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Fuzzy statistical tests based on fuzzy confidence intervals

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Abstract: A general method is proposed to construct fuzzy tests for testing statistical hypotheses about an unknown fuzzy parameter, when the data available are observations of a fuzzy random variable. The proposed method to construct such tests is essentially based on the one-to-one correspondence between the acceptance region of any level α test and the $1-\alpha$ level confidence interval for the same parameter. By using this equivalency in the case of fuzzy environment, we construct the fuzzy test showing the degree of acceptability of the null and alternative hypotheses of interest. A numerical example in the field of lifetime study is given to clarify the theoretical results.
Keywords: Fuzzy confidence interval, Fuzzy parameter, Fuzzy random variable, Fuzzy test, Lifetime data

1 INTRODUCTION AND BACKGROUND

In the classical theory of parametric statistical inference there is a one-to-one relationship between a subset of the parameter space for which the null hypothesis is accepted and the structure of the confidence set for the same parameter. Namely, the level α acceptance region for a statistical test about the parameter of interest is equivalent to a certain confidence set for that parameter on the confidence level $1-\alpha$ [1].

For instance, consider the problem of testing hypothesis $H_0(\theta_0) : \theta = \theta_0$. Let $A(\theta_0)$ denote the level α acceptance region of a test for testing such a hypothesis. If $S(\mathbf{x}) = \{\theta : \mathbf{x} \in A(\theta)\}$, then

$$\theta \in S(\mathbf{x}) \iff \mathbf{x} \in A(\theta),$$

and hence

$$P_\theta \{\theta \in S(\mathbf{X})\} \geq 1-\alpha, \qquad \text{for all } \theta. \qquad (1)$$

Thus, any family of level α acceptance regions, leads to a family of confidence sets at confidence level $1-\alpha$. Conversely, given any class of confidence sets $S(\mathbf{x})$ satisfying (1), let $A(\theta) = \{\mathbf{x} : \theta \in S(\mathbf{x})\}$. Then, the sets $A(\theta_0)$ are level α acceptance regions for testing the hypothesis $H_0(\theta_0) : \theta = \theta_0$.

The same arguments can be applied if the set $A(\theta_0)$ is the acceptance region for the hypotheses $H_0(\theta_0) : \theta \leq \theta_0$ or $H_0(\theta_0) : \theta \geq \theta_0$.

So, a confidence set can be viewed as a rule, which exhibits the values for which the hypothesis is completely accepted, i.e. $\{\theta : \theta \in S(\mathbf{x})\}$ (or $\{\theta : I_{S(\mathbf{x})}(\theta) = 1\}$, where, $I_A(\cdot)$ stands for the indicator function of a set A) and those for which it is completely rejected, i.e. $\{\theta : \theta \notin S(\mathbf{x})\}$ (or $\{\theta : 1 - I_{S(\mathbf{x})}(\theta) = 1\}$). Therefore, we can summarize the test function for testing the hypothesis $H_0(\theta_0) : \theta = \theta_0$ against some H_1 as

$$\varphi(\mathbf{x}) = \left\{ \frac{I_{S(\mathbf{x})}(\theta_0)}{\text{Accept } H_0(\theta_0)}, \frac{1 - I_{S(\mathbf{x})}(\theta_0)}{\text{Reject } H_0(\theta_0)} \right\}.$$

The above test function is just an equivalent case of the general rule stating that the null hypothesis should be rejected if the confidence interval does not contain the hypothesized value of the parameter and should not be rejected if the interval contains the hypothesized value.

Example 1.1 *Let $X_1, \ldots, X_n$ be i.i.d. from the normal distribution $N(\theta, 1)$ with an unknown mean θ. The usual two-sided $1-\alpha$ confidence interval for θ is of the form $S(\boldsymbol{X}) = [\bar{X} - \frac{1}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X} + \frac{1}{\sqrt{n}} z_{1-\frac{\alpha}{2}}]$, where z_α is the $\alpha-$quintile of the standard normal distribution, i.e. $\Phi(z_\alpha) = \alpha$. Assume that in a random sample of size $n = 25$, $\bar{x} = 0.75$ is observed and we want to test $H_0 : \theta = 0.5$ against $H_1 : \theta \neq 0.5$ at level* 0.05*. Since $S(\boldsymbol{x}) = [0.358, 1.142]$, therefore, the test function is obtained as*

$$\begin{aligned} \varphi(\boldsymbol{x}) &= \left\{ \frac{I_{[0.358,1.142]}(0.5)}{Accept\ H_0(0.5)}, \frac{1 - I_{[0.358,1.142]}(0.5)}{Reject\ H_0(0.5)} \right\} \\ &= \left\{ \frac{1}{Accept\ H_0(0.5)}, \frac{0}{Reject\ H_0(0.5)} \right\}. \end{aligned}$$

Concerning the above discussion, the main propose of this study is to investigate such relationship between the methods of testing statistical hypotheses and confidence intervals in fuzzy environment. It should be mentioned that, over the last decades, the two topics of testing hypotheses in fuzzy environment and confidence intervals with fuzzy information have been considered independently by some authors. Let us briefly review some works on these topics.

Grzegorzewski [2, 3] suggested some fuzzy tests for testing statistical hypotheses with vague data in parametric and non-parametric cases. Montenegro et al. [4, 5], using a generalized metric for fuzzy numbers, proposed a method to test about the fuzzy mean of a fuzzy random variable (FRV). Gil et al. [6] and González-Rodríguez [7] introduced a bootstrap approach to the one-sample and multi-sample test of means for imprecisely valued sample data. Wu [8] investigated a procedure to accept or reject statistical hypotheses about a fuzzy parameter by introducing the concepts of degrees of optimism and pessimism, based on FRVs. Parchami et al. [9] studied a fuzzy version of some process capability indices when specification limits are fuzzy rather than precise, and obtained fuzzy confidence intervals for such indices. Najafi et al. [10] introduced a likelihood ratio procedure to test statistical hypotheses for fuzzy data. Arefi and Taheri [11] developed an approach to test statistical hypotheses upon fuzzy test statistic where for rejecting or accepting the null hypothesis an evidential point of view was proposed. Wu [12] and Chachi and Taheri [13] proposed some approaches to construct fuzzy confidence intervals for the unknown fuzzy mean of a FRV. Viertl [14] investigated some methods to construct confidence intervals and statistical tests for fuzzy data.

It is noticeable that, the common approaches to the problem of testing hypothesis in fuzzy environment are done independently from the problem of interval estimation. The exception is the work by Grzegorzewski [2], in which, using the one-to-one correspondence between confidence intervals and testing statistical hypotheses, he proposed a fuzzy test for testing classical hypotheses about a crisp parameter. His work was relied on a fuzzy random sample as a fuzzy perception of the related crisp random sample.

In this paper, the problem of testing hypothesis about a fuzzy parameter is investigated based on a fuzzy confidence interval for the parameter of interest using fuzzy random sample. To do this, we determine the degree of membership by which the fuzzy parameter is contained in the fuzzy confidence interval. Then, the related hypothesis is accepted with the same degree of certainty. In such a way, the obtained test, contrary to the classical crisp test, does not lead to a binary decision, i.e. to accept or reject the null hypothesis, but to a fuzzy decision.

To provide the suitable procedure, the rest of this paper is organized as follows: in the next section, some basic concepts, that will be used in the sequel, are recalled. Section 3 provides the proposed approach to the problem of testing hypotheses by using fuzzy confidence intervals. A numerical example is provided in Section 4 to clarify the proposed method. In the final section, we make some concluding remarks.

2 Preliminaries

2.1 Fuzzy arithmetic

In this paper let the real line $\mathbb{R}$ be the universal set. A fuzzy set $\tilde{A}$ of $\mathbb{R}$ is defined by its membership function $\tilde{A}:\mathbb{R}\to[0,1]$. For each $h\in(0,1]$, the $h-$level set of $\tilde{A}$ is defined by $\tilde{A}_h=\{x\in\mathbb{R}:\tilde{A}(x)\geq h\}$, and $\tilde{A}_0$ is the closure of the set $\{x\in\mathbb{R}:\tilde{A}(x)>0\}$. The fuzzy set $\tilde{A}$ is called a fuzzy number if each $\tilde{A}_h$ is a nonempty closed interval for all $h\in(0,1]$. The h-level set of each fuzzy number $\tilde{A}$ is denoted by the interval $\tilde{A}_h=[a^l_h,a^u_h]$, where $a^l_h=\inf\{x\in\mathbb{R}:A(x)\geq h\}$ and $a^u_h=\sup\{x\in\mathbb{R}:A(x)\geq h\}$. $\tilde{A}$ is called a fuzzy point (crisp number) with the value m if its membership function is $\tilde{A}(x)=I_{\{m\}}(x)$. We denote by $\mathcal{F}(\mathbb{R})$ the class of all fuzzy numbers on $\mathbb{R}$.

A special kind of fuzzy numbers are the triangular fuzzy numbers denoted by $\tilde{A}=(a,a_l,a_r)_T$, where a, a_l, and a_r are the center, the left and right spreads of $\tilde{A}$, respectively. The membership function and the $h-$level set of the triangular fuzzy number $\tilde{A}$ are as follows

$$\begin{aligned}\tilde{A}(x) &= \frac{x-a+a_l}{a_l}I_{[a-a_l,a)}(x)+\frac{a+a_r-x}{a_r}I_{[a,a+a_r]}(x),\\ \tilde{A}_h &= [a^l_h,a^u_h]=[a-(1-h)a_l,a+(1-h)a_r],\end{aligned}$$

A well-known ordering on fuzzy numbers, which will be used in the definition of hypotheses about an unknown fuzzy parameter, is as follows [8]

1. $\tilde{A}=(\neq)\tilde{B}$, if $a^l_h=(\neq)b^l_h$ and $a^u_h=(\neq)b^u_h$ for any $h\in(0,1]$.
2. $\tilde{A}\preceq(\prec)\tilde{B}$, if $a^l_h\leq(<)b^l_h$ and $a^u_h\leq(<)b^u_h$ for any $h\in(0,1]$.
3. $\tilde{A}\succeq(\succ)\tilde{B}$, if $a^l_h\geq(>)b^l_h$ and $a^u_h\geq(>)b^u_h$ for any $h\in(0,1]$.

2.2 Fuzzy random variables

Let $(\Omega,\mathcal{A},\mathcal{P})$ be a probability space, $\mathcal{X}:\Omega\to\mathcal{F}(\mathbb{R})$ be a fuzzy-valued function, X be a random variable having distribution $f_{\underline{\theta}}$ with parameter $\underline{\theta}=(\theta_1,\ldots,\theta_p)\in\Theta^p$, i.e. $X\sim f_{\underline{\theta}}$, and $\Theta^p\subset\mathbb{R}^p$ be the parameter space, where $p\geq 1$. Throughout this paper, we assume that all random variables have the same probability space $(\Omega,\mathcal{A},\mathcal{P})$.

The fuzzy-valued function $\mathcal{X}:\Omega\to\mathcal{F}(\mathbb{R})$ is called a FRV if $X^l_h:\Omega\to\mathbb{R}$ and $X^u_h:\Omega\to\mathbb{R}$ be two real valued random variables for all $h\in(0,1]$ (where $\forall\omega\in\Omega;\ \mathcal{X}(\omega)_h=[X^l_h(\omega),X^u_h(\omega)]$) [15].

FRVs $\mathcal{X}$ and $\mathcal{Y}$ are called identically distributed if X^l_h and Y^l_h are identically distributed, and X^u_h and Y^u_h are identically distributed, for all $h\in(0,1]$, and they are called independent if each random variable in the set $\{X^l_h,\,X^u_h:\ h\in(0,1]\}$ is independent of each random variable in the set $\{Y^l_h,\,Y^u_h:\ h\in(0,1]\}$ [12].

FRV $\mathcal{X}$ is said to have the same distribution as X with fuzzy parameter $\tilde{\underline{\theta}}=(\tilde{\theta}_1,\ldots,\tilde{\theta}_p)$ if for all $h\in(0,1]$, $X^l_h\sim f_{\underline{\theta}^l_h}$ and $X^u_h\sim f_{\underline{\theta}^u_h}$, where $\underline{\theta}^l_h=(\theta^l_{1h},\ldots,\theta^l_{ph})$, $\underline{\theta}^u_h=(\theta^u_{1h},\ldots,\theta^u_{ph})$, and $\tilde{\theta}_{jh}=[\theta^l_{jh},\theta^u_{jh}]$, $j=1,\ldots,p$ [12].

For example, $\mathcal{X}$ is normally distributed with fuzzy parameters $\tilde{\theta}$ and $\tilde{\sigma}^2$ if and only if $X^l_h\sim N(\theta^l_h,\sigma^{2l}_h)$ and $X^u_h\sim N(\theta^u_h,\sigma^{2u}_h)$, for all $h\in(0,1]$.

Definition 2.1 ([12]) $\underline{\mathcal{X}}=(\mathcal{X}_1,\cdots,\mathcal{X}_n)$ *is said to be a fuzzy random sample of size n from a normal distribution with fuzzy parameters $\tilde{\theta}$ and $\tilde{\sigma}^2$, if $\mathcal{X}_i$'s are independent and identically distributed normal FRVs with fuzzy parameters $\tilde{\theta}$ and $\tilde{\sigma}^2$ for all $i=1,\ldots,n$. In this case, we write $\mathcal{X}_1,\cdots,\mathcal{X}_n\overset{i.i.d.}{\sim}N(\tilde{\theta},\tilde{\sigma}^2)$.*

Corollary 2.1 ([12]) *Let $\mathcal{X}_1,\cdots,\mathcal{X}_n\overset{i.i.d.}{\sim}N(\tilde{\theta},\tilde{\sigma}^2)$, then $X^l_{1h},\cdots,X^l_{nh}\overset{i.i.d.}{\sim}N(\theta^l_h,\sigma^{2l}_h)$ and $X^u_{1h},\cdots,X^u_{nh}\overset{i.i.d.}{\sim}N(\theta^u_h,\sigma^{2u}_h)$ for all $h\in(0,1]$. In the case of $\tilde{\sigma}^2=I_{\{\sigma^2\}}$, i.e. $\tilde{\sigma}^2$ is a crisp parameter, we also see that for all $h\in(0,1]$, $X^l_{1h},\ldots,X^l_{nh}\overset{i.i.d.}{\sim}N(\theta^l_h,\sigma^2)$ and $X^u_{1h},\ldots,X^u_{nh}\overset{i.i.d.}{\sim}N(\theta^u_h,\sigma^2)$. In this case, we write $\mathcal{X}_1,\ldots,\mathcal{X}_n\overset{i.i.d.}{\sim}N(\tilde{\theta},\sigma^2)$.*

3 Constructing fuzzy tests based on fuzzy confidence intervals

Now, we use the relationship between confidence intervals and testing hypotheses in fuzzy environment, when the parameter of interest, data available, and confidence intervals are fuzzy and the statistical hypotheses are statements about the fuzzy parameter. First, let us define the different types of statistical hypotheses about a fuzzy parameter.

Definition 3.1 *Let $\tilde{\Theta} = \mathcal{F}(\Theta)$ be the class of all fuzzy numbers on the parameter space Θ and $\tilde{\theta}_0 \in \tilde{\Theta}$ be a known fuzzy number. Then:*

1. *Any hypothesis of the form "$H : \tilde{\theta} = \tilde{\theta}_0$" is called a simple hypothesis.*
2. *Any hypothesis of the form "$H : \tilde{\theta} \neq \tilde{\theta}_0$" is called a two-sided hypothesis.*
3. *Any hypothesis of the form "$H : \tilde{\theta} \succ \tilde{\theta}_0$" is called a right one-sided hypothesis.*
4. *Any hypothesis of the form "$H : \tilde{\theta} \prec \tilde{\theta}_0$" is called a left one-sided hypothesis.*

Example 3.1 *Let $\tilde{\Theta} = \mathcal{F}(\mathbb{R})$ be the fuzzy parameter space for the fuzzy mean of a normal FRV. Then:*

1. *The hypothesis "$H : \tilde{\theta} = (1, 1, 2)_T$" is a simple hypothesis.*
2. *The hypothesis "$H : \tilde{\theta} \neq (1, 1, 2)_T$" is a two-sided hypothesis. This hypothesis is equivalent to "$H : \tilde{\theta} \in \tilde{\Theta}_1 = \{\tilde{\theta} \in \tilde{\Theta} \mid \theta^l_h \neq h,\ \theta^u_h \neq 3 - 2h;\ \forall h \in (0, 1]\}$", where, $[h, 3-2h]$ is the $h-$level set of the triangular fuzzy number $(1, 1, 2)_T$.*
3. *The hypothesis "$H : \tilde{\theta} \succ (1, 1, 2)_T$" is a right one-sided hypothesis. This hypothesis is equivalent to "$H : \tilde{\theta} \in \tilde{\Theta}_1 = \{\tilde{\theta} \in \tilde{\Theta} \mid \theta^l_h > h,\ \theta^u_h > 3 - 2h;\ \forall h \in (0, 1]\}$".*
4. *The hypothesis "$H : \tilde{\theta} \prec (1, 1, 2)_T$" is a left one-sided hypothesis. This hypothesis is equivalent to "$H : \tilde{\theta} \in \tilde{\Theta}_1 = \{\tilde{\theta} \in \tilde{\Theta} \mid \theta^l_h < h,\ \theta^u_h < 3 - 2h;\ \forall h \in (0, 1]\}$".*

The main problem: Let $\underline{\mathcal{X}} = (\mathcal{X}_1, \ldots, \mathcal{X}_n)$ denotes a fuzzy sample from the population with the distribution $f_{\tilde{\theta}}$ including the fuzzy parameter $\tilde{\theta} \in \tilde{\Theta}$. We wish to test the following hypotheses about the fuzzy parameter $\tilde{\theta}$ at level α

$$\begin{cases} H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0 \\ H_1(\tilde{\theta}_0) : \tilde{\theta} \neq \tilde{\theta}_0 \end{cases} \tag{2}$$

using fuzzy confidence intervals, where $\tilde{\theta}_0 \in \tilde{\Theta}$ is a known fuzzy number.

The proposed procedure: In order to derive the grades of acceptability of the above null and alternative hypotheses, we consider the following procedure. We explain this procedure for the case when the data available are observations from the normal distribution $N(\tilde{\theta}, \sigma^2)$, with unknown fuzzy mean $\tilde{\theta}$ and known crisp variance σ^2, i.e. $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\tilde{\theta}, \sigma^2)$.

Note that the following procedure is general and a similar approach can be developed for testing one-sided alternative hypotheses with any kind of FRVs.

Step 1. First, we transform the original testing problem (2) into a set of crisp testing problems using the h-levels of the fuzzy parameter. In the other words, we test the following crisp hypotheses

$$H_0(\theta^l_{0h}) : \theta^l_h = \theta^l_{0h} \qquad \text{v.s} \qquad H_1 : \theta^l_h \neq \theta^l_{0h}, \tag{3}$$
$$H_0(\theta^u_{0h}) : \theta^u_h = \theta^u_{0h} \qquad \text{v.s} \qquad H_1 : \theta^u_h \neq \theta^u_{0h}, \tag{4}$$

at level α, based on the samples $\mathbf{X}^l_h = (X^l_{1h}, \ldots, X^l_{nh})$ and $\mathbf{X}^u_h = (X^u_{1h}, \ldots, X^u_{nh})$, respectively, where $\tilde{\theta}_h = [\theta^l_h, \theta^u_h]$ and $\tilde{\theta}_{0h} = [\theta^l_{0h}, \theta^u_{0h}]$ (see Corollary 2.1).

Step 2. We obtain the $1 - \alpha$ confidence intervals for the crisp parameters θ^l_h and θ^u_h, for any $h \in (0, 1]$, denoted by $[L_1(\mathbf{X}^l_h), L_2(\mathbf{X}^l_h)]$ and $[U_1(\mathbf{X}^u_h), U_2(\mathbf{X}^u_h)]$, respectively.

Step 3. We test the hypotheses (3) and (4), based on the confidence intervals $[L_1(\mathbf{X}^l_h), L_2(\mathbf{X}^l_h)]$ and $[U_1(\mathbf{X}^u_h), U_2(\mathbf{X}^u_h)]$, respectively. In fact, the test functions can be shown in the following way

$$\varphi(\mathbf{X}^l_h) = \left\{ \frac{I_{[L_1(\mathbf{X}^l_h), L_2(\mathbf{X}^l_h)]}(\theta^l_{0h})}{\text{Accept } H_0(\theta^l_{0h})}, \frac{1 - I_{[L_1(\mathbf{X}^l_h), L_2(\mathbf{X}^l_h)]}(\theta^l_{0h})}{\text{Reject } H_0(\theta^l_{0h})} \right\}$$
$$\varphi(\mathbf{X}^u_h) = \left\{ \frac{I_{[U_1(\mathbf{X}^u_h), U_2(\mathbf{X}^u_h)]}(\theta^u_{0h})}{\text{Accept } H_0(\theta^u_{0h})}, \frac{1 - I_{[U_1(\mathbf{X}^u_h), U_2(\mathbf{X}^u_h)]}(\theta^u_{0h})}{\text{Reject } H_0(\theta^u_{0h})} \right\}.$$

Example 3.2 *Let $\mathcal{X}_1, \ldots, \mathcal{X}_n \overset{i.i.d.}{\sim} N(\tilde{\theta}, \sigma^2)$, with known crisp variance σ^2. Then, from Corollary 2.1, we have $X^l_{1h}, \ldots, X^l_{nh} \overset{i.i.d.}{\sim} N(\theta^l_h, \sigma^2)$ and $X^u_{1h}, \ldots, X^u_{nh} \overset{i.i.d.}{\sim} N(\theta^u_h, \sigma^2)$. The known two-sided symmetric $1 - \alpha$ confidence intervals for parameters θ^l_h and θ^u_h can be easily derived as follows, respectively,*

$$\begin{aligned} S_T(\boldsymbol{X}^l_h) &= \left\{ \theta^l_h : \left| \frac{\sqrt{n}(\bar{X}^l_h - \theta^l_h)}{\sigma} \right| \leq z_{1-\frac{\alpha}{2}} \right\} \\ &= \left[\bar{X}^l_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}^l_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \right], \quad (5) \\ S_T(\boldsymbol{X}^u_h) &= \left\{ \theta^u_h : \left| \frac{\sqrt{n}(\bar{X}^u_h - \theta^u_h)}{\sigma} \right| \leq z_{1-\frac{\alpha}{2}} \right\} \\ &= \left[\bar{X}^u_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}^u_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \right], \quad (6) \end{aligned}$$

where, $\bar{X}^i_h = n^{-1} \sum_{j=1}^n X^i_{jh}$ for $i = l, u$.

So, the test functions for testing hypotheses (3) and (4) are obtained, respectively, as

$$\varphi(\boldsymbol{X}^l_h) = \left\{ \frac{I_{S_T(\boldsymbol{X}^l_h)}(\theta^l_{0h})}{\textit{Accept } H_0(\theta^l_{0h})}, \frac{1 - I_{S_T(\boldsymbol{X}^l_h)}(\theta^l_{0h})}{\textit{Reject } H_0(\theta^l_{0h})} \right\}$$
$$\varphi(\boldsymbol{X}^u_h) = \left\{ \frac{I_{S_T(\boldsymbol{X}^u_h)}(\theta^u_{0h})}{\textit{Accept } H_0(\theta^u_{0h})}, \frac{1 - I_{S_T(\boldsymbol{X}^u_h)}(\theta^u_{0h})}{\textit{Reject } H_0(\theta^u_{0h})} \right\}.$$

Step 4. Now, we aggregate the results in Step 3, in order to construct a fuzzy test. To do this, first, the values of h for which the null hypotheses (3) and (4) are accepted or rejected, need to be categorized. For simplicity, we utilize Fig. 1 to depict those values of h in their categories. To construct this plot, by beginning from 0 to 1, we put together each confidence interval in (5) and (6) in such a way that every confidence intervals receive h as its height.

Now, by considering the membership function of the fuzzy parameter $\tilde{\theta}_0$ and comparing it with the bounds obtained from ordinary confidence intervals (see Fig. 1), we can determine the values of h for which the null hypotheses in (3) and (4) are accepted or rejected.

To do this, for the fuzzy parameter $\tilde{\theta}_0$ we need to determine the degree of acceptability of the hypothesis $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$, which will be described in the next step.

Step 5. In this step, we employ the procedure introduced by Chachi and Taheri [13] to construct a fuzzy confidence interval for fuzzy parameter $\tilde{\theta}$. Upon this procedure we obtain the fuzzy set $\tilde{C}_T = \{(\tilde{\theta}, C_T(\tilde{\theta})) : \tilde{\theta} \in \mathcal{F}(\Theta)\}$, as a fuzzy confidence interval for the fuzzy parameter $\tilde{\theta}$.

The degree of membership of $\tilde{\theta}$ in the two-sided fuzzy confidence interval $\tilde{C}_T$ is defined as $C_T(\tilde{\theta}) = \frac{W}{W+S}$, where $W = W^l + W^u$, $S = S^l + S^u$, and

$$K^l_{\tilde{\theta};\alpha} = \left\{h : \theta^l_h \in \left[\bar{X}^l_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}^l_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right]\right\}$$

$$C^l_{1;\tilde{\theta};\alpha} = \left\{h : \theta^l_h < \bar{X}^l_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right\}$$

$$C^l_{2;\tilde{\theta};\alpha} = \left\{h : \theta^l_h > \bar{X}^l_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right\}$$

$$K^u_{\tilde{\theta};\alpha} = \left\{h : \theta^u_h \in \left[\bar{X}^u_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}, \bar{X}^u_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right]\right\}$$

$$C^u_{1;\tilde{\theta};\alpha} = \left\{h : \theta^u_h < \bar{X}^u_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right\}$$

$$C^u_{2;\tilde{\theta};\alpha} = \left\{h : \theta^u_h > \bar{X}^u_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right\}$$

$$S^l = \int_{C^l_{1;\tilde{\theta};\alpha}} \left[\bar{X}^l_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} - \theta^l_h\right] dh + \int_{C^l_{2;\tilde{\theta};\alpha}} \left[\theta^l_h - \left(\bar{X}^l_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)\right] dh$$

$$S^u = \int_{C^u_{1;\tilde{\theta};\alpha}} \left[\bar{X}^u_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} - \theta^u_h\right] dh + \int_{C^u_{2;\tilde{\theta};\alpha}} \left[\theta^u_h - \left(\bar{X}^u_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)\right] dh$$

$$W^l = \int_{K^l_{\tilde{\theta};\alpha}} \left[\theta^l_h - \left(\bar{X}^l_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)\right] dh + \int_{K^l_{\tilde{\theta};\alpha}} \left[\bar{X}^l_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} - \theta^l_h\right] dh = \frac{2\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \mathcal{L}\left(K^l_{\tilde{\theta};\alpha}\right),$$

$$W^u = \int_{K^u_{\tilde{\theta};\alpha}} \left[\theta^u_h - \left(\bar{X}^u_h - \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right)\right] dh + \int_{K^u_{\tilde{\theta};\alpha}} \left[\bar{X}^u_h + \frac{\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} - \theta^u_h\right] dh = \frac{2\sigma}{\sqrt{n}} z_{1-\frac{\alpha}{2}} \mathcal{L}\left(K^u_{\tilde{\theta};\alpha}\right)$$

where, $\mathcal{L}(A)$ is the Lebesgue measure of set A (see Fig. 2) [13].

Step 6. Finally, we use the result in Step 5 to define the fuzzy test function for testing hypotheses (2).

Definition 3.2 *The function* $\tilde{\varphi}(\underline{\mathcal{X}}) : \mathcal{F}^n(\mathbb{R}) \to \mathcal{F}\{Accept\ H_0, Reject\ \ H_0\}$ *is called a fuzzy test for testing the hypothesis* $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ *versus* $H_1(\tilde{\theta}_0) : \tilde{\theta} \neq \tilde{\theta}_0$*, at level* α*, if*

$$\tilde{\varphi}(\underline{\mathcal{X}}) = \left\{\frac{C_T(\tilde{\theta}_0)}{Accept\ \ H_0(\tilde{\theta}_0)}, \frac{1 - C_T(\tilde{\theta}_0)}{Reject\ \ H_0(\tilde{\theta}_0)}\right\}.$$

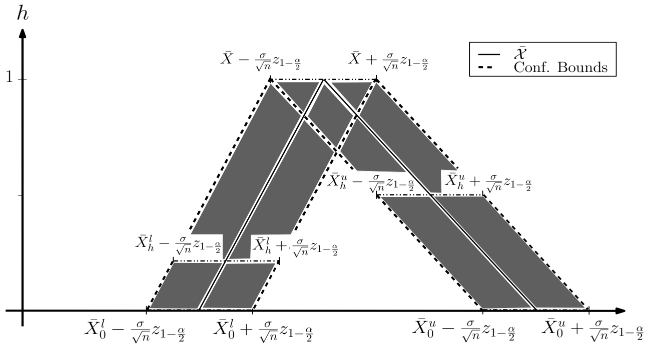

Fig. 1: The class of two-sided $(1-\alpha)$ confidence intervals in Step 4

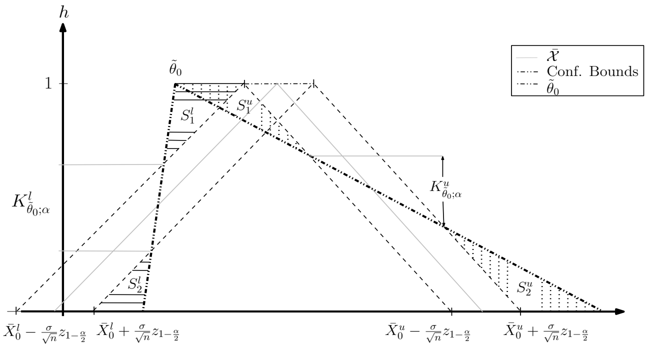

Fig. 2: The hypothesis $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ is partially accepted

Remark 3.1 *It is easy to see that the fuzzy test in Definition 3.2, contrary to the classical crisp test, does not lead to a binary decision, i.e. to accept or to reject the null hypothesis, but to a fuzzy decision. However, we may also get the situation of binary decision for vague data and imprecise parameter. For example, suppose we want to test a hypothesis about the fuzzy parameter* $\tilde{\theta}_0$ *as shown in Fig. 3. It is clear that for each* $h \in [0,1]$, $\theta^l_{0h} \in S_T(\boldsymbol{X}^l_h)$ *and* $\theta^u_{0h} \in S_T(\boldsymbol{X}^u_h)$. *Therefore, we should accept the crisp null hypotheses (3) and (4) for each* $h \in [0,1]$*, so it is reasonable that the null hypothesis* $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ *should be completely accepted. On the other hand, by using the method in Step 5, we also get* $C_T(\tilde{\theta}_0) = 1$. *Similarly, for testing the fuzzy parameter* $\tilde{\theta}_0$ *as shown in Fig. 4, since for each* $h \in [0,1]$, $\theta^l_{0h} \notin S_T(\boldsymbol{X}^l_h)$ *and* $\theta^u_{0h} \notin S_T(\boldsymbol{X}^u_h)$, *we reject the crisp null hypotheses (3) and (4) for each* $h \in [0,1]$. *Therefore, the null hypothesis* $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ *should be completely rejected. By the way, by using the method in Step 5, we also obtain* $C_T(\tilde{\theta}_0) = 0$. *But, for arbitrary* $\tilde{\theta}_0 \in \mathcal{F}(\Theta)$*, the relations* $\theta^l_{0h} \in S_T(\boldsymbol{X}^l_h)$ *and* $\theta^u_{0h} \in S_T(\boldsymbol{X}^u_h)$ *may be satisfied for some values of* $h \in [0,1]$*, so it is reasonable that* $\tilde{\theta}_0$ *belongs to* $\tilde{C}_T$ *partially, i.e.* $0 \leq C_T(\tilde{\theta}) \leq 1$ *(see Fig. 2).*

4 A NUMERICAL EXAMPLE IN LIFETIME TESTING

The following practical example illustrates the application of the study above by means of a lifetime data set.

Example 4.1 (See also [8, 13]) *The marketing department for a tire and rubber company wants to estimate the average life of a tire that the company recently developed. Only 24 new tires were tested because the tests are destructive and take*

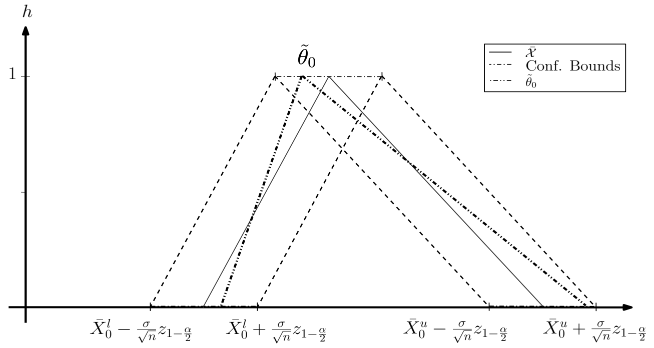

Fig. 3: The hypothesis $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ is completely accepted

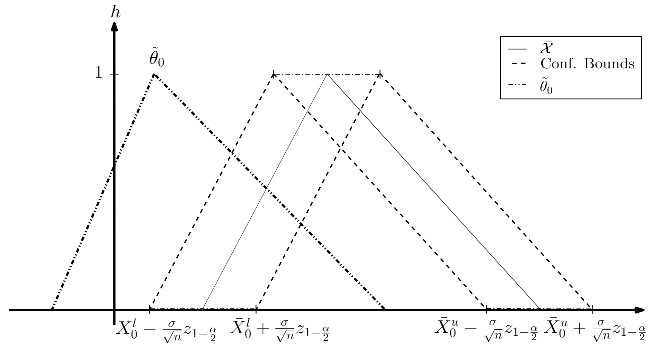

Fig. 4: The hypothesis $H_0(\tilde{\theta}_0) : \tilde{\theta} = \tilde{\theta}_0$ is completely rejected

considerable time to complete. Six cars, all the same model and brand, were used to test the tires. Since, under some unexpected situations, we cannot measure the tire life precisely. We can just obtain the tire life around a number. Therefore the tire life numbers are taken to be triangular fuzzy numbers as in Table i. It is assumed that the data are observations from normally distributed FRVs with variance 747000 *[8].*

Suppose that the company wants to investigate that the average life of the tires the company recently developed exceeds the well-known average tire life of a competitive brand, which is known to be around 32000 miles with triangular fuzzy number $\theta_0 = (32000, 2000, 2000)_T$. *So, we wish to test the following hypotheses*

$$\begin{cases} H_0 : \tilde{\theta} = (32000, 2000, 2000)_T \\ H_1 : \tilde{\theta} \succ (32000, 2000, 2000)_T \end{cases}.$$

Using the procedure introduced in Section 3 and at level $\alpha =$

TABLE I: TRIANGULAR FUZZY NUMBERS AS RECORDED DATA FOR TIRE LIFETIME

$(33978, 712, 911)_T$	$(32617, 524, 638)_T$
$(33052, 467, 735)_T$	$(32611, 891, 886)_T$
$(33418, 612, 490)_T$	$(32455, 478, 579)_T$
$(33463, 368, 668)_T$	$(32466, 523, 746)_T$
$(31624, 881, 836)_T$	$(33070, 901, 898)_T$
$(33127, 712, 945)_T$	$(33543, 643, 792)_T$
$(33224, 537, 684)_T$	$(30881, 554, 564)_T$
$(32597, 412, 589)_T$	$(31565, 378, 672)_T$
$(34036, 613, 735)_T$	$(34053, 845, 823)_T$
$(32584, 945, 958)_T$	$(31838, 893, 901)_T$
$(32290, 779, 774)_T$	$(32800, 866, 645)_T$
$(33844, 784, 605)_T$	$(34157, 693, 817)_T$

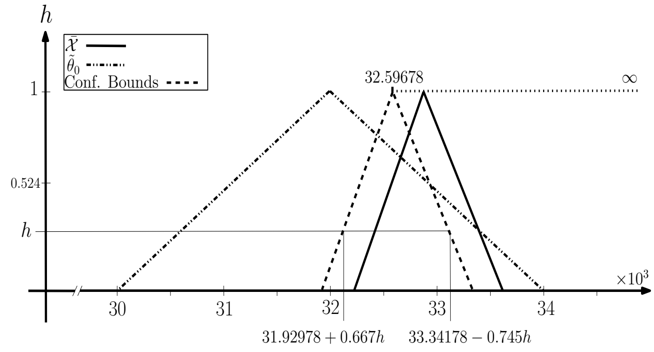

Fig. 5: The membership function of fuzzy numbers in Example 4.1

0.05, *for any* $h \in (0, 1]$ *we have*

$$\begin{aligned} &\bar{\mathcal{X}} = (32887, 667, 745)_T, \\ &\tilde{\theta}_{0h} = [30000 + 2000h, 32000 - 2000h], \\ &H_0^l : \theta_h^l = 30000 + 2000h \quad v.s \quad H_1^l : \theta_h^l > 30000 + 2000h, \\ &H_0^u : \theta_h^u = 32000 - 2000h \quad v.s \quad H_1^u : \theta_h^u > 32000 - 2000h. \end{aligned}$$

The right one-sided 0.95 *confidence intervals for parameters* θ_h^l *and* θ_h^u *are* $[31929.78 + 667h, \infty)$ *and* $[33341.78 - 745h, \infty)$, *respectively. The membership functions of the fuzzy parameter* $\tilde{\theta}_0$, *fuzzy mean* $\bar{\mathcal{X}}$, *and confidence bounds (obtained from confidence intervals) are shown in Fig. 5.*

Based on the proposed procedure, we have (for more details see [13])

$$\begin{aligned} &K^l_{\tilde{\theta};0.05} = \emptyset, & &C^l_{\tilde{\theta};0.05} = [0, 1], & &S = 1405.17, \\ &K^u_{\tilde{\theta};0.05} = [0, 0.524], & &C^u_{\tilde{\theta};0.05} = (0.524, 1], & &W = 172.61. \end{aligned}$$

Therefore, at level $\alpha = 0.05$, *the hypothesis* $H_0 : \tilde{\theta} = (32000, 2000, 2000)_T$ *is accepted against the hypothesis* $H_0 : \tilde{\theta} \succ (32000, 2000, 2000)_T$ *with degree of acceptability* $C_R(\tilde{\theta}_0) = 0.109$, *and thus the fuzzy test function is*

$$\tilde{\varphi}(\underline{\mathcal{X}}) = \left\{ \frac{0.109}{Accept \;\; H_0(\tilde{\theta}_0)}, \frac{0.891}{Reject \;\; H_0(\tilde{\theta}_0)} \right\}.$$

5 CONCLUSION

In the present study, we introduced a novel approach to the problem of testing hypotheses for fuzzy parameter by introducing and applying fuzzy confidence intervals. In this approach the available data are assumed to be the observations of FRVs. We used the degree of membership of hypothesized fuzzy parameter in the fuzzy confidence interval to make inference about the hypotheses of interest.

Contrary to the classical approach, our fuzzy test leads not to a binary decision: to reject or to accept the null hypothesis, but to a fuzzy decision showing the grades of acceptability of the null and the alternative hypotheses which is more flexible than the traditional significance tests. The proposed fuzzy test is a natural generalization of the traditional test, in the sense that if the data and the parameter of interest are precise, then we get a classical statistical test with a binary decision.

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