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CONTINUITY OF SEPARATELY CONTINUOUS MAPPINGS

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ABSTRACT. By means of a topological game, a class of topological spaces which contains compact spaces, q-spaces and W-spaces was defined in [BOUZIAD, A.: *The Ellis theorem and continuity in groups*, Topology Appl. **50** (1993), 73–80]. We will show that if Y belongs to this class, every separately continuous function $f\colon X \times Y \to Z$ is jointly continuous on a dense subset of $X \times Y$ provided that X is σ-β-unfavorable and Z is a regular weakly developable space.

1. Introduction

Let X, Y and Z be topological spaces and $f\colon X \times Y \to Z$ be a function. The function f is called *separately continuous* if for each $(a, b) \in X \times Y$, the functions $x \mapsto f(x, b)$ and $y \mapsto f(a, y)$ are continuous. If f is continuous with respect to the product topology on $X \times Y$, we say that f is *jointly continuous*.

Let $(a, b) \in X \times Y$, if for each neighborhood W of $f(a, b)$ in Z and for each product of open sets $U \times V \subset X \times Y$ containing (a, b), there is a non-void open set $U_1 \subset U$ and a neighborhood $V_1 \subset V$ of b such that $f(U_1 \times V_1) \subset W$, then f is called *strongly quasi-continuous* at (a, b). The function f is called strongly quasi-continuous if f is strongly quasi-continuous at each point of $X \times Y$.

It is natural to ask when a separately continuous mapping $f\colon X \times Y \to Z$, where X is a Baire space and Z is a topological space, is strongly quasi-continuous?

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Hansel and Troallic [9] proved that every separately continuous mapping $f\colon X \times Y \to Z$ is strongly quasi-continuous provided that X and Y are countably Čech complete regular spaces and Z is a metric space. They also proved that in such a situation, for any $b \in Y$, the set of $x \in X$ such that f is (jointly) continuous at (x, b) is a dense G_δ subset of X. Piotrowski in [17: Theorem A] (see also [16: Theorem 2]) improved the latter result as follows.

Theorem 1.1. *Let X be a Baire space, Y be a topological space and Z be a developable space, then the set of joint continuity of a strongly quasi-continuous mapping $f\colon X \times Y \to Z$ is a dense G_δ subset of $X \times \{y\}$ for each $y \in Y$.*

Bouziad [2: Lemma 2.6] proved the following.

Theorem 1.2. *Let X be a σ-β-unfavorable space, Y a q-space, Z a completely regular space and $f\colon X \times Y \to Z$ a separately continuous mapping. Then f is strongly quasi-continuous.*

Several other partial results to the above question were obtained under some topological restrictions (see for example [4, 10, 11, 13, 14]).

In this paper, we will show that in order to investigate strong quasi-continuity of a separately continuous mapping $f\colon X \times Y \to Z$, where Z is completely regular, we may assume that $Z = \mathbb{R}$. So that, it is natural to find another candidate in place of complete regularity of Z in Theorem 1.2. We extend the above results by showing that the same conclusions hold if we merely assume that Z is a weakly developable T_3 space. Bouziad in [3] introduced a topological game on a topological space Y between two players $\mathcal{O}$ and $\mathcal{P}$ to obtain the points of joint continuity of a separately continuous mapping. It follows from the definition that compact spaces, W-space and p-spaces [7] belong to the class of $\mathcal{O}$-favorable spaces (see the next section for definitions). We will show that if X is σ-β-unfavorable and Y is $\mathcal{O}$-favorable, every separately continuous function $f\colon X \times Y \to Z$ is strongly quasi-continuous, provided that Z is regular weakly developable space. This extends [3: Theorems 1.2, 2.3].

2. Results

In order to state the main results of this paper, we need to introduce two classes of topological spaces, which are defied via topological games. The first game, which was introduced by Christensen in [6], is similar to the Banach-Mazur game [15].

The Christensen's topological game $\mathcal{CH}(X)$

Let X be a topological space. The game $\mathcal{CH}(X)$ is played by two players α and β. The player β starts a game by selecting a nonempty open subset U_1 of X. In return, α-player replies by selecting a pair (V_1, a_1), where V_1 is nonempty open subset of U_1 and $a_1 \in X$. In general, at the nth stage of the game, $n > 1$, the player β chooses a nonempty open subset $U_n \subset V_{n-1}$ and α answers by choosing a pair (V_n, a_n) where V_n is a nonempty open subset of U_n and $a_n \in X$. Proceeding in this fashion, the players generate an infinite sequence $(U_n, (V_n, a_n))_{n=1}^{\infty}$ which is called a *play*. The player α is said to have *won* the play if $\big(\bigcap_{n\geq 1} U_n\big) \cap \overline{\{a_1, a_2, \dots\}} \neq \emptyset$; otherwise the player β is said to have won this play. A *partial play* is a finite sequence of sets consisting of the first few moves of a play. A *strategy* of a player is a function which prescribes the next move of him provided that a (finite) sequence of his opponent's moves is given. A strategy of a player is a *winning* one if, applying that one, he wins each play. We will say that X is σ-β-favorable, provided player β has a winning strategy in $\mathcal{CH}(X)$. Otherwise, X is called σ-β-unfavorable.

The following topological game was first introduced by Bouziad in [3].

The topological game $\mathcal{G}(Y, b)$

Let Y be a topological space and $b \in Y$, the topological game $\mathcal{G}(Y, b)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows. Player $\mathcal{O}$ goes first by selecting an open neighborhood H_1 of b. $\mathcal{P}$ answers by choosing a point $y_1 \in H_1$.

In general, in step n, if selections $H_1, y_1, \dots, H_n, y_n$ have already been specified, $\mathcal{O}$ selects an open set H_{n+1} with $b \in H_{n+1}$ and then $\mathcal{P}$ chooses a point $y_{n+1} \in H_{n+1}$. We say $\mathcal{O}$ wins the game $g = (H_n, y_n)_{n\geq 1}$ if (y_n) has a cluster point in Y. Otherwise $\mathcal{P}$ wins this game.

The terms strategy and winning strategy for one of the players are defined similar to that of the game $\mathcal{CH}(X)$.

We say that a topological space Y is $\mathcal{O}$-favorable for the game $\mathcal{G}(Y, b)$ if the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, b)$, where $b \in Y$. The topological space Y is called $\mathcal{O}$-favorable if for each $b \in Y$, the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, b)$. Clearly each compact space is $\mathcal{O}$-favorable. In [8], the same topological game, but with a different winning rule, has been defined to introduce a class of topological spaces, called W-spaces. It follows from the definition that in a W-space Y, for each $y \in Y$, the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, y)$ but the converse is not true in general. Hence the class of $\mathcal{O}$-favorable spaces contains all W-spaces.

Definition 2.1. A topological space Z is said to be *weakly developable* [1] if there is a sequence of open coverings $(\mathcal{G}_n)$ of Z such that $\big\{\bigcap_{k\leq n} G_k :\ n\geq 1\big\}$ is a base at z provided that $z\in\bigcap_{n\geq 1} G_n$ where $G_n\in\mathcal{G}_n$ for each $n\geq 1$. In this case, $\{\mathcal{G}_n :\ n\geq 1\}$ is called a *weak development* on Z.

Theorem 2.2. *Let X be σ-β-unfavorable space and Y be $\mathcal{O}$-favorable. If f is a separately continuous mapping from $X\times Y$ into a weakly developable T_3 space Z, then f is strongly quasi-continuous.*

P r o o f. Let us suppose that the result is not true, then there is some $a\in X$ and neighborhoods U, H and W of a, b and $f(a,b)$ respectively such that for each open subset U' of U and every neighborhood H' of b with $H'\subset H$, $f(U'\times H')$ is not contained in W. Since Z is regular, there is an open set W' in Z such that

$$f(a,b)\in W'\subset\overline{W'}\subset W.$$

Let $\{\mathcal{G}_n :\ n\geq 1\}$ be a weakly development for Z, and $\hat{s}$ be a winning strategy for the player $\mathcal{O}$ in $\mathcal{G}(Y,b)$. We simultaneously define a strategy τ for β in $\mathcal{CH}(X)$, and a strategy s for $\mathcal{P}$ in $\mathcal{G}(Y,b)$ as follows.

By continuity of $x\mapsto f(x,b)$, there is an open subset U_1 of U such that $f(U_1\times\{b\})\subset W'$. Define $\tau(\emptyset)=U_1$. Let H_1 be the first choice of the player $\mathcal{O}$ and let (V_1,a_1) be the response of α to U_1. Choose some $G_{1,1}\in\mathcal{G}_1$ such that $f(a_1,b)\in G_{1,1}$ and define

$$H_1'=\big\{y\in H_1\cap H :\ f(a_1,y)\in G_{1,1}\big\}.$$

Then H_1' is a neighborhood of b which contained in H. By our hypothesis $f(V_1\times H_1')$ is not contained in W. Therefore we can find some $(x_1,y_1)\in V_1\times(H_1\cap H)$ such that $f(x_1,y_1)\notin W$. Define $s(H_1)=y_1$.

In general, in step $n>1$, let the partial plays $p_n=(H_1,\dots,H_n)$ and $g_n=(U_1,\dots,(V_{n-1},a_{n-1}))$ together with $(x_i,y_i)\in V_i\times H_i$, $1\leq i\leq n-1$, with $f(x_i,y_i)\notin W$ be selected, where the sets $H_1,\dots,H_n$ are chosen according to the winning strategy $\hat{s}$. Moreover, for each $1\leq i\leq n-1$, choose some $G_{i,n}\in\mathcal{G}_n$ such that $f(a_i,b)\in\mathcal{G}_{i,n}$ and define

$$H_n'=\bigcap_{1\leq i<n}\Big\{y\in H_n\cap H :\ f(a_i,y)\in\bigcap_{i\leq k\leq n} G_{i,k}\Big\}.$$

Then $H_n'\subset H$ is a neighborhood of b. Therefore, there is some $(x_n,y_n)\in V_n\times H_n'$ such that $f(x_n,y_n)\notin W$. Now choose an open subset U_n of V_{n-1} such that

$$f(U_n\times\{y_n\})\subset Z\setminus\overline{W'}. \tag{2.1}$$

Define $\tau(g_n) = U_n$ and $s(p_n) = y_n$. In this way, by induction on n strategies τ and s respectively for players β and $\mathcal{P}$ are defined. Since X is σ-β-unfavorable, there is a plays $p = \big(U_n, (V_n, a_n)\big)_{n\geq 1}$ in $\mathcal{CH}(X)$ that α wins, so there exists $t \in \big(\bigcap_{n\geq 1} U_n\big) \cap \overline{\{a_1, a_2, \dots\}}$. Moreover, $(H_n, y_n)_{n\geq 1}$ is a play of $\mathcal{G}(Y, b)$ compatible with the strategy $\hat{s}$, so player $\mathcal{O}$ wins it; thus, the sequence $(y_n)_n$ has some cluster point of z (without loss of generality, assume that $(y_n)_n$ converges to z). It follows from (2.1) that $f(t, y_n) \in Z \setminus \overline{W'}$ for each $n \geq 1$, therefore

$$f(t,z) \in \overline{Z \setminus \overline{W'}} \subset Z \setminus W'.$$

On the other hand for each $i \geq 1$ and $n \geq i$ we have $f(a_i, b) \in G_{i,n}$, hence $\big\{ \bigcap_{i\leq k\leq n} G_{i,k} : \ n \geq i \big\}$ is a base at $f(a_i, b)$. Since $f(a_i, y_n) \in \bigcap_{i\leq k\leq n} G_{i,k}$, we have $f(a_i, b) = \lim_{n\to\infty} f(a_i, y_n)$. It follows that $f(a_i, b) = f(a_i, z)$ for each $i \geq 1$. Therefore $f(t, b) = f(t, z)$. However, by the definition of U_1, $f(t, b) \in W'$, which is a contradiction. □

In the next result we will show that Theorem 2.2 holds when Z is assumed to be a completely regular space.

Theorem 2.3. *Let X and Y be topological spaces. Then for each $b \in Y$, the following assertions are equivalent:*

(1) *Every separately continuous mapping from $X \times Y$ to $\mathbb{R}$ is strongly quasi-continuous on $X \times \{b\}$.*

(2) *Every separately continuous mapping from $X \times Y$ to a completely regular space Z is strongly quasi-continuous on $X \times \{b\}$.*

Proof. Clearly (1) follows from (2). Suppose that (1) holds, $b \in Y$ and $f\colon X \times Y \to Z$, where Z is a completely regular space, is a separately continuous mapping. If for some $a \in X$, f is not strongly quasi-continuous at (a, b), we can find neighborhoods U, V and W of a, b and $f(a, b)$ respectively such that for each nonempty pair of open sets (U', V') with $U' \times V' \subset U \times V$ and $b \in V$, the set $f(U \times V)$ is not contained in W. Since Z is completely regular, there is a continuous function $g\colon Z \to [0,1]$ such that $g(f(a,b)) = 1$ and $g(z) = 0$ for each $z \in Z \setminus W$. Let $G = \big\{z \in Z : \ g(z) > \frac{1}{2}\big\} \subset W$. Then G is open in Z and $f(a, b) \in G$. Applying (1) for the separately continuous mapping $g \circ f\colon X \times Y \to \mathbb{R}$, one can find nonempty open sets U' and V' such that $b \in V'$, $U' \times V' \subset U \times V$ and $g \circ f(U' \times V') \subset \big(\frac{1}{2}, 1\big]$. However, by our assumption, there is some $(x_0, y_0) \in U' \times V'$ such that $f(x_0, y_0) \in Z \setminus W$. It follows that $\frac{1}{2} < g \circ f(x_0, y_0) = 0$. This contradiction proves our result. □

Definition 2.4. Let Y be a topological space and $y_0 \in Y$, then y_0 is called a q-point [12] if there is a sequence $(U_n)_{n\geq 1}$ of neighborhoods of y_0 such that $(y_n)_{n\geq 1}$ has a cluster point in Y provided that $y_n \in U_n$ for each $n \in \mathbb{N}$. The space Y is called a q-space if each $y \in Y$ is a q-point.

It follows from the definition that if b is a q-point, then the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, b)$. Therefore we have the following generalization of Theorem 1.1.

Corollary 2.5. *Let X be a σ-β-unfavorable space and Y be $\mathcal{O}$-favorable. Then every separately continuous mapping $f\colon X \times Y \to Z$ is strongly quasi-continuous provided that Z is a completely regular space.*

Theorem 2.6. *Let X be a Baire space, Y be a topological space, $b \in Y$ and Z a weakly developable space. Then the set of joint continuity of a strongly quasi-continuous function $f\colon X \times Y \to Z$ is a dense G_δ subset of $X \times \{b\}$ for each $b \in Y$.*

Proof. Let $(\mathcal{G}_k)_{k\geq 1}$ be a weak development of Y and let A_n be the set of all $x \in X$ such that for some $W_n \in \mathcal{G}_n$ with $f(x, b) \in W_n$, there are open sets $U_n \subset X$ and $H_n \subset Y$ such that $(x, b) \in U_n \times H_n$ and $f(U_k \times H_k) \subset W_n$. By the definition, each A_n is open in X. It follows from the definition of strong quasi-continuity that each A_n is dense in X. Let $D_b = \bigcap_{n\geq 1} A_n$. If $x \in D_b$, then $f(x, b) \in W_n$ for each $n \in \mathbb{N}$. Therefore $\{\bigcap_{k\leq n} W_k\}_{n\geq 1}$ is a base for $f(x, b)$. It follows that for each open neighborhood G of $f(x, b)$, there is some $n_0 \in \mathbb{N}$ such that $\bigcap_{k\leq n_0} W_k \subset G$. Therefore $f(x, b) \in \bigcap_{k\leq n_0} f(U_k \times H_k) \subseteq G$. This proves that f is continuous on $D_b \times \{b\}$.

Conversely, if f is continuous at (x, b), then for each $n \geq 1$ and $W_n \in \mathcal{G}_n$ with $f(x, b) \in W_n$, there exist open neighborhoods U_n, H_n of x and b, respectively, so that $f(U_n \times H_n) \subseteq W_n$, which implies that $x \in D_b$. □

The following result, which follows immediately from Theorems 2.2 and 2.6, generalizes [3: Theorem 2.3].

Corollary 2.7. *Let X be σ-β-unfavorable, Y be $\mathcal{O}$-favorable for the game $\mathcal{G}(Y, b)$ for each $b \in Y$, and Z be a weakly developable T_3 space. Then the set of points of joint continuity of a separately continuous function $f\colon X \times Y \to Z$ is a dense G_δ subset of $X \times \{b\}$ for each $b \in Y$.*

Definition 2.8. Given two groups $(X,*)$ and $(Z,\cdot)$, a group homomorphism h from $(X,*)$ to $(Z,\cdot)$ is a function such that for all $x,x' \in X$ it holds that $h(x*x') = h(x)\cdot h(x')$. From this property, one can deduce that h maps the identity element e_X of X to the identity element e_Z of Z, and it also maps inverses to inverses in the sense that $h(x^{-1}) = h(x)^{-1}$ for each $x \in X$.

A topological group is a group together with a topology on the space such that the group's binary operation and the group's inverse function are continuous functions with respect to the topology.

The following theorem follows from Corollary 2.5. We omit its proof since it is similar to that of [5: Theorem 2.5].

Theorem 2.9. *Let $(X,*)$ and $(Z,\cdot)$ be topological groups. Let X be σ-β-unfavorable, $b \in Y$ and Y be $\mathcal{O}$-favorable for the game $\mathcal{G}(Y,b)$. Let $f\colon X\times Y \to Z$ be separately continuous and $x \mapsto f(x,b)$ be a group homomorphism. Then f is jointly continuous at each point of $X\times\{b\}$. In particular, if Y is $\mathcal{O}$-favorable, then f is jointly continuous on $X\times Y$.*

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