A fuzzy-based approach to testing statistical hypotheses

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Abstract

An approach to test the crisp statistical hypotheses upon the fuzzy test statistic is developed. The proposed approach is investigated for two cases: the case without nuisance parameter, and the case with a nuisance parameter. For rejecting or accepting the null hypothesis of interest, an evidential point of view is proposed, based on a concept called degree of consistency of observations with the hypothesis. The approach is applied to test the parameters of the normal, Bernoulli, Poisson, and exponential distributions, as well as to test the difference of means and the ratio of variances of two normal distributions.

Keywords: Evidential inference; Fuzzy test statistic; Degree of consistency (DC); Degree of inconsistency (DI); Testing hypothesis

1 Introduction

In classical statistics as well as in traditional probability all parameters of the underlying model, possible observations, and decision rules, should be well defined. Very often, such assumptions appear too rigid for real problems. To relax this rigidity, fuzzy methods are incorporated into probability and statistics. This paper regards the problem of testing statistical hypotheses using a fuzzy-based approach.

The problem of testing hypotheses in fuzzy environment has been studied by some authors. Here, there is a brief review of literatures in this topic. Casals and Gil (1989), Son et al. (1992), Taheri and Behboodian (1999), and Torabi et al. (2006) considered the Neyman-Pearson type testing hypotheses when the available data and/or the hypotheses are fuzzy rather than crisp. Arnold (1996) proposed the fuzzification of usual statistical hypotheses and considered the hypotheses test under fuzzy constraints on the type I and II errors. Körner (2000) presented an asymptotic test for expectation of random fuzzy variables. Montenegro et al. (2001) considered some two-sample hypothesis tests for means concerning a fuzzy random variable in two populations. Taheri and Behboodian (2001, 2006), using a Bayesian approach, considered and analyzed the problem of testing fuzzy hypotheses, with crisp and also with fuzzy data, on the basis of a Bayesian approach. Kahraman et al. (2004) present some algorithms for testing fuzzy parametric and nonparametric test. Filzmoser and Viertl (2004), and Parchami et al. (2010,2011) investigated some p-value-based approach to the problem of testing hypothesis in the fuzzy environments. Wu (2007) investigated a class of fuzzy statistical decision process in which testing hypothesis can be formulated in terms of interval-valued statistics. Torabi and Behboodian (2007) studied the likelihood ratio method for testing fuzzy hypotheses, and Najafi et al. (2010) investigated such method for testing crisp hypotheses based on fuzzy data. Niwitpong et al. (2008) investigated a method of testing hypotheses with interval data. Taheri and Arefi (2009) studied a new approach for testing fuzzy hypotheses based on fuzzy test statistic. A bootstrap approach for testing fuzzy hypotheses based on fuzzy data is introduced by Akbari and Rezaei (2010). Recently, Arefi and Taheri (2011) introduced a procedure to test the fuzzy hypotheses based on a fuzzy test statistic when the available data are fuzzy. For more studies about fuzzy statistics and non-precise (fuzzy) data see Taheri (2003) and Viertl (1996, 2008).

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In this paper, we extend, and in some sense we modify, the method proposed by Buckley (2004) for testing statistical hypotheses. To do this, we put ourselves in the framework of the pivotal approach to construct a fuzzy test statistic. In this context, we provide a triplet procedure for testing hypotheses of interest.

It should be mentioned that, concerning the works by Buckley (2004), Taheri and Arefi (2009), and Arefi and Taheri (2011), the present work has some advantages as follows:

I) The Buckley's method is improved and simplified by removing the fuzzy critical region from the procedure of testing, to achieve a more simply procedure.

II) The cases of existence the nuisance parameter are considered in the present work.

III) The method of evaluating the hypotheses of interest are performed based on the evidential statistics (see Royall (1997,2000)), by introducing a new concept called degree of consistency (DC).

The paper is organized as follows: In Section 2, we recall some preliminary concepts about fuzzy numbers and interval arithmetic. In Section 3, a general method for testing parametric hypotheses based on fuzzy test statistic, when the underlying model has no any nuisance parameter, is investigated. In this section, we apply the proposed method to test the parameter of a normal distribution (mean and variance), the parameter of a Bernoulli distribution, the mean of a Poisson distribution, the mean of an exponential distribution, as well as to test the difference of means of two normal distributions (known variances), and finally to test the ratio of variances of two normal distributions. In Section 4, we focus on the case in which there exists a nuisance parameter in the underlying model. In this section, we apply the proposed method, to test the mean of a normal distribution with unknown variance and also to test the difference between means of two normal distributions with unknown (but equal) variances. Finally, in Section 5 a brief conclusion is provided.

2 Preliminary Concepts

In this section, we provide some preliminary concepts and some notations about fuzzy numbers and interval arithmetic. For details, the reader is referred to standard texts, e.g. Klir and Yuan (1995) and Buckley (2005).

We place a "tilde" over a symbol to denote a fuzzy set. So, $\widetilde{A}: X \to [0,1]$ represents the membership function of the fuzzy set $\widetilde{A}$. An α-cut of $\widetilde{A}$, written $\widetilde{A}[\alpha] = \widetilde{A}_\alpha$, is defined as $\widetilde{A}[\alpha] = \{x | \widetilde{A}(x) \geq \alpha\}$, for $0 < \alpha \leq 1$. A fuzzy number $\widetilde{N}$ is a fuzzy subset of the real numbers satisfying:

i) $\widetilde{N}(x) = 1$ for some x,

ii) $\widetilde{N}[\alpha]$ is a closed bounded interval for $0 < \alpha \leq 1$.

A triangular fuzzy number $\widetilde{T}$, denoted by $\widetilde{T} = (a_1/a_2/a_3)$, is defined as

$$\widetilde{T}(t) = \begin{cases} \frac{t-a_1}{a_2-a_1} & a_1 < t \leq a_2, \\ \\ \frac{t-a_3}{a_2-a_3} & a_2 < t \leq a_3, \\ \\ 0 & otherwise. \end{cases}$$

Let $I = [a, b]$ and $J = [c, d]$ be two closed intervals. Then, based on the interval arithmetic, we have

$$I + J = [a + c, b + d],$$
$$I - J = [a - d, b - c],$$
$$I.J = [\alpha_1, \beta_1], \quad \alpha_1 = \min\{ac, ad, bc, bd\}, \quad \beta_1 = \max\{ac, ad, bc, bd\},$$
$$I \div J = [\alpha_2, \beta_2], \quad \alpha_2 = \min\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\}, \quad \beta_2 = \max\{\frac{a}{c}, \frac{a}{d}, \frac{b}{c}, \frac{b}{d}\},$$

where zero does not belong to $J = [c, d]$ in the last case.

3 Testing hypotheses based on fuzzy test statistic (one parameter case)

3.1 Statement of the method

Let $X_1, X_2, ..., X_n$ be a random sample of size n, from a probability density function (or probability mass function) $f(x;\theta)$, with observed values $x_1, x_2, ..., x_n$, where the parameter of interest θ is unknown. We want to test the following hypotheses

$$\begin{cases} H_0 : \theta = \theta_0, \\ H_1 : \theta > \theta_0, \end{cases} \qquad (1)$$

Let $\theta^* = u(x_1, x_2, ..., x_n)$ be a point estimation for the parameter θ. Suppose that the function $Q(\theta, \theta^*)$ of parameter θ and θ^* is a pivotal quantity. The usual decision rule for testing the above hypotheses, at the significance level β, is

$$\begin{cases} Q_0 \geq Q_{1-\beta} \Rightarrow \ RH_0 \ (\textit{Reject } H_0), \\ Q_0 < Q_{1-\beta} \Rightarrow \ AH_0 \ (\textit{Accept } H_0), \end{cases}$$

where, Q_0 is the crisp test statistic (under H_0), and $Q_{1-\beta}$ is the $(1-\beta)$-quantile of the Q_0.

Now, we state a modification of Buckley's approach for testing the above hypotheses in which we emphasize on the role of the pivotal quantity (see also, Falsafain and Taheri (2011) for an improved method for the fuzzy estimation of parameters on Buckley's approach).

i) First, we find a confidence interval for θ, at the confidence level $1-\alpha$, based on pivotal quantity $Q(\theta, \theta^*)$, for all $0.01 \leq \alpha < 1$. This confidence interval is considered to be the α-cuts of a triangular shaped fuzzy number, $\widetilde{X}[\alpha]$ (so called as the fuzzy estimation of parameter of interest).

ii) By substituting the bounds of confidence interval $\widetilde{X}[\alpha]$ instead of θ^* in the crisp test statistic (Q_0) and using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$.

Now, we introduce an approach for evaluating the statistical hypotheses (1), based on fuzzy test statistic and the following triplet procedure (see Fig. 1)

a) We calculate the total area under the graph of $\widetilde{Z}$, denoted by A_T.

b) We obtain the area under the graph of $\widetilde{Z}$, but to the right of the vertical line through $Q_{1-\beta}$, denoted by A_R.

c) Finally, the hypotheses of interest are evaluated by two indices: degree of consistency $DC = \frac{A_R}{A_T}$ that measure the consistency of data with the hypothesis H_0, and degree of inconsistency $DI = 1 - \frac{A_R}{A_T} = 1 - DC$ that measure the inconsistency of data with the hypothesis H_0 (the consistency of data with the hypothesis H_1).

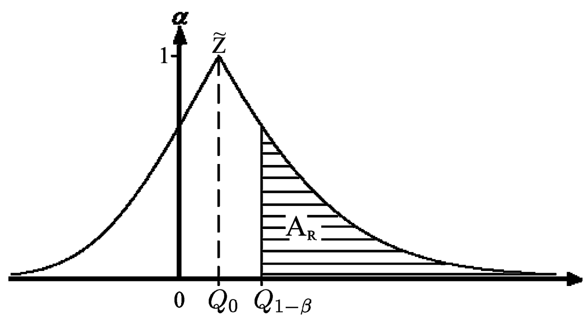

Figure 1. The fuzzy test statistic and A_R in testing hypotheses (1).

Remark 1 *We can apply the above procedure for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta < \theta_0$ (see Fig. 2), and for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ (see Fig. 3) in a similar way.*

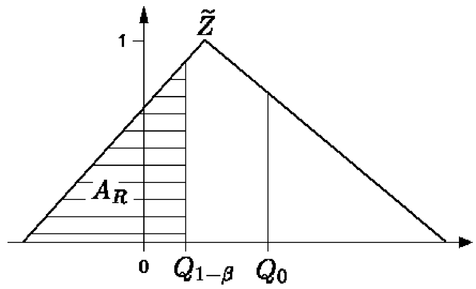

Figure 2. $H_1 : \theta < \theta_0$.

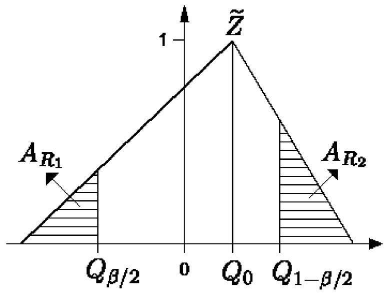

Figure 3. $H_1 : \theta \neq \theta_0$ ($A_R = A_{R_1} + A_{R_2}$).

3.2 Testing hypothesis for the mean of a normal distribution with known variance

Based on a random sample of size n from $N(\theta, \sigma^2)$ (σ^2 known), we want to test the hypotheses (1), at the significance level β. Under H_0, the crisp test statistic $Z_0 = \frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}}$ is distributed by the standard normal distribution, with the observed value $z_0 = \frac{\overline{x}-\theta_0}{\sigma/\sqrt{n}}$. The usual confidence interval for θ, at the confidence level $1-\alpha$, is

$$\tilde{X}[\alpha] = \left[\overline{x} - z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right].$$

By substituting the bounds of the confidence interval instead of $\overline{x}$ in z_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as

$$\tilde{Z}[\alpha] = \left[\frac{\overline{x} - z_{1-\alpha/2}\sigma/\sqrt{n} - \theta_0}{\sigma/\sqrt{n}}, \frac{\overline{x} + z_{1-\alpha/2}\sigma/\sqrt{n} - \theta_0}{\sigma/\sqrt{n}}\right] = [z_0 - z_{1-\alpha/2}, z_0 + z_{1-\alpha/2}].$$

Now, using the above fuzzy test statistic, we can apply the triplet procedure for evaluating the hypotheses (1).

Example 1 *Suppose that, based on a random sample of size $n = 100$ from $N(\theta, \sigma^2 = 4)$, we obtain $\overline{x} = 1.08$. Now, we want to test the following statistical hypotheses, at the significance level $\beta = 0.05$,*

$$\begin{cases} H_0 : \theta = 1, \\ H_1 : \theta > 1. \end{cases}$$

Here, $z_0 = 0.40$, so that we can calculate A_R and A_T in the following way

$$A_R = \int_{z_{1-\beta}}^{z_0+z_{1-\alpha/2}} \Phi(x)dx = \int_{1.645}^{2.975} \Phi(x)dx = 0.1023,$$

$$A_T = 2\int_{z_0}^{z_0+z_{1-\alpha/2}} \Phi(x)dx = 2\int_{0.400}^{2.975} \Phi(x)dx = 1.5958,$$

where $\Phi(x) = 2(1 - F_Z(x - z_0))$ and F_Z is CDF of the standard normal distribution, (we use MATLAB software for calculations). Hence, the degree of consistency of data with the hypothesis H_0 is equal to $DC = \frac{A_R}{A_T} = 0.0641$ (see Fig. 4).

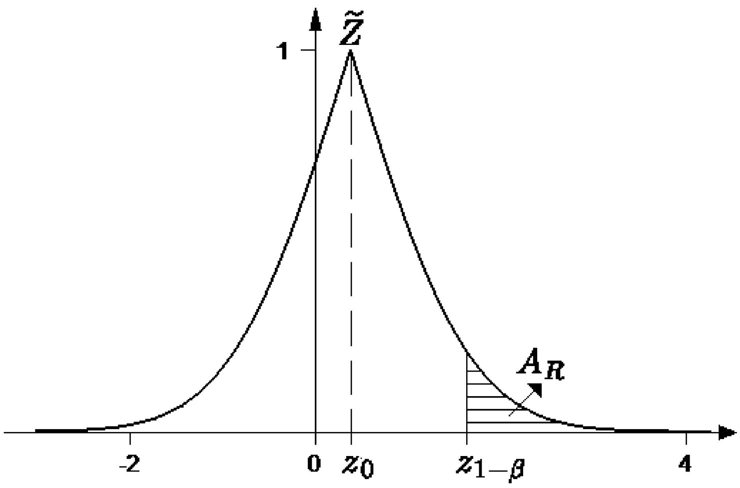

Figure 4. The fuzzy test statistic and A_R in Example 1.

3.3 Testing hypothesis for variance of a normal distribution

Suppose that, we have taken a random sample of size n from $N(\mu,\theta)$, and we want to test the hypotheses (1), at the significance level β. The pivotal quantity based on the estimator $S^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$ is $\frac{(n-1)S^2}{\theta}$. Under H_0, the crisp test statistic $\frac{(n-1)S^2}{\theta_0}$ is distributed according to χ^2_{n-1} with $Q_0=\frac{(n-1)\theta^*}{\theta_0}=\frac{(n-1)s^2}{\theta_0}$ as its observed value. The related confidence interval for θ, at the confidence level $1-\alpha$, is

$$\tilde{X}[\alpha]=\left[\frac{(n-1)\theta^*}{\chi^2_{n-1,1-\alpha/2}},\frac{(n-1)\theta^*}{\chi^2_{n-1,\alpha/2}}\right].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of θ^* in Q_0, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are calculated as

$$\tilde{Z}[\alpha]=\frac{(n-1)\tilde{X}[\alpha]}{\theta_0}=\left[\frac{(n-1)Q_0}{\chi^2_{n-1,1-\alpha/2}},\frac{(n-1)Q_0}{\chi^2_{n-1,\alpha/2}}\right].$$

Now, based on the above fuzzy test statistic, we can evaluate the hypotheses (1) by the proposed triplet procedure.

Example 2 *Suppose that, based on a random sample of size $n=75$ from $N(\mu,\theta)$, we observe $s^2=2.2635$. We want to test the following hypotheses, at the significance level $\beta=0.05$,*

$$\begin{cases} H_0:\theta=2,\\ H_1:\theta>2.\end{cases}$$

In this case, we can calculate A_R and A_T in the following way ($Q_0=83.7495$)

$$A_R=\int_{\chi^2_{n-1,1-\beta}}^{\frac{(n-1)Q_0}{\chi^2_{n-1,\alpha/2}}}\Phi^*(x)dx=\int_{95.0812}^{133.5172}\Phi^*(x)dx=5.2559,$$

$$A_T=\int_{\frac{(n-1)Q_0}{\chi^2_{n-1,0.995}}}^{\frac{(n-1)Q_0}{\chi^2_{n-1,0.5}}}\Phi(x)dx+\int_{\frac{(n-1)Q_0}{\chi^2_{n-1,0.5}}}^{\frac{(n-1)Q_0}{\chi^2_{n-1,0.005}}}\Phi^*(x)dx=9.7842+12.9166=22.7008,$$

where $\Phi^*(x)=2(1-F_X(\frac{(n-1)Q_0}{x}))$, $\Phi(x)=2F_X(\frac{(n-1)Q_0}{x})$ and F_X is the CDF of Chi-square distribution with $n-1$ degree of freedom. Hence, the degree of consistency of data with the hypothesis H_0 is equal to $DC=\frac{A_R}{A_T}=0.2315$, (also, the degree of inconsistency is $DI=1-DC=0.7685$) (see Fig. 5).

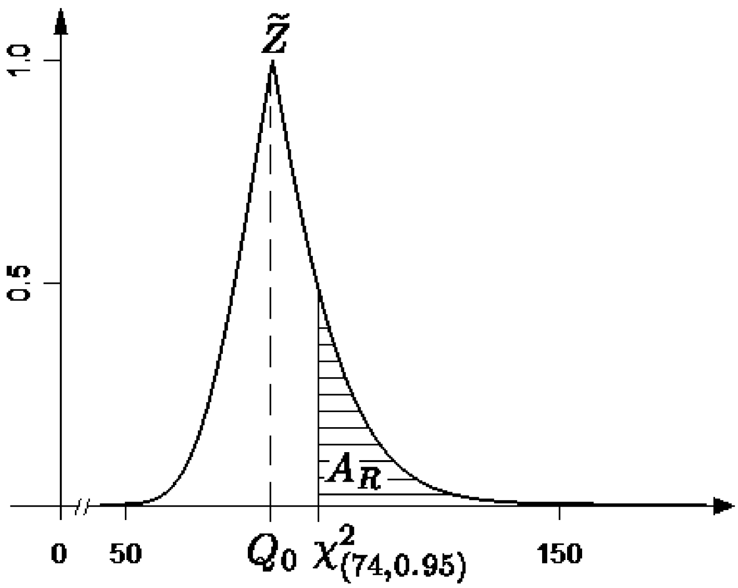

Figure 5. The fuzzy test statistic and A_R in Example 2.

3.4 Testing hypothesis for the proportion of a Bernoulli distribution

Assume that, based on a random sample of size n from $Bernoulli(\theta)$, we want to test the hypotheses (1). The usual point estimation for θ is $\widehat{\theta} = \overline{x}$. Under H_0, for large n, the crisp test statistic $\frac{\overline{X}-\theta_0}{\sqrt{\theta_0(1-\theta_0)/n}}$ is distributed approximately as $N(0,1)$. The confidence interval for θ, at the confidence level $1-\alpha$, is

$$\tilde{X}[\alpha] = \left[\widehat{\theta} - z_{1-\alpha/2}\sqrt{\widehat{\theta}(1-\widehat{\theta})/n}, \widehat{\theta} + z_{1-\alpha/2}\sqrt{\widehat{\theta}(1-\widehat{\theta})/n}\right].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of $\widehat{\theta}$ in $z_0 = \frac{\widehat{\theta}-\theta_0}{\sqrt{\theta_0(1-\theta_0)/n}}$, and using the interval arithmetic, we derive the α-cuts of the fuzzy test statistic as

$$\tilde{Z}[\alpha] = \frac{\tilde{X}[\alpha]-\theta_0}{\sqrt{\theta_0(1-\theta_0)/n}} = \left[z_0 - z_{1-\alpha/2}\sqrt{\frac{\widehat{\theta}(1-\widehat{\theta})}{\theta_0(1-\theta_0)}}, z_0 + z_{1-\alpha/2}\sqrt{\frac{\widehat{\theta}(1-\widehat{\theta})}{\theta_0(1-\theta_0)}}\right].$$

Now, we can test the hypotheses (1) by the proposed triplet procedure.

Example 3 *Suppose that, based on a random sample of size $n = 100$ from $Bernoulli(\theta)$, we obtain $\widehat{\theta} = \overline{x} = 0.54$. We want to test the following hypotheses, at the significance level $\beta = 0.05$,*

$$\begin{cases} H_0 : \theta = 0.5, \\ H_1 : \theta > 0.5. \end{cases}$$

Here, A_R and A_T are obtained as ($z_0 = 0.80$)

$$A_R = \int_{z_{1-\beta}}^{z_0+z_{1-\alpha/2}\sqrt{\frac{\widehat{\theta}(1-\widehat{\theta})}{\theta_0(1-\theta_0)}}} \Phi(x)dx = \int_{1.645}^{3.368} \Phi(x)dx = 0.2202,$$

$$A_T = 2\int_{z_0}^{z_0+z_{1-\alpha/2}\sqrt{\frac{\widehat{\theta}(1-\widehat{\theta})}{\theta_0(1-\theta_0)}}} \Phi(x)dx = 2\int_{0.800}^{3.368} \Phi(x)dx = 1.5906,$$

where $\Phi(x) = 2(1 - F_Z((x-z_0)\sqrt{\frac{\theta_0(1-\theta_0)}{\widehat{\theta}(1-\widehat{\theta})}}))$ and F_Z is the CDF of standard normal distribution. Since $\frac{A_R}{A_T} = 0.1384$, the degree of consistency of data with H_0 is $DC = 0.1384$ (see Fig. 6).

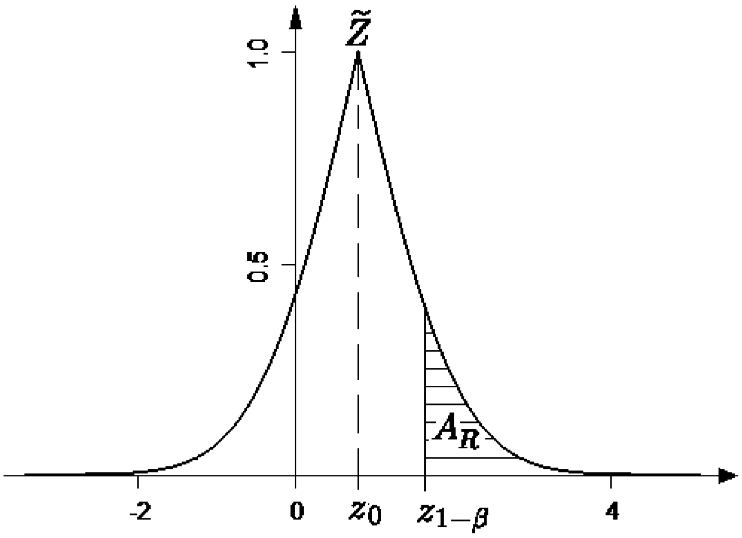

Figure 6. The fuzzy test statistic and A_R in Example 3.

3.5 Testing hypothesis for mean of a Poisson distribution

Assume that, based on a random sample of size n from $Poisson(\theta)$, we want to test the hypotheses (1). The usual point estimation for θ is $\widehat{\theta} = \overline{x}$. Under H_0, for sufficiently large n, the crisp test statistic $Z_0 = \frac{\overline{X}-\theta_0}{\sqrt{\theta_0/n}}$ is distributed approximately as $N(0,1)$. By solving the quadratic equation $(\overline{x}-\theta)^2 = \frac{\theta}{n} z^2_{1-\alpha/2}$ in terms of θ, the Score confidence interval (see Brown et al. 2003) for θ, at the confidence level $1-\alpha$, is obtained as

$$\tilde{X}[\alpha] = \left[\overline{x} + \frac{z_{1-\alpha/2}}{2n}\left(z_{1-\alpha/2} - \sqrt{z^2_{1-\alpha/2} + 4n\overline{x}}\right), \overline{x} + \frac{z_{1-\alpha/2}}{2n}\left(z_{1-\alpha/2} + \sqrt{z^2_{1-\alpha/2} + 4n\overline{x}}\right)\right].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of $\widehat{\theta}$ in $z_0 = \frac{\widehat{\theta}-\theta_0}{\sqrt{\theta_0/n}}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as follows, by which we can test the hypotheses of interest

$$\tilde{Z}[\alpha] = \left[z_0 + \frac{z_{1-\alpha/2}}{2\sqrt{n\theta_0}}\left(z_{1-\alpha/2} - \sqrt{z^2_{1-\alpha/2} + 4n\overline{x}}\right), z_0 + \frac{z_{1-\alpha/2}}{2\sqrt{n\theta_0}}\left(z_{1-\alpha/2} + \sqrt{z^2_{1-\alpha/2} + 4n\overline{x}}\right)\right].$$

Example 4 *Suppose that, we have taken a random sample of size $n = 100$ from $Poisson(\theta)$, and we obtain $\overline{x} = 2.03$, Now, we want to test the following hypotheses, at the significance level $\beta = 0.05$,*

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

In this case, the upper bound of the related $\tilde{Z}[\alpha]$ is ($z_0 = 0.2121$)

$$x = z_0 + \frac{z_{1-\alpha/2}}{2\sqrt{n\theta_0}}\left(z_{1-\alpha/2} + \sqrt{z^2_{1-\alpha/2} + 4n\overline{x}}\right) = 0.2121 + \frac{z_{1-\alpha/2}}{20\sqrt{2}}\left(z_{1-\alpha/2} + \sqrt{z^2_{1-\alpha/2} + 812}\right).$$

But, we can not analytically solve the above equation, in terms of α. Using a numerical method, e.g. the trapezoidal rule (Finney and Thomas 1994), we obtain $A_T = 1.6097$ and $A_R = 0.0919$. Hence, the degree of consistency of data with the hypothesis H_0 is $DC = \frac{A_R}{A_T} = 0.0571$, and the degree of inconsistency is $DI = 0.9429$ (see Fig. 7).

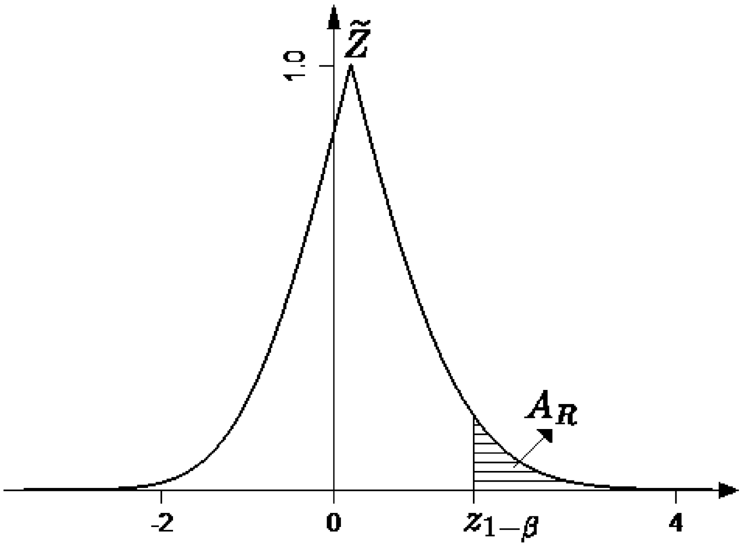

Figure 7. The fuzzy test statistic and A_R in Example 4.

3.6 Testing hypothesis for mean of an exponential distribution

Suppose that, based on a random sample of size n from $Exp(\theta)$ with the following density

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x > 0, \quad \theta > 0,$$

we want to test the hypotheses (1). The usual point estimation for θ is $\theta^* = \overline{x}$. Under H_0, the crisp test statistic $\frac{2n\overline{X}}{\theta_0}$ is distributed as χ^2_{2n}. The confidence interval for θ, at the confidence level $1-\alpha$, is

$$\tilde{X}[\alpha] = \left[\frac{2n\theta^*}{\chi^2_{2n,1-\alpha/2}}, \frac{2n\theta^*}{\chi^2_{2n,\alpha/2}}\right].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of θ^* in $Q_0 = \frac{2n\theta^*}{\theta_0}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are calculated as

$$\tilde{Z}[\alpha] = \frac{2n\tilde{X}[\alpha]}{\theta_0} = \left[\frac{2nQ_0}{\chi^2_{2n,1-\alpha/2}}, \frac{2nQ_0}{\chi^2_{2n,\alpha/2}}\right].$$

Now, based on the above fuzzy test statistic, we can evaluate the hypotheses of interest by the proposed triplet procedure.

Example 5 *Suppose that, based on a random sample of size $n = 50$ from $Exp(\theta)$, we obtain $\overline{x} = 2.33$. We want to test the following hypotheses, at the significance level $\beta = 0.05$,*

$$\begin{cases} H_0 : \theta = 2, \\ H_1 : \theta > 2. \end{cases}$$

Here, $Q_0 = 116.5$, so that we can calculate A_R and A_T in the following way

$$A_R = \int_{\chi^2_{2n,1-\beta}}^{\frac{2nQ_0}{\chi^2_{2n,0.005}}} \Phi(x)dx = \int_{124.3421}^{173.0346} \Phi(x)dx = 9.0576,$$

$$A_T = \int_{\frac{2nQ_0}{\chi^2_{2n,0.995}}}^{\frac{2nQ_0}{\chi^2_{2n,0.5}}} \Phi(x)dx + \int_{\frac{2nQ_0}{\chi^2_{2n,0.5}}}^{\frac{2nQ_0}{\chi^2_{2n,0.005}}} \Phi^*(x)dx = 11.8697 + 15.0628 = 29.9385,$$

where $\Phi(x) = 2(1-F_X(\frac{2nQ_0}{x}))$, $\Phi^*(x) = 2F_X(\frac{2nQ_0}{x})$, and F_X is the CDF of Chi-square distribution with $2n$ degree of freedom. Hence, the degree of consistency of data with the null hypothesis is $DC = \frac{A_R}{A_T} = 0.3025$, and the degree of inconsistency of data with H_0 is $DI = 1-DC = 0.6975$ (see Fig. 8).

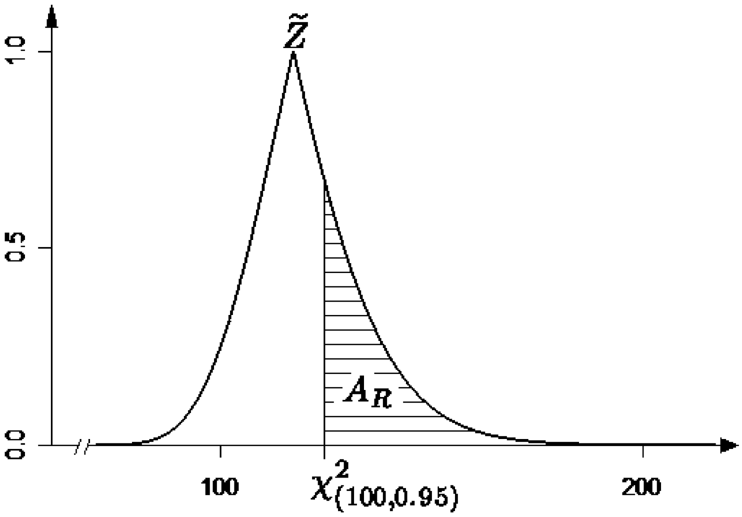

Figure 8. The fuzzy test statistic and A_R in Example 5.

3.7 Testing hypothesis for difference between means of two normal distributions with known variances

Let $X_1, ...X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be two independent random samples of sizes n_1 and n_2 from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively (with σ_1^2 and σ_2^2 known). Now, we want to test the following hypotheses

$$\begin{cases} H_0 : \mu_1 - \mu_2 = \theta_0, \\ H_1 : \mu_1 - \mu_2 > \theta_0. \end{cases}$$

The usual point estimation for $\theta = \mu_1 - \mu_2$ is $\theta^* = \overline{x} - \overline{y}$. Under H_0, the crisp test statistic is distributed as

$$Z_0 = \frac{(\overline{X} - \overline{Y}) - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1).$$

The confidence interval for θ, at the confidence level $1 - \alpha$, is

$$\tilde{X}[\alpha] = \left[\theta^* - z_{1-\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, \theta^* + z_{1-\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of θ^* in $z_0 = \frac{\theta^* - \theta_0}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as follows, by which we can evaluate the hypotheses of interest

$$\tilde{Z}[\alpha] = \frac{\tilde{X}[\alpha] - \theta}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} = [z_0 - z_{1-\alpha/2}, z_0 + z_{1-\alpha/2}].$$

Example 6 *Suppose that based on two independent random samples from $N(\mu_1, \sigma_1^2 = 9)$ and $N(\mu_2, \sigma_2^2 = 4)$, we obtain the following results*

Sample size	*Mean*
$n_1 = 121$	$\overline{x} = 7.27$
$n_2 = 100$	$\overline{y} = 4.76$

Now, assume that we wish to test the following hypotheses, at the significance level $\beta = 0.05$,

$$\begin{cases} H_0 : \mu_1 - \mu_2 = 2, \\ H_1 : \mu_1 - \mu_2 > 2. \end{cases}$$

The A_R and A_T are obtained as follows ($z_0 = 1.508$)

$$A_R = \int_{z_{1-\beta}}^{z_0+z_{1-\alpha/2}} \Phi(x)dx = \int_{1.645}^{4.084} \Phi(x)dx = 0.6685,$$

$$A_T = 2\int_{z_0}^{z_0+z_{1-\alpha/2}} \Phi(x)dx = 2\int_{1.508}^{4.084} \Phi(x)dx = 1.5958,$$

where $\Phi(x) = 2(1 - F_Z(x - z_0))$ and F_Z is the CDF of standard normal distribution. Hence, the degree of consistency of data with H_0 is $DC = \frac{A_R}{A_T} = 0.4189$, and the degree of inconsistency of data with H_0 is $DI = 1 - DC = 0.5811$ (see Fig. 9).

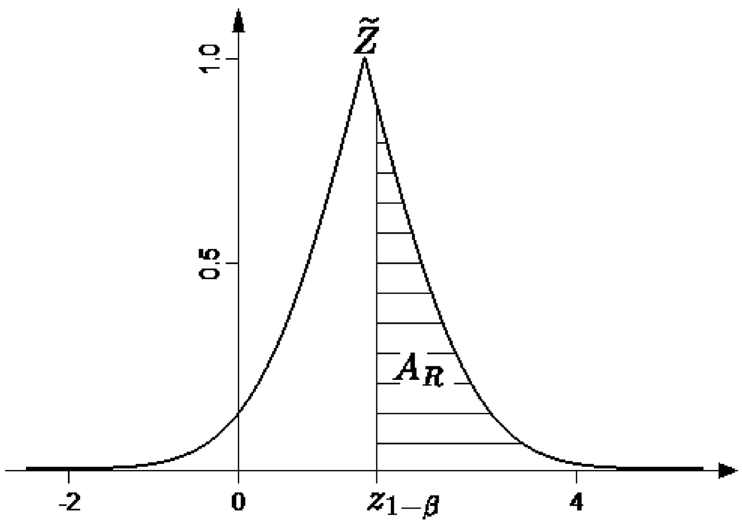

Figure 9. The fuzzy test statistic and A_R in Example 6.

3.8 Testing hypothesis for $\frac{\sigma_1^2}{\sigma_2^2}$ of two normal distributions

Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively. Now, we want to test the following hypotheses

$$\begin{cases} H_0 : \frac{\sigma_1^2}{\sigma_2^2} = \theta_0, \\ H_1 : \frac{\sigma_1^2}{\sigma_2^2} > \theta_0. \end{cases}$$

The usual point estimation for $\theta = \frac{\sigma_1^2}{\sigma_2^2}$ is $\theta^* = \frac{s_1^2}{s_2^2} = \frac{\frac{1}{n_1-1}\sum_{i=1}^{n_1}(x_i-\overline{x})^2}{\frac{1}{n_2-1}\sum_{j=1}^{n_2}(y_i-\overline{y})^2}$. Under H_0, the crisp test statistic is distributed as

$$\frac{S_2^2/\sigma_2^2}{S_1^2/\sigma_1^2} = \frac{S_2^2}{S_1^2}\frac{\sigma_1^2}{\sigma_2^2} \sim F_{n_2-1,n_1-1}.$$

The related confidence interval for θ, at the confidence level $1 - \alpha$, is

$$\tilde{X}[\alpha] = [\theta^* F_{n_2-1,n_1-1,\alpha/2}, \theta^* F_{n_2-1,n_1-1,1-\alpha/2}].$$

Substituting the bounds of $\tilde{X}[\alpha]$ instead of θ^* in $Q_0 = \frac{\theta_0}{\theta^*}$, and using the interval arithmetic, we derive the α-cuts of the fuzzy test statistic as

$$\tilde{Z}[\alpha] = \frac{\theta_0}{\tilde{X}[\alpha]} = \left[\frac{Q_0}{F_{n_2-1,n_1-1,1-\alpha/2}}, \frac{Q_0}{F_{n_2-1,n_1-1,\alpha/2}}\right] = [Q_0 F_{n_1-1,n_2-1,\alpha/2}, Q_0 F_{n_1-1,n_2-1,1-\alpha/2}].$$

Now, based on the above fuzzy test statistic, we can evaluate the hypotheses of interest by the proposed triplet procedure.

Example 7 *Assume that based on two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, we obtain the following results*

Sample size	*Mean*	*Variance*
$n_1 = 121$	$\overline{x} = 6.19$	$s_1^2 = 89.47$
$n_2 = 101$	$\overline{y} = 5.07$	$s_2^2 = 23.02$

We want to test the following hypotheses, at the significance level $\beta = 0.05$,

$$\begin{cases} H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 2, \\ H_1 : \frac{\sigma_1^2}{\sigma_2^2} > 2. \end{cases}$$

Here, $Q_0 = 0.51$ and so

$$Q_0 F_{n_1-1,n_2-1,1-\alpha/2} = 0.51 F_{120,100,0.995} = 0.8423 < 1.3685 = F_{100,120,0.95} = F_{n_2-1,n_1-1,1-\beta}.$$

We obtain $A_R = 0$, which is indicated that the data are completely inconsistent with H_0 (and are completely consistent with H_1) (see Fig. 10).

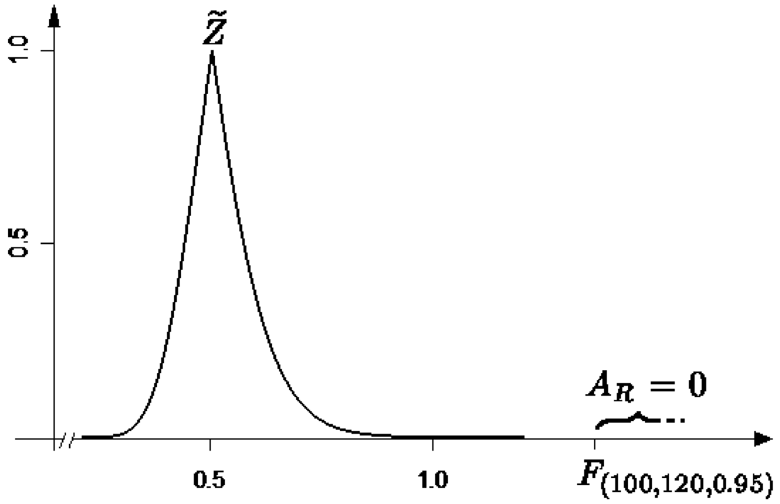

Figure 10. The fuzzy test statistic and A_R in Example 7.

4 Testing hypotheses based on fuzzy test statistic in models with nuisance parameter

4.1 Statement of the method

Let $X_1, X_2, ..., X_n$ be a random sample of size n, from a population with the probability density function (or probability mass function) $f(x; \theta_1, \theta_2)$, with observed values $x_1, x_2, ..., x_n$, where θ_1 is a unknown original parameter and θ_2 is a unknown nuisance parameter. Now, consider the null hypothesis as follows

$$H_0 : \theta_1 = \theta_1^*,$$

where, θ_1^* is a known constant. Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ be the point estimations for θ_1 and θ_2, respectively. Suppose that the functions Q_1 and Q_2 are two pivotal quantities for parameters θ_1 and θ_2.
An approach to test the above hypotheses based on the fuzzy-based approach is introduced in the following way:

i) First, we find the confidence intervals, at the confidence level $1-\alpha$, based on pivotal quantities Q_1 and Q_2 for θ_1 and θ_2 with $0.01 \leq \alpha < 1$. These confidence intervals are considered to be the α-cuts of two triangular shaped fuzzy numbers $\widetilde{X}$ and $\widetilde{\Sigma}$.

ii) By substituting the bounds of the confidence intervals $\widetilde{X}[\alpha]$ and $\widetilde{\Sigma}[\alpha]$ instead of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, respectively, in the crisp test statistic Q_0 and using the interval arithmetic, we obtain the α-cuts of the so-called fuzzy test statistic $\widetilde{Z}$.

Now, based on fuzzy test statistic, the statistical hypothesis H_0 is evaluated by the following triplet procedure.

a) We calculate the total area under the graph of $\widetilde{Z}$, denoted by A_T.

b) For the alternative $H_1 : \theta_1 > \theta_1^*$, we obtain the area under the graph of $\widetilde{Z}$, but to the right of the vertical line through $Q_{1-\beta}$, denoted by A_R. For the alternative $H_1 : \theta_1 < \theta_1^*$, we obtain the area under the graph of $\widetilde{Z}$, but to the left of the vertical line through Q_β, denoted by A_R. For the alternative $H_1 : \theta_1 \neq \theta_1^*$, We obtain the area under the graph of $\widetilde{Z}$, but to the right of the vertical line through $Q_{1-\beta/2}$ and to the left of the vertical line through $Q_{\beta/2}$, denoted by A_R..

c) Finally, the hypotheses of interest are evaluated by two indices: degree of consistency $DC = \frac{A_R}{A_T}$ that measure the consistency of data with the hypothesis H_0, and degree of inconsistency $DI = 1 - \frac{A_R}{A_T} = 1 - DC$ that measure the inconsistency of data with the hypothesis H_0 (the consistency of data with the hypothesis H_1).

4.2 Testing hypothesis for mean of a normal distribution with unknown variance

Suppose that, based on a random sample of size n from $N(\mu, \sigma^2)$ (σ^2 unknown), we want to test the following hypotheses, at the significance level β

$$\begin{cases} H_0 : \mu = \mu_0, \\ H_1 : \mu \neq \mu_0. \end{cases}$$

The usual point estimations for μ and σ^2 are $\widehat{\mu} = \overline{x}$ and $\widehat{\sigma^2} = s^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \overline{x})^2$, respectively. The pivotal quantities for μ and σ^2 are $Q_1 = \frac{\overline{X}-\mu}{s/\sqrt{n}}$ and $Q_2 = \frac{(n-1)S^2}{\sigma^2}$. Under H_0, the crisp test statistic is distributed as

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}.$$

The confidence intervals for μ and σ^2, at the confidence level $1 - \alpha$, are

$$\mu : \qquad \tilde{X}[\alpha] = \left[\overline{x} - t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}}, \overline{x} + t_{n-1,1-\alpha/2}\frac{s}{\sqrt{n}}\right],$$

$$\sigma^2 : \qquad \tilde{\Sigma}[\alpha] = \left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}\right].$$

By substituting the bounds of $\tilde{X}[\alpha]$ and $\tilde{\Sigma}[\alpha]$ instead of $\widehat{\mu} = \overline{x}$ and $\widehat{\sigma^2} = s^2$ in $t_0 = \frac{\overline{x}-\mu_0}{s/\sqrt{n}}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic is obtained as

$$\tilde{Z}[\alpha] = \frac{\tilde{X}[\alpha] - \mu_0}{\sqrt{\tilde{\Sigma}[\alpha]/n}} = [\Pi_1(t_0 - t_{n-1,1-\alpha/2}), \Pi_2(t_0 + t_{n-1,1-\alpha/2})],$$

where $\Pi_1 = \sqrt{\frac{\chi^2_{n-1,\alpha/2}}{n-1}}$ and $\Pi_2 = \sqrt{\frac{\chi^2_{n-1,1-\alpha/2}}{n-1}}$. Now, we can employ the proposed triplet procedure for testing the hypotheses of interest.

Example 8 *Suppose that based on a random sample of size $n = 101$ from $N(\mu, \sigma^2)$, we obtain $\overline{x} = 3.94$ and $s^2 = 1.47$. We want to test the following hypotheses, at the significance level $\beta = 0.10$,*

$$\begin{cases} H_0 : \mu = 4, \\ H_1 : \mu > 4. \end{cases}$$

In this case, we can't solve the equation $x = \Pi_2(t_0 + t_{n-1,1-\alpha/2})$ in terms of α, analytically (here $t_0 = -0.4973$). But, using the trapezoidal rule (Finney and Thomas 1994), we obtain $A_T = 1.5492$ and $A_R = A_{R_1} + A_{R_2} = 0.0639 + 0.0272 = 0.0911$. Hence, the degree of consistency of data with H_0 is $DC = \frac{A_R}{A_T} = 0.0588$, and the degree of inconsistency of data with H_0 is $DI = 1 - DC = 0.9412$ (see Fig. 11).

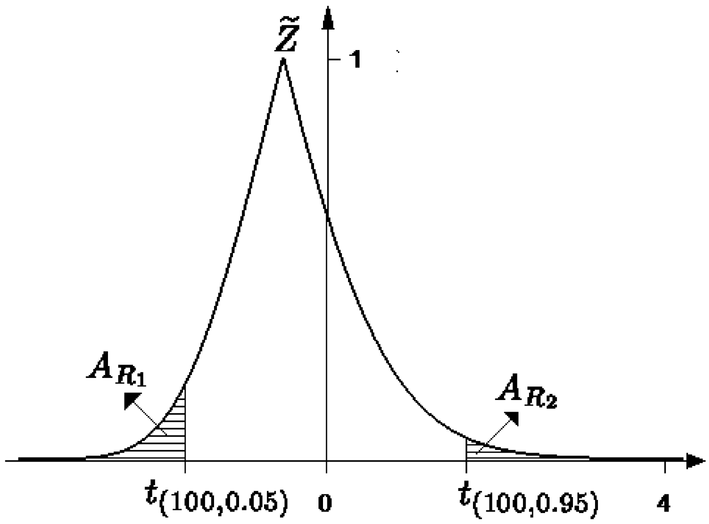

Figure 11. The fuzzy test statistic and $A_R = A_{R_1} + A_{R_2}$ in Example 8.

4.3 Testing hypothesis for difference between means of two normal distributions with equal (but unknown) variances

Let $X_1, ..., X_{n_1}$ and $Y_1, ..., Y_{n_2}$ be two independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, respectively (where μ_1, μ_2 and σ^2 are unknown). We want to test the following hypotheses, at the significance level β

$$\begin{cases} H_0 : \mu_1 - \mu_2 = \delta_0, \\ H_1 : \mu_1 - \mu_2 > \delta_0. \end{cases}$$

The usual point estimations for $\theta_1 = \mu_1 - \mu_2$ and $\theta_2 = \sigma^2$ are $\widehat{\theta}_1 = \overline{x} - \overline{y}$ and $\widehat{\theta}_2 = s_p^2 = \frac{(n_1-1)s_x^2 + (n_2-1)s_y^2}{n_1+n_2-2}$, respectively. The pivotal quantities for θ_1 and θ_2 are $Q_1 = \frac{(\overline{X}-\overline{Y})-\theta_1}{s_p\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$ and $Q_2 = \frac{(n_1+n_2-2)S_p^2}{\sigma^2}$, respectively. Under H_0, the crisp test statistic is distributed as

$$\frac{(\overline{X} - \overline{Y}) - \delta_0}{S_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}.$$

The related confidence interval for θ_1 and θ_2, at the confidence level $1 - \alpha$, are as follows

$$\theta_1 : \quad \tilde{X}[\alpha] = \left[(\overline{x} - \overline{y}) - t_{n_1+n_2-2,1-\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, (\overline{x} - \overline{y}) + t_{n_1+n_2-2,1-\alpha/2}s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}\ \right],$$

$$\theta_2 : \quad \tilde{\Sigma}[\alpha] = \left[\frac{(n_1+n_2-2)s_p^2}{\chi^2_{n_1+n_2-2,1-\alpha/2}}, \frac{(n_1+n_2-2)s_p^2}{\chi^2_{n_1+n_2-2,\alpha/2}}\right].$$

By substituting the bounds of $\tilde{X}[\alpha]$ and $\tilde{\Sigma}[\alpha]$ instead of $\widehat{\theta}_1$ and $\widehat{\sigma^2}$ in $t_0 = \frac{\widehat{\theta}_1 - \delta_0}{\widehat{\sigma}\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}$, and using the interval arithmetic, the α-cuts of the fuzzy test statistic are obtained as

$$\tilde{Z}[\alpha] = \frac{\tilde{X}[\alpha] - \delta_0}{\sqrt{\tilde{\Sigma}[\alpha](\frac{1}{n_1} + \frac{1}{n_2})}} = [\Pi_1(t_0 - t_{n_1+n_2-2,1-\alpha/2}), \Pi_2(t_0 + t_{n_1+n_2-2,1-\alpha/2})],$$

where $\Pi_1 = \sqrt{\frac{\chi^2_{n_1+n_2-2,\alpha/2}}{n_1+n_2-2}}$ and $\Pi_2 = \sqrt{\frac{\chi^2_{n_1+n_2-2,1-\alpha/2}}{n_1+n_2-2}}$. Finally, the hypotheses of interest could be evaluated by the triplet procedure.

Example 9 *Based on two independent random samples from $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$, the following results are obtained*

Sample size	*Mean*	*Variance*
$n_1 = 81$	$\overline{x} = 2.84$	$s_1^2 = 3.87$
$n_2 = 101$	$\overline{y} = 0.17$	$s_2^2 = 5.07$

Suppose that, we want to test the following hypotheses, at the significance level $\beta = 0.05$,

$$\begin{cases} H_0 : \mu_1 - \mu_2 = 2.5, \\ H_1 : \mu_1 - \mu_2 > 2.5\cdot \end{cases}$$

Here, $t_0 = 0.54$, and using the trapezoidal rule (Finney and Thomas 1994), we obtain $A_T = 1.6464$ and $A_R = 0.1895$. Hence, the degree of consistency of data with H_0 is $DC = \frac{A_R}{A_T} = 0.1151$, and also, the degree of inconsistency of data with H_0 is $DI = 1 - DC = 0.8849$ (see Fig. 12).

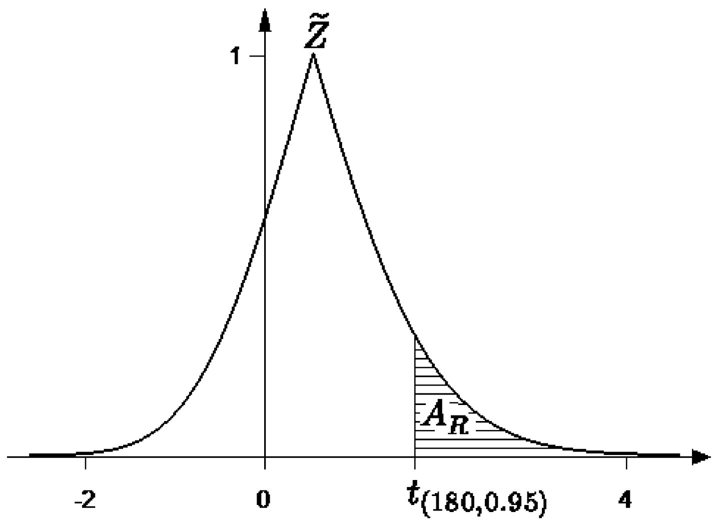

Figure 12. The fuzzy test statistic and A_R in Example 9.

5 Conclusion

The problem of testing crisp statistical hypotheses was investigated, using a fuzzy-based approach. The proposed approach has some advantages regards to more simplicity (with respect to some other approaches) and because of considering the case with nuisance parameter. Another performance of the introduced approach is that it proposed an evidential inference for evaluating the hypotheses of interest. Contrary to the classical approach (in which the decision to accept or reject a null hypothesis depends just on the significance level), the proposed procedure is based on two criteria: a significance level and a degree of consistency.

Further research may be concerned with extending the results to test the fuzzy hypotheses based on fuzzy data. In addition, employing the proposed approach in a Bayesian framework may be a suitable topic for more research.

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References

[1] M. GH. Akbari and A. Rezaei (2010). Bootstrap testing fuzzy hypotheses and observations on fuzzy statistic, *Expert Systems with Applications,* 37, 5782-5787.

[2] M. Arefi and S.M. Taheri (2011). Testing fuzzy hypotheses using fuzzy data based on fuzzy test statistic, *Journal of Uncertain Systems,* 5, 45-61.

[3] B.F. Arnold (1996). An approach to fuzzy hypothesis testing, *Metrika,* 44, 119-126.

[4] L.D. Brown, T. Cai, and A. DusGupta (2003). Interval estimation in exponential families, *Statistica Sinica,* 13, 19-49.

[5] J.J. Buckley (2004). Fuzzy statistics: hypothesis testing, *Soft Computing,* 9, 512-518.

[6] J.J. Buckley (2005). *Fuzzy Statistics,* Springer, Berlin.

[7] M.R. Casals and M.A. Gil (1989). A note on the operativeness of Neyman-Pearson tests with fuzzy information, *Fuzzy Sets and Systems,* 30, 215-220.

[8] A. Falsafain and S.M. Taheri (2011). On Buckley's approach to fuzzy estimation, *Soft computing,* (to appear).

[9] P. Filzmoser and R. Viertl (2004). Testing hypotheses with fuzzy data: the fuzzy p-value, *Metrika,* 59, 21-29.

[10] R.L. Finney and G.B. Thomas (1994). *Calculus,* 2nd ed. Addison Wesley, New York.

[11] C. Kahraman, C.E. Bozdag, D. Ruan, and A.F. Ozok (2004). Fuzzy sets approaches to statistical parametric and nonparametric tests, *International Journal of Intelligent Systems,* 19, 1-19.

[12] G.J. Klir and B. Yuan (1995). *Fuzzy Sets and Fuzzy Logic, Theory and Applications.* Prentic-Hall, Englewood Cliffs, NJ.

[13] R. Körner (2000). An asymptotic α-test for the expectation of random fuzzy variables, *Journal of Statistical Planning and Inference,* 83, 331-346.

[14] M. Montenegro, M.R. Casals, M.A. Lubiano, and M.A. Gil (2001). Two sample hypothesis tests of a fuzzy random variable, *Information Sciences,* 133, 89-100.

[15] Z. Najafi, S.M. Taheri, and M. Mashinchi (2010). Likelihood ratio test based on fuzzy data, *International Journal of Intelligent Technologies and Applied Statistics,* 3, 285-301.

[16] S. Niwitpong, H.T. Nguyen, V. Kreinvich, and I. Neumann (2008). Hypothesis testing with interval data: case of regulatory conditions, *International Journal of Intelligent Technologies and Applied Statistics,* 1, 19-41.

[17] A. Parchami, S.M. Taheri, and M. Mashinchi (2010). Fuzzy p-value in testing fuzzy hypotheses with crisp data, *Statistical Papers,* 51, 209-226.

[18] A. Parchami, S.M. Taheri, and M. Mashinchi (2011). Testing fuzzy hypotheses based on vague observations: a p-value approach, *Statistical Papers,* (to appear).

[19] R.M. Royall (1997). *Statistical Evidence: A Likelihood Paradigm,* Chapman and Hall, London.

[20] R.M. Royall (2000). On the probability of observing misleading statistical evidence, *Journal of the American Statistical Association,* 95, 760-780.

[21] J.C. Son, I. Song, and H.Y. Kim (1992). A fuzzy decision problem based on the generalized Neyman-Pearson criterion, *Fuzzy Sets and Systems,* 47, 65-75.

[22] S.M. Taheri (2003). Trends in fuzzy statistics, *Austrian Journal of Statistics,* 32, 239-257.

[23] S.M. Taheri and M. Arefi (2009). Testing fuzzy hypotheses based on fuzzy test statistic, *Soft Computing,* 13, 617-625.

[24] S.M. Taheri and J. Behboodian (1999). Neyman-Pearson Lemma for fuzzy hypothesis testing, *Metrika,* 49, 3-17.

[25] S.M. Taheri and J. Behboodian (2001). A Bayesian approach to fuzzy hypotheses testing, *Fuzzy Sets and Systems,* 123, 39-48.

[26] S.M. Taheri and J. Behboodian (2006). On Bayesian approach to fuzzy hypotheses testing with fuzzy data, *Italian Journal of Pure and Applied Mathematics,* 19, 139-154.

[27] H. Torabi and J. Behboodian (2007). Likelihood ratio test for fuzzy hypotheses testing, *Statistical Papers,* 48, 509-522.

[28] H. Torabi, J. Behboodian, and S.M. Taheri (2006). Neyman-Pearson Lemma for fuzzy hypotheses testing with vague data, *Metrika,* 64, 289-304.

[29] R. Viertl (1996). *Statistical Methods for Non-Precise Data,* CRC Press, Boca Raton.

[30] R. Viertl (2008). Fuzzy models for precision measurements, *Mathematics and Computers in Simulation,* 79, 874-878.

[31] B. Wu and Sh.K. Chang (2007). On testing hypothesis of fuzzy sample mean, *Japan Journal of Industrial and Applied Mathematics,* 24, 197-209.