We get thus

$$CP|K^0\rangle = -|\overline{K}^0\rangle, \quad CP|\overline{K}^0\rangle = -|K^0\rangle. \tag{1.13}$$

Clearly, $|K^0\rangle$ and $|\overline{K}^0\rangle$ are not *CP*-eigenstates, but the following combinations

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\overline{K}^0\rangle), \quad |K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\overline{K}^0\rangle) \tag{1.14}$$

are *CP*-eigenstates.

$$CP|K_1\rangle = -|K_1\rangle, \quad CP|K_2\rangle = +|K_2\rangle. \tag{1.15}$$

In the absence of matter, kaons disintegrate through weak interactions [14]. Actually, $K^0$ and $\overline{K}^0$ are distinguished by their mode of *production*. $K_1$ and $K_2$ are the decay modes of kaons. In absence of *CP*-violation, the weak disintegration process distinguishes the $K_1$ states which decay only into "$2\pi$" while the $K_2$ states decay into "$3\pi, \pi e\nu, \ldots$" [15]. The lifetime of the $K_1$ kaon is short ($\tau_S \approx 8.92 \times 10^{-11}$ s), while the lifetime of the $K_2$ kaon is quite longer ($\tau_L \approx 5.17 \times 10^{-8}$ s).

*CP-violation* was discovered by Christenson et al. [16]. *CP*-violation means that the long-lived kaon can also decay to "$2\pi$". Then, the *CP* symmetry is slightly violated (by a factor of $10^{-3}$) by weak interactions so that the *CP*-eigenstates $K_1$ and $K_2$ are not exact eigenstates of the decay interaction. That $K_S$ (*S* = short-lived ) and $K_L$ (*L* = long-lived ) can be expressed as a superpositions of the $K_1$ and $K_2$ eigenstates. Then

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}[\epsilon|K_1\rangle + |K_2\rangle], \quad |K_S\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}[|K_1\rangle + \epsilon|K_2\rangle], \tag{1.16}$$

where $\epsilon$ is a complex *CP*-violation parameter, $|\epsilon| \ll 1$ and $\epsilon$ does not have to be real. $K_L$ and $K_S$ are the eigenstates of the Hamiltonian for the mass-decay matrix [14,15], which has in the basis $|K^0\rangle$ and $|\overline{K}^0\rangle$ has the following form:

$$H = M - \frac{\mathrm{i}}{2}\Gamma \equiv \begin{pmatrix} M_{11} - \frac{\mathrm{i}}{2}\Gamma_{11} & M_{12} - \frac{\mathrm{i}}{2}\Gamma_{12} \\ M_{21} - \frac{\mathrm{i}}{2}\Gamma_{21} & M_{22} - \frac{\mathrm{i}}{2}\Gamma_{22} \end{pmatrix}, \tag{1.17}$$

where $M$ and $\Gamma$ are individually Hermitian since they correspond to observable (mass and lifetime). The corresponding eigenvalues of the mass-decay matrix are equal to

$$m_L - \frac{\mathrm{i}}{2}\Gamma_L, \quad m_S - \frac{\mathrm{i}}{2}\Gamma_S. \tag{1.18}$$

The *CP*-violation was established by the observation that $K_L$ decays not only via three-pion, which has natural *CP* parity, but also via the two-pion ("$2\pi$") mode with a $|\epsilon|$ of order $10^{-3}$, which is truly unexpected. The experimental value of $\epsilon$ is [17]

$$|\epsilon| = (2.232 \pm 0.007) \times 10^{-3}, \quad \arg(\epsilon) = 43.4^\circ. \tag{1.19}$$

The LOY model uses the Weisskopf–Wigner treatment of the time-dependent Schödinger equation of the amplitudes of the particles $K^0$ and $\overline{K}^0$ as follows:

$$\mathrm{i}\frac{d\psi(t)}{dt} = H\psi(t) = (M - \mathrm{i}\Gamma)\psi(t), \tag{1.20}$$

where $M$ and $\Gamma$ are both $2 \times 2$ Hermitian matrices. They consider weak interaction as a perturbation in the total Hamiltonian, i.e. $H = H_{\mathrm{st}} + H_{\mathrm{el}} + H_{\mathrm{wk}}$, where $H_{\mathrm{st}}, H_{\mathrm{el}}$ and $H_{\mathrm{wk}}$ are the strong, electromagnetic and weak interactions Hamiltonian, respectively. Since $H_{\mathrm{st}} + H_{\mathrm{el}}$ commute with *CPT*, then $|K^0\rangle$ and $|\overline{K}^0\rangle$ are the eigenstates of $H_{\mathrm{st}} + H_{\mathrm{el}}$ with degenerate $m_k$ eigenvalue. Weak interaction connects $K^0$ and $\overline{K}^0$ with the other continuum states such as $2\pi, 3\pi, \pi e\nu$, etc. Thus, various decay modes removes their degeneracy [15]. Lee gave theorems which explain the mass-decay matrix [15]:

(i) If *CPT* invariance holds, then independently of *T* symmetry $\Gamma_{11} = \Gamma_{22}, \; M_{11} = M_{22}$.
(ii) If *T* invariance holds, then independently of *CPT* symmetry $\frac{\Gamma^*_{12}}{\Gamma_{12}} = \frac{M^*_{12}}{M_{12}}$.
(iii) If *CPT* invariance holds, then independently of *T* invariance, $\mathscr{E} = \langle K_S|K_L\rangle$ is a real number.
(iv) If *T* invariance holds, then independently of *CPT* invariance, $\mathscr{E}$ is an imaginary number.

*CP-violation:* If both *CPT* and *T* were exact symmetries, then *CP* must be conserved and then $\mathscr{E} = 0$. Thus, $\mathscr{E}$ can be a good *CP*-violation parameter. Now, the eigenvalue equation of Eq. (1.20) is written as

$$(M - \mathrm{i}\Gamma)\psi_\pm = \omega_\pm\psi_\pm, \tag{1.21}$$

where $\psi_- = |K_S\rangle$ and $\psi_+ = |K_L\rangle$ are the eigenstates of the kaons decay and the $\omega_\pm$ are the corresponding eigenvalues. The fractional number of kaon that decay at time $t$ after production is given by

$$N(t)dt = -d[\psi^\dagger\psi]. \tag{1.22}$$

Using Eq. (1.20), one obtain

$$N(t) = -\frac{d}{dt}[\psi^\dagger\psi] = \psi^\dagger \Gamma \psi. \tag{1.23}$$

Finally, the above equation becomes [9]

$$N(t) = \frac{1}{2}(1+\mathscr{E})^{-1}\{\gamma_+ e^{-\Gamma_S t} + \Gamma_L e^{-\Gamma_L t} + \mathscr{E} e^{-\frac{1}{2}(\Gamma_S+\Gamma_S)t}[(\Gamma_S+\Gamma_L)\cos\Delta mt - 2\Delta m \sin \Delta mt]\}, \tag{1.24}$$

where $\mathscr{E} = \langle K_S|K_L\rangle = \psi_+^\dagger\psi_-$ is a real number that represents the non-orthogonality of these two eigenstates. The four real numbers $\Gamma_S, \Gamma_L, \Delta m, \mathscr{E}$ characterize the decay of the kaon and satisfy the inequalities [9]

$$\Gamma_{S,L} \geqslant 0, \quad |\mathscr{E}|^2 \leqslant \frac{4\Gamma_S\Gamma_L}{(\Gamma_S+\Gamma_L)^2 + 4(\Delta m)^2}, \tag{1.25}$$

which follow from $\Gamma$ is positive Hermitian matrix. LOY considered the indications of the experimental values which showed that $\Gamma_L/\Gamma_S > 100$ then, from Eq. (1.25) they showed $|\mathscr{E}|^2 < 4\Gamma_S/\Gamma_L < 0.04$ [9].

In the 1980s, Khalfin (see [18]) pointed out to some theoretical deficiencies of the LOY theory due to the un physical energy interval $]-\infty, \infty[$ and gave estimates of the departure from this theory. His estimate of the *CP*-violation parameter has 30 times greater magnitude order than the experimental measurement. Later on, Chiu and Sudarshan [7] and Jankiewicz and Urbanowski [10] presented a new solution which improves the Khalfin computation.

Although some aspects of the questions involved in the theory of decaying phenomena may be formulated in a general manner, it has been proved very useful to study such phenomena using simple Hamiltonian models that permit explicit calculations. As a matter of fact, the first systematic discussion of the validity of the results of Weisskopf and Wigner was done in a basic paper of Friedrichs [20] where he analyzed the evolution of a solvable model. He considered a "free" Hamiltonian $H_0$ having a simple absolutely continuous spectrum and a point eigenvalue embedded in it. The eigenvalue is coupled to the continuum through a bounded interaction $\lambda V$. Because of the interaction, the eigenvalue disappears and the total Hamiltonian $H(= H_0 + \lambda V)$ has no point spectrum (at least for small values of the coupling parameter $\lambda$). Friedrichs has shown that the exponentially decaying solution of Weisskopf and Wigner becomes exact in the so-called weak coupling limit, that is, when $|\lambda| \to 1,\ t \to 0$, while $\lambda^2 t$ is kept finite [19].

We shall consider such a model for the problem of an estimation of the *CP*-violation parameter.

## 2. Dissipative evolution from Hamiltonian dynamics of kaons

The Friedrichs interaction Hamiltonian between two discrete modes and continuous degree of freedom is the following [20–23]:

$$H_{\text{Friedrich}} = \begin{pmatrix} \omega_1 & 0 & \lambda_1 v^*(\omega) \\ 0 & \omega_2 & \lambda_2 v^*(\omega) \\ \lambda_1 v(\omega) & \lambda_2 v(\omega) & \omega \end{pmatrix}, \tag{2.26}$$

where $\omega_{1,2}$ represent the energies of the discrete levels, and the factors $\lambda_i v(\omega)$ $(i = 1, 2)$ represent the couplings to the continuous degree of freedom. The energies $\omega$ of the different modes of the continuum range from 0 to $+\infty$, we are free to tune the coupling $v(\omega)$ in order to introduce a selective cut off to extreme energy modes. Let us now solve the Schrödinger equation in order to derive the master equation for the two-level system. The Schrödinger equation associated to the corresponding Friedrichs model with $\hbar = 1$ is formally written as

$$\begin{pmatrix} \omega_1 & 0 & \lambda_1 v^*(\omega) \\ 0 & \omega_2 & \lambda_2 v^*(\omega) \\ \lambda_1 v(\omega) & \lambda_2 v(\omega) & \omega \end{pmatrix} \begin{pmatrix} f_1(t) \\ f_2(t) \\ g(\omega,t) \end{pmatrix} = \mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix} f_1(t) \\ f_2(t) \\ g(\omega,t) \end{pmatrix}. \tag{2.27}$$

That is,

$$\omega_1 f_1(t) + \lambda_1 \int d\omega\, v^*(\omega) g(\omega,t) = \mathrm{i}\frac{\partial f_1(t)}{\partial t}, \tag{2.28}$$

$$\omega_2 f_2(t) + \lambda_2 \int d\omega\, v^*(\omega) g(\omega,t) = \mathrm{i}\frac{\partial f_2(t)}{\partial t} \tag{2.29}$$

and

$$\lambda_1 v(\omega) f_1(t) + \lambda_2 v(\omega) f_2(t) + \omega g(\omega,t) = \mathrm{i}\frac{\partial g(\omega,t)}{\partial t}. \tag{2.30}$$

Integrating the last equation, we obtain $g(\omega,t)$ assuming $g(\omega, t=0) = 0$:

$$g(\omega,t) = -\mathrm{i}e^{-\mathrm{i}\omega t}\int_0^t d\tau[\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau)]v(\omega)e^{\mathrm{i}\omega\tau}, \tag{2.31}$$

then, we substitute $g(\omega,t)$ in the above equation (2.28), we obtain

$$\mathrm{i}\frac{\partial f_1(t)}{\partial t} = \omega_1 f_1(t) - \mathrm{i}\lambda_1 \int d\omega |v(\omega)|^2 e^{-\mathrm{i}\omega t}\int_0^t d\tau[\lambda_1 f_1(\tau) + \lambda_2 f_2(\tau)]e^{\mathrm{i}\omega\tau}, \tag{2.32}$$

we also obtain the same relation for $f_2(t)$. Now, we shall make use of the Friedrichs model in order to simulate interesting properties of the kaonic systems.

In what follows, we shall identify the discrete modes of the Friedrichs model with the $K_1$ and $K_2$ states. This is our basic postulate according to which we can now make use of the Friedrichs model in order to establish a phenomenology for the kaonic behavior. More precisely, we shall assume that

$$|K_1\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |K_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.33}$$

Let us consider the solution of the two-level Friedrichs model Schrödinger equation (2.27). According to this equation, the state is at time $t$ superposition of two components that correspond to the two (complex) eigenvalues of the effective Hamiltonian. In order to avoid confusion, we shall use different parameters when we deal with the "real" kaons that are associated with experimental data and when we deal with the "theoretic" ones in the framework of the Friedrichs model.

–The masses $m_S$ and $m_L$ and the lifetime $\tau_S$ and $\tau_S$ will remain attributed to the real objects.
–The parameters $\omega_1$, $\omega_2$, $\lambda_1$, $\lambda_2$, $\omega_+$ and $\omega_-$ will refer to the theoretic quantities.

In order to present the idea of the method for obtaining the *CP*-violation, we consider first the case where $\omega \in ]-\infty,+\infty[$. If we substitute $v(\omega) = e^{-\alpha\omega^2/2},\ \alpha > 0,\ \alpha \to 0$ in Eqs. (2.32) and integrate from $-\infty$ to $\infty$, we can easily derive the effective master equation

$$\mathrm{i}\frac{\partial}{\partial t}\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix} = \begin{pmatrix} \frac{\omega_1 - \mathrm{i}\pi\lambda_1^2}{1-2\sqrt{\pi\alpha}\lambda_1^2} & \lambda_1\lambda_2\left(-\frac{\mathrm{i}\pi}{(1-2\sqrt{\pi\alpha}\lambda_1^2)} + \frac{2\sqrt{\pi\alpha}\omega_2}{1-2\sqrt{\pi\alpha}(\lambda_1^2+\lambda_2^2)}\right) \\ \lambda_1\lambda_2\left(-\frac{\mathrm{i}\pi}{(1-2\sqrt{\pi\alpha}\lambda_2^2)} + \frac{2\sqrt{\pi\alpha}\omega_1}{1-2\sqrt{\pi\alpha}(\lambda_1^2+\lambda_2^2)}\right) & \frac{\omega_2 - \mathrm{i}\pi\lambda_2^2}{1-2\sqrt{\pi\alpha}\lambda_2^2}\end{pmatrix}\begin{pmatrix}f_1(t)\\f_2(t)\end{pmatrix}, \tag{2.34}$$

in which we neglect the $O(\lambda^4)$ contributions. The eigenvalues of the above effective Hamiltonian, here denoted $H_{\text{eff}}$, are

$$\omega_+ = \frac{\omega_1 - \mathrm{i}\pi\lambda_1^2}{1-2\sqrt{\pi\alpha}\lambda_1^2} + O(\lambda^4) \approx (1+2\sqrt{\pi\alpha}\lambda_1^2)\omega_1 - \mathrm{i}\pi\lambda_1^2 \tag{2.35}$$

and

$$\omega_- \approx (1+2\sqrt{\pi\alpha}\lambda_2^2)\omega_2 - \mathrm{i}\pi\lambda_2^2. \tag{2.36}$$

In this approximation, the eigenvectors of the effective Hamiltonian are obtained as follows:

$$|f_+\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |f_-\rangle = \begin{pmatrix}0\\1\end{pmatrix}. \tag{2.37}$$

Comparing the eigenvalues in Eqs. (2.35) and (2.36) with the equations in (1.18), we obtain

$$\begin{aligned} 2\pi\lambda_1^2 &= \Gamma_S, \quad \omega_1 = \frac{m_S}{1+2\sqrt{\pi\alpha}\Gamma_S} \approx m_S(1-2\sqrt{\pi\alpha}\Gamma_S), \\ 2\pi\lambda_2^2 &= \Gamma_L, \quad \omega_2 = \frac{m_L}{1+2\sqrt{\pi\alpha}\Gamma_L} \approx m_L(1-2\sqrt{\pi\alpha}\Gamma_L). \end{aligned} \tag{2.38}$$

*CP-violation:* If the effective Hamiltonian (Eq. (2.34)) acts on $|K_L\rangle$ vector states as an eigenstate corresponding to the eigenvalue $\omega_- = (1+2\sqrt{\pi\alpha}\lambda_2^2)\omega_2 - \mathrm{i}\pi\lambda_2^2$ according to Eq. (1.16), we must impose that $H_{\text{eff}}\begin{pmatrix}-\epsilon\\1\end{pmatrix} = \omega_-\begin{pmatrix}-\epsilon\\1\end{pmatrix}$, after some algebraic computation by replacing $\lambda$'s and $\omega$'s by their corresponding values from Eq. (2.38), we have

$$\epsilon \approx \frac{\frac{\mathrm{i}}{2}\sqrt{\Gamma_L\Gamma_S}(1+2\mathrm{i}\sqrt{\frac{\alpha}{\pi}}m_S)}{(m_L - m_S) - \frac{\mathrm{i}}{2}(\Gamma_L - \Gamma_S)}. \tag{2.39}$$

In the zeroth approximation of $\alpha$ [24], we obtain thus

$$\epsilon \approx \sqrt{(1.82\times 10^{-3})/2}e^{\mathrm{i}(43.4)^\circ}, \tag{2.40}$$

which shows that our estimation of the modulus of $\epsilon$ is $\sim 14$ times greater than its experimental value while the estimated phase is correct. Now, in the case $\alpha \neq 0$, $\epsilon$ is given as

$$\epsilon \approx \sqrt{(1.82\times 10^{-3})/2}e^{\mathrm{i}(43.4)^\circ}\left(1+2\mathrm{i}\sqrt{\frac{\alpha}{\pi}}m_S\right), \tag{2.41}$$

we see that $\alpha > 0$ both changes the argument of $\epsilon$ and increases its modulus. Henceforth, a Gaussian test function in $]-\infty,\infty[$ is not a good choice if we aim at improving the fit with the experimental *CP*-violation.

In the realistic case $\omega \in [0,+\infty[$. If we substitute $\nu(\omega) = e^{-\alpha\omega^2/2}$ in Eqs. (2.32) and integrate from 0 to $\infty$, we obtain [25]

$$\epsilon \approx \frac{\mathrm{i}\sqrt{\Gamma_L\Gamma_S}\left[\left(\frac{1}{2}-\mathrm{i}\frac{1}{\pi}\right)+\left(-2+\mathrm{i}\frac{\pi+2}{\pi}\right)m_S\sqrt{\frac{\alpha}{\pi}}\right]}{(m_L-m_S)-\frac{\mathrm{i}}{2}(\Gamma_L-\Gamma_S)} \approx \sqrt{2(1.82\times10^{-3})}e^{\mathrm{i}(43.4)^\circ}\left[\left(\frac{1}{2}-\mathrm{i}\frac{1}{\pi}\right)+\left(-2+\mathrm{i}\frac{\pi+2}{\pi}\right)m_S\sqrt{\frac{\alpha}{\pi}}\right]. \tag{2.42}$$

We see that if $m_S\sqrt{\frac{\alpha}{\pi}} = \frac{1}{2+\pi}$ the imaginary part in the bracket of above equation is zero and the real part is equal to 0.111, which corresponds to the estimation

$$\epsilon = 6.69\times10^{-3}e^{\mathrm{i}(43.4)^\circ}. \tag{2.43}$$

So, in this case, $|\epsilon| = 6.69\times10^{-3}$ which is only $\sim 3$ times greater than the experimental value while the estimated phase is correct.

## 3. Decoherence and open systems

The unitary evolution condemns the closed quantum system to "purity". Yet if the outcomes of a measurement are to become independent, a way must be found to dispose of the excess information. This disposal can be caused by interaction with the degrees of freedom external to the system, which we shall summarily refer to as "the environment".

If the relative phase between two wave function ($|\phi_1\rangle$ and $|\phi_2\rangle$), is constant or it does not fluctuate randomly with time, these wave function are *coherent*. Now, the superposition of two coherent wave functions, i.e. $|\phi\rangle = \alpha_1|\phi_1\rangle + \alpha_2|\phi_2\rangle$, gives the interference effect in the probability density $|\phi|^2 = \langle\phi|\phi\rangle$. If this system is coupled to an environment, the relative phase between $|\phi_1\rangle$ and $|\phi_2\rangle$ will typically fluctuate with time, and the interference terms will rapidly average to zero. Their vanishing is called *decoherence*, i.e. the different components of wave function lose their ability to interfere.

We can also use the density matrix to describe the probability distribution for the alternative outcome, by taking the pure state density matrix

$$\rho = |\alpha_1|^2|\phi_1\rangle\langle\phi_1| + \alpha_1^*\alpha_2|\phi_2\rangle\langle\phi_1| + \alpha_2^*\alpha_1|\phi_1\rangle\langle\phi_2| + |\alpha_2|^2|\phi_2\rangle\langle\phi_2|. \tag{3.44}$$

Thus, the decay of off-diagonal elements in the density matrix gives the reduced density as

$$\rho_{red} = |\alpha_1|^2|\phi_1\rangle\langle\phi_1| + |\alpha_2|^2|\phi_2\rangle\langle\phi_2|. \tag{3.45}$$

Reduction of the state from $\rho$ to $\rho_{red}$ decreases the information available to the observer about the composite system (system + detector). Thus, its entropy increases as it must

$$\triangle\mathscr{S} = \mathscr{S}(\rho_{red}) - \mathscr{S}(\rho) = |\alpha_1|^2\log|\alpha_1|^2 + |\alpha_2|^2\log|\alpha_2|^2. \tag{3.46}$$

The initial state described by $\rho$ was pure, and the reduced state, $\rho_{red}$ is mixed [26–28].

It might appear as if accelerated decoherence is an inevitable fact, a fundamental natural law. This is, however, not the case. It is well known by now that certain subspaces of Hilbert space might be completely decoherence free. Such a situation arises if the coupling to the environment has a certain symmetry, in the sense that the interaction Hamiltonian, $H_{\mathrm{int}}$ has degenerate eigenvalues. Denoting $H = H_s + H_{\mathrm{int}}$, if $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle$ are eigenstates of $H_{\mathrm{int}}$ with the same eigenvalue, then there is no accelerated decoherence in the subspace they span.

The physical principle behind this is very simple. If the system Hamiltonian can be neglected, the states of the system are propagated by $e^{-\mathrm{i}H_{\mathrm{int}}t/\hbar}$. States that are eigenstates of $H_{\mathrm{int}}$ with the same eigenvalue acquire exactly the same phase factors as a function of time. Therefore, the phase coherence between such states remains intact.

Recently, decoherence-free subspaces have attracted renewed attention in quantum computing. It has been shown that general Markovian master equations for a density matrix, $\rho$, has the Lindblad form [27]

$$\frac{\partial\rho}{\partial t} = -\frac{\mathrm{i}}{\hbar}[H_s,\rho] + L_D[\rho], \tag{3.47}$$

$$L_D[\rho] = \frac{1}{2}\sum_{\alpha,\beta}^{M} a_{\alpha\beta}L_{\alpha\beta}[\rho], \tag{3.48}$$

$$L_{\alpha\beta}[\rho] = [F_\alpha, \rho F_\beta^\dagger] + [F_\alpha\rho, F_\beta^\dagger], \tag{3.49}$$

where $H_s$ is the Hamiltonian of the system and the coefficients $a_{\alpha\beta}$ form a Hermitian matrix. The operators $F_\alpha$ are known as "coupling agents" or, in the context of quantum computing, as "error generators". They span an $M$-dimensional Lie algebra. A decoherence-free subspace is defined as all states $\rho$ with $L_D[\rho] = 0$, since then only the unitary evolution according to the first terms in (3.47) remains [27].

In the usual formulation of the Friedrichs model, the border line between system and environment is ill defined because the Hilbert spaces associated to those degrees of freedom is not the tensorial product of their respective Hilbert spaces but is rather their direct sum. Nevertheless it is possible, as we showed in a previous paper [25], to imbed the direct sum of the

Hilbert spaces associated to the discrete and continuous degrees of freedom into a larger space in which those subspaces (tensorially) factorize, and to formulate an equivalent Hamiltonian dynamics that contains as a special subset of solutions all the solutions of the original model.

In this modified Friedrichs model, instead of representing the state of the system at time $t$ by a direct sum of the Hilbert spaces associated to the discrete and continuous degrees of freedom, we imbed it into the tensorial products of a three-dimensional Hilbert space $\mathbb{C}^3$ (that corresponds to the two discrete levels plus their decay product) and of a Fock space; $\mathbb{C}^3 \otimes \mathscr{H}_{photon}$, $\psi_{kaon} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$ and $\psi_{photon} = \begin{pmatrix} f^0 \\ f^1(\omega') \\ f^2(\omega', \omega'') \\ \cdots \end{pmatrix}$ and the state is given by

$$\Psi_{0,1,2,\omega^i} = \begin{pmatrix} f_0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ f^1(\omega') \\ f^2(\omega', \omega'') \\ \vdots \\ f^2(\omega', \omega'', \ldots, \omega^{(n)}) \\ \cdots \end{pmatrix} + \begin{pmatrix} 0 \\ f_1 \\ f_2 \end{pmatrix} \otimes \begin{pmatrix} f^0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \tag{3.50}$$

where $f_0$ represents the amplitude of a new discrete state $|0\rangle$ that is assumed to contain the "decay products" resulting from the disintegration of the two discrete kaonic states $|1\rangle$ and $|2\rangle$; besides, $f^n(\omega', \omega'', \ldots, \omega^{(n)})$ $(n = 1, 2, \ldots)$ represents the amplitude of the $n$ environment particles. We also define adequately the Hamiltonian on this space as in the Friedrichs model.

In analogy with quantum optics, this interaction represents the decay of the kaonic "excited" states (1 and 2) to the "ground" state (0), with excitation of a mode of energy $(\omega)$ while by unitarity the inverse process is also possible (diminution of the energy of a continuous mode by one quantum of energy $\omega$ (here $\hbar = 1$), and repopulation of the discrete states $|1\rangle$ and $|2\rangle$). If the initial state is such that no continuous mode is excited $(f^i(\omega', \omega'', \ldots, \omega^{(i)}, t = 0) = 0\ \forall i > 0)$, then, the dynamics of the state $\Psi_{0,1,2,\omega}(t)$ is considerably simplified because there will never occur more than one excitation.

In that case $f_1(t), f_2(t)$ and $f^1(\omega, t)$ obey a closed system of three equations of evolution that can be shown to be rigorously identical to the system of Eqs. (2.28)–(2.30) derived in the framework of the Friedrichs model [25].

The coherence between the decay products $|0\rangle$ at one side and the space spanned by the kaonic modes $|1\rangle$ and $|2\rangle$ at the other side is not preserved under the partial trace. In fact, this comes out from the computation of the partial trace over the biorthogonal decomposition of the full state (3.50). Then, taking a partial trace of this state over continuous degrees of freedom, it is easy to check that the reduced density matrix of the discrete (tripartite) degrees of freedom is equal to

$$\rho^{kaons} = \begin{pmatrix} \|f^1\|^2 & 0 & 0 \\ 0 & |f_1|^2 & f_1 f_2^* \\ 0 & f_1^* f_2 & |f_2|^2 \end{pmatrix}, \tag{3.51}$$

where $\|f^1\|^2 = \int |f^1(\omega)|^2 d\omega$ and $\|f^1\|^2 + |f_1|^2 + |f_2|^2 = 1$. This is clearly the incoherent sum of the decay products and a pure state that is coherent superposition of the $K_1$ and $K_2$ modes. The coherence between the decay products $|0\rangle$ and the space spanned by the kaonic modes $|1\rangle$ and $|2\rangle$ is not preserved. The entanglement measure between continuous modes and the kaons is given by the von Neumann entropy

$$S = \|f^1(t)\|^2 \log \|f^1(t)\| + (1 - \|f^1(t))\|^2 \log(1 - \|f^1(t)\|), \tag{3.52}$$

which is not zero. Tracing out the continuous degrees of freedom, the density matrix is the incoherent sum of the non-decayed kaonic states at one side and the decay products at the other side.

We shall remark that our model at the present stage does not fit exactly in the general decoherence framework [28], in the sense that asymptotically in time the state of the system (the three discrete modes) and the state of the environment (the continuous mode) factorize. This is due to the fact that if we consider the kaonic decay as a measurement process, it is a destructive process during which the original state is not preserved.

## 4. Conclusion and perspectives

We have shown that the framework of the Friedrichs model is relevant in order to grasp, despite of its simplicity, essential features of kaons decay. This model allows us to describe complex temporal evolutions (such as kaonic oscillations, generation and regeneration) and to simulate with a good approximation *CP*-violation. We also recover the experimental value of the phase, 43.4° as a results of Eqs. (2.39), (2.40) and (2.43).

We have shown that in the framework of the Friedrichs model the main feature that is responsible for the derivation of an irreversible in time master equation for the discrete system is the energy continuum.

To the contrary of entanglement, the localization of energy plays perfectly the role of time arrow pointer because during the decay process, the continuous modes are irreversibly excited. The same phenomenon occurs in first approximation during the disexcitation of a metastable excited electronic state accompanied by the emission of a photon although in that case a refined analysis shows that the final (ground) state of the atom is degenerate and remains generally entangled with the electro-magnetic field due to general conservation laws (for instance conservation of impulsion). It is out of the scope of the present paper to carry out exact computations but we could enrich our model by incorporating such features for instance by introducing two ground states and two continuous modes, one coupled to the $K_1$ state and one coupled to the $K_2$ state in which case the ground states as well as the emitted "$\omega$ particles" keep track of their particle of origin. This model fits more closely to experiments in which the $K_1$ and $K_2$ particles can be distinguished through their decay products. Moreover, such a model fits in the decoherence program because it predicts an irreversible increase of the entanglement with the environment during the measurement (here also decay) process.

Such a model could also be adapted in order to describe the strong coupling between kaons and hadrons in the presence of matter, which makes it possible to distinguish $K^0$ and $\overline{K}^0$ particles.

It is worth noting that even for this more sophisticated model where we would introduce two ground states in order to simulate the fact that the decay products are different, depending on which kaonic mode was at their origin, the superposition principle remains valid for what concerns the non-decayed kaons, because the non-decayed kaonic sector factorizes with the continuous mode (due to energy conservation: no decay means no excitation).

It has been shown that the departure from pure exponential decay for short time is characterized by a pure quadratic term. This is the so-called Zeno regime for some time called the Zeno time [29]. It is out of the scope of the present paper but it would worth to study Zeno effect in kaons specially on account of the shortness of the lifetime.

It would be also very interesting to study the properties of the Friedrichs model and of kaonic oscillations in terms of the time operator approach [30]. This can be done for the one-level Friedrichs model [13,31–33] but higher level systems present more subtle and involved temporal behavior [29] so that it is worth studying the time-operator in this context.

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