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An Outer Commutator Capability of Finitely Generated Abelian Groups

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Abstract: We use the explicit structure for the Baer invariant of a finitely generated abelian group with respect to the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$, for all $c_2 \leq c_1 \leq 2c_2$, to determine necessary and sufficient conditions for such groups to be $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. We also show that if $c_1 \neq 1 \neq c_2$, then a finitely generated abelian group is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if and only if it is capable.

Keywords: Capability; Outer commutator variety; Finitely generated abelian groups.

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1. Introduction

Determining capable groups or more general varietal capable groups in a class of groups is an interesting problem. In 1938 Baer [1] classified all capable groups among the direct sums of cyclic groups and in particular among the finitely generated abelian groups. In 1998 Burns and Ellis [2], extended the result for $\mathfrak{N}_c$-capability and recently the authors in a joint paper [6] with S. Kayvanfar classified all finitely generated abelian groups that are polynilpotent capable.

2. Main Results

First, we introduce capable and varietal capable groups and stste some properties of them.

Definition 1. *Let $\mathfrak{V}$ be any variety and G be any group. Define $V^{**}(G)$ as follows:*

$$V^{**}(G) = \cap\{\psi(V^*(E)) \mid \psi : E \overset{onto}{\longrightarrow} G \ , \ ker\psi \subseteq V^*(E)\}.$$

Note that if $\mathfrak{V}$ is the variety of abelian groups, then the above notion is denoted by $Z^*(G)$ and called epicenter in [2]. Also the above notion has been studied in [2], for the variety $\mathfrak{N}_c$.

Theorem 2. *With the above notations and assumptions $G/V^{**}(G)$ is the largest quotient of G which is $\mathfrak{V}$-capable, and hence G is $\mathfrak{V}$-capable if and only if $V^{**}(G) = 1$.*

The following theorem and its conclusion state a relationship between $\mathfrak{V}$-capability and Baer invariants.

Theorem 3. *Let $\mathfrak{V}$ be any variety of groups, G be any group, and N be a normal subgroup of G contained in the marginal subgroup with respect to $\mathfrak{V}$. Then the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is injective if and only if $N \subseteq V^{**}(G)$, where $\mathfrak{V}M(G)$ is the Baer invariant of G with respect to $\mathfrak{V}$.*

Proof. See [4] □

In the finite case the following theorem is easier to use than the proceeding ones.

Theorem 4. *Let $\mathfrak{V}$ be any variety and G be any group with $V(G) = 1$. If $\mathfrak{V}M(G)$ is finite, and N is a normal subgroup of G such that $\mathfrak{V}M(G/N)$ is also finite, then the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is injective if and only if $|\mathfrak{V}M(G/N)| = |\mathfrak{V}M(G)|$.*

Proof. It is easy to see that with the assumption of the theorem we have $\mathfrak{V}M(G) \cong V(F)/[RV^*F]$ and $\mathfrak{V}M(G/N) \cong V(F)/[SV^*F]$ in which $G \cong F/R$ is a free presentation for G and $N \cong S/R$. Therefore the kernel of the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/N)$ is the group $[SV^*F]/[RV^*F]$. Considering the finiteness of $\mathfrak{V}M(G)$ and $\mathfrak{V}M(G/N)$, the result easily follows. □

As a useful consequence of Theorem 3 we have:

Corollary 5. *An abelian group G is $\mathfrak{V}$-capable if and only if the natural homomorphism $\mathfrak{V}M(G) \longrightarrow \mathfrak{V}M(G/\langle x\rangle)$ has a non-trivial kernel for all non-identity elements x in $V^*(G)$.*

The following fact is used in the last section [7].

Theorem 6. *Let u and v be two words in independent variables and $w = [u, v]$. Then, in any group G,*
(i) $w(G) = [u(G), v(G)]$
(ii) *if $A = C_G(u(G))$, $B = C_G(v(G))$, $L/A = v^*(G/A)$, and $M/B = u^*(G/B)$, then $w^*(G) = L \cap M$.*

We use the following theorem which is in [5] to determine the capability of the mentioned groups.

Theorem 7. *Let $G \cong \mathbf{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group with $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$. If $c_2 \leq c_1 \leq 2c_2$, then*

$$[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \mathbb{Z}_{n_2}^{(b_{k+2}-b_{k+1})} \oplus \ldots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$$

where $b_i = \chi_{c_1+1}(i)\chi_{c_2+1}(i)$, *if* $c_1 > c_2$ *and* $b_i = \chi_2(\chi_{c_1+1}(i))$ *if* $c_1 = c_2$.

To use the above theorems we need some lemmas as stated.

Lemma 8. *Let G be a finitely generated abelian group and $H \leq G$. Then $r_0(G) = r_0(G/H)+r_0(H)$, where $r_0(X)$ is the torsion free rank of a finitely generated abelian group X.*

Proof. See [3]. □

In the case of p-groups the following theorem has an important role in our investigation.

Theorem 9. *Let $G \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \mathbb{Z}_{p^{\alpha_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_k}}$ be a finite abelian p-group, where $\alpha_{i+1} \leq \alpha_i$ for all $1 \leq i \leq k-1$, and let H be a subgroup of G. Then $H \cong \mathbb{Z}_{p^{\beta_1}} \oplus \mathbb{Z}_{p^{\beta_2}} \oplus \cdots \oplus \mathbb{Z}_{p^{\beta_k}}$ where $\beta_{i+1} \leq \beta_i$ for all $1 \leq i \leq k-1$ and $0 \leq \beta_i \leq \alpha_i$ for $1 \leq i \leq k$.*

Proof. See [3]. □

Theorem 10. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for $1 \leq i \leq t-1$, and let H be a finite subgroup of G. Then $H \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ for all $1 \leq i \leq t-1$ and $m_i \mid n_i$ for all $1 \leq i \leq t$.*

Proof. Trivially $H \leq t(G)$, the maximal torsion subgroup of G, so without loss of generality we may assume that G is finite. It is well known that $G \cong S_{p_1} \oplus \cdots \oplus S_{p_t}$, where S_{p_i} is the p_i-Sylow subgroup of G. One may easily show that if $H \cong S'_{p_1} \oplus \cdots \oplus S'_{p_t}$ is the same decomposition for H, then $S'_{p_i} \leq S_{p_i}$ for all $1 \leq i \leq t$. Therefore it is enough to consider finite abelian p-groups. Now Theorem 9 completes the proof. □

Proceeding now to $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability, note that $[\mathfrak{N}_1, \mathfrak{N}_1] = \mathfrak{S}_2$ is the variety of metabelian groups (that is groups of solvability length at most 2) and, according to Theorem 2.6, $\mathfrak{S}_2 M(G) = 0$ whenever G has at most two generators. But if $c_2 < c_1 \leq 2c_2$ or $c_1 = c_2 > 1$, then the Baer invariant is trivial only if G is cyclic. This suggests dealing with the two cases separately, and so we assume first that $c_2 < c_1 \leq 2c_2$ or $c_1 = c_2 > 1$. The method we use here implies separating the cases which G is finite or infinite.

Case one: G is a finite abelian group.

Theorem 11. *Let $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finite abelian group, where $n_{i+1} \mid n_i$ for $1 \leq i \leq t-1$, then G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable if and only if $t \geq 2$ and $n_1 = n_2$.*

Proof. We will establish the necessity by contrapositive. If $t = 1$, then G and all its quotients are cyclic abelian groups so by Theorem 7 $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N) = 0$ for any normal subgroup N of G, hence by Corollary 2 G is not $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. On the other hand if $n_1 \neq n_2$, then let $x = (\bar{n}_2, \bar{0}, \ldots, \bar{0})$, since $G/\langle x\rangle \cong \mathbb{Z}_{n_2} \oplus \mathbb{Z}_{n_2} \cdots \oplus \mathbb{Z}_{n_t}$, Theorem 7 shows the Baer invariants for G and $G/\langle x\rangle$ have the same size. This shows G is not $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable in this case by Corollary .

For sufficiency, assume $t \geq 2$ and $n_1 = n_2$. By Corollary 2 it is enough to show that if $N < G$ and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$ is injective, then N is trivial.

In finite abelian groups each quotient is isomorphic to a subgroup and vice versa. Now let $N < G$, then G/N is isomorphic to a subgroup of G, H say; so by Theorem 10 $H \cong \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ for all $1 \leq i \leq t-1$ and $m_i \mid n_i$ for all $1 \leq i \leq t$. Computing $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$ and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(H)$ using Theorem 7 shows that $|[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)| = |[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(H)|$ if and only if $m_i = n_i$ for all $2 \leq i \leq t$, but $n_1 = n_2$ by hypothesis which implies $n_1 = m_1$ which implies $H = G$ and hence $N = 0$. Therefore G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable. □

Now we consider the infinite case.

Theorem 12. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be an infinite finitely generated abelian group, where $n_{i+1} \mid n_i$ for $1 \leq i \leq t-1$, then G is $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capable, if and only if $k \geq 2$.*

Proof. We first show that if $k = 1$, then there exists a nontrivial element x of G for which the natural homomorphism $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/\langle x\rangle)$ is injective, proving the necessity by contrapositive.

Suppose $k = 1$, then $G \cong \mathbb{Z} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$. Let $x = (n_1, \bar{0}, \ldots, \bar{0})$, so $G/\langle x\rangle \cong \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$. Now by Theorem 7 we have $|[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)| = |[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/\langle x\rangle)|$, so the result follows.

For sufficiency, assume that $k \geq 2$. It is enough to show that there is no nontrivial subgroup N of G for which $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \longrightarrow [\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$ is injective. If N is an infinite subgroup, then $r_0(G/N) < r_0(G)$, so by Theorem 7 the torsion free rank of the Baer invariant of G/N is strictly smaller than that of the invariant for G, so no injection is possible. On the other hand if N is contained in the torsion subgroup of G, then $G/N \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_t}$, where $m_{i+1} \mid m_i$ and $m_i \mid n_i$ for all $1 \leq i \leq t-1$, so by Theorem 7 we have $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$ and $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N) \cong \mathbb{Z}^{(b_k)} \oplus \mathbb{Z}_{m_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{m_t}^{(b_{k+t}-b_{k+t-1})}$. It is easy to show that

$$t([\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)) = \mathbb{Z}_{n_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{n_t}^{(b_{k+t}-b_{k+t-1})}$$

and

$$t([\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)) = \mathbb{Z}_{m_1}^{(b_{k+1}-b_k)} \oplus \cdots \oplus \mathbb{Z}_{m_t}^{(b_{k+t}-b_{k+t-1})}.$$

The image of the torsion subgroup of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G)$ under the natural homomorphism must lie in the torsion subgroup of $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]M(G/N)$, so if the homomorphism is injective, then we must have $t(G) = t(G/N)$; $t(G/N) = t(G)/N$, this proves that if the map is injective then $N = 0$, completing the proof. □

Remark 13. Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, with $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$. Baer's result Baer (1938), implies that G is capable if and only if $k \geq 2$ or $k = 0$, $t \geq 2$ and $n_1 = n_2$. Burns and Ellis (1998), proved that G is $\mathfrak{N}_c$-capable if and only if it is capable. We now see that this also holds for $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$-capability when $c_1 \neq 1 \neq c_2$.

In the case $c_1 = c_2 = 1$ we only state the characterization of the $\mathfrak{S}_2$-capable groups among finitely generated abelian groups. The proofs are similar to those of Theorems 11 and 12. The needed lemmas and their proofs can be restated with necessary changes similar to Theorems 11 and 12. Note that in this case the variety $[\mathfrak{N}_{c_1}, \mathfrak{N}_{c_2}]$ is actually the variety of metabelian groups $\mathfrak{S}_2$.

Theorem 14. *Let $G \cong \mathbb{Z}^{(k)} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_t}$ be a finitely generated abelian group, where $n_{i+1} \mid n_i$ for all $1 \leq i \leq t-1$. Then G is $\mathfrak{S}_2$-capable if and only if $k \geq 3$, or $k = 0$, $t \geq 3$, and $n_1 = n_2 = n_3$.*

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