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ON VARIETAL CAPABILITY OF DIRECT PRODUCTS OF GROUPS AND PAIRS OF GROUPS

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ABSTRACT. In this paper we give some conditions in which a direct product of groups is $\mathcal{V}-$capable if and only if each of its factors is $\mathcal{V}-$capable for some varieties $\mathcal{V}$.

1. Introduction and Preliminaries

R. Baer [1] initiated an investigation of the question which conditions a group G must fulfill in order to be the group of inner automorphisms of a group E, that is $(G \cong E/Z(E))$. Following M. Hall and J. K. Senior [5], such a group G is called capable. Baer [1] determined all capable groups which are direct sums of cyclic groups. As P. Hall [4] mentioned, characterizations of capable groups are important in classifying groups of prime-power order.

F. R. Beyl, U. Felgner and P. Schmid [2] proved that every group G possesses a uniquely determined central subgroup $Z^*(G)$ which is minimal subject to being the image in G of the center of some central extension of G. This $Z^*(G)$ is the smallest central subgroup of G whose factor group is capable $[2, Corollary\ 2.2]$. Hence G is capable if and only if $Z^*(G) = 1$ $[2, Corollary\ 2.3]$. They showed that the class of all capable groups is closed under the direct products $[2, Proposition\ 6.1]$. Also, they presented a condition in which the capability of a direct product of finitely many of groups implies the capability of each of the factors $[2, Proposition\ 6.2]$. Moreover, they proved that if

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N is a central subgroup of G, then $N \subseteq Z^*(G)$ if and only if the mapping $M(G) \to M(G/N)$ is monomorphic [2, *Theorem* 4.2].

Then M. R. R. Moghadam and S. Kayvanfar [8] generalized the concept of capability to $\mathcal{V}-$capability for a group G. They introduced the subgroup $(V^*)^*(G)$ which is associated with the variety $\mathcal{V}$ defined by the set of laws V and a group G in order to establish a necessary and sufficient condition under which G can be $\mathcal{V}-$capable [8, *Corollary* 2.4]. They also showed that the class of all $\mathcal{V}-$capable groups is closed under the direct products [8, *Theorem* 2.6]. Moreover, they exhibited a close relationship between the groups $\mathcal{V}M(G)$ and $\mathcal{V}M(G/N)$, where N is a normal subgroup contained in the marginal subgroup of G with respect to the variety $\mathcal{V}$. Using this relationship, they gave a necessary and sufficient condition for a group G to be $\mathcal{V}-$capable [8, *Theorem* 4.4].

In this note, we present some conditions in which the $\mathcal{V}-$capablity of a direct product of finitely many groups implies the $\mathcal{V}-$capablity of each of its factors.

2. Main results

Suppose that $\mathcal{V}$ is a variety of groups defined by the set of laws V. A group G is said to be $\mathcal{V}-$capable if there exists a group E such that $G \cong E/V^*(E)$. If $\psi : E \to G$ is a surjective homomorphism with $ker\psi \subseteq V^*(E)$, then the intersection of all subgroups of the form $\psi(V^*(E))$ is denoted by $(V^*)^*(G)$. It is obvious that $(V^*)^*(G)$ is a characteristic subgroup of G contained in $V^*(G)$. If $\mathcal{V}$ is the variety of abelian groups, then the subgroup $(V^*)^*(G)$ is the same as $Z^*(G)$ and in this case $\mathcal{V}-$capability is equal to capability [8].

Theorem 2.1. [8] *(i) A group G is $\mathcal{V}-$capable if and only if $(V^*)^*(G) = 1$.*
(ii) $(V^*)^*(\prod_{i\in I} G_i) \subseteq \prod_{i\in I}(V^*)^*(G_i)$.

As a consequence, if G_i's are $\mathcal{V}-$capable groups, then $G = \prod_{i\in I} G_i$ is also $\mathcal{V}-$capable. In the above theorem, the equality does not hold in general (Example 2.3).

Theorem 2.2. [8] *Let N be a normal subgroup contained in the marginal subgroup of G, $V^*(G)$. Then $N \subseteq (V^*)^*(G)$ if and only if the homomorphism induced by the natural map $\mathcal{V}M(G) \to \mathcal{V}M(G/N)$ is a monomorphism.*

In this section we verify the equation $(V^*)^*(A \times B) = (V^*)^*(A) \times (V^*)^*(B)$ for some famous varieties.

In general, for an arbitrary variety of groups $\mathcal{V}$, and groups A and B, $\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)\times T$, where T is an abelian group [7]. For some particular varieties, the group T is trivial with some conditions. For instance, some famous varieties as variety of abelian groups [7], variety of nilpotent groups [3], and some varieties of polynilpotent groups [6] have the property that: *for any two groups A and B with $(|A^{ab}|,|B^{ab}|)=1$ the isomorphism $\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)$ $(*)$ holds.*

Now, suppose that $\mathcal{V}$ is a variety, A and B are two groups with the property

$$\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B).$$

By Theorem 2.2, we have the following monomorphism

$$\mathcal{V}M(A)\times\mathcal{V}M(B)\hookrightarrow \mathcal{V}M(\frac{A}{(V^*)^*(A)})\times\mathcal{V}M(\frac{B}{(V^*)^*(B)}).$$

Moreover, we have the following inclusion

$$\mathcal{V}M(\frac{A}{(V^*)^*(A)})\times\mathcal{V}M(\frac{B}{(V^*)^*(B)})\hookrightarrow \mathcal{V}M(\frac{A}{(V^*)^*(A)}\times\frac{B}{(V^*)^*(B)}).$$

Finally, we get the monomorphism

$$\mathcal{V}M(A\times B)\hookrightarrow \mathcal{V}M(\frac{A\times B}{(V^*)^*(A)\times(V^*)^*(B)}).$$

Thus, by Theorem 2.2, we conclude that

$$(V^*)^*(A)\times(V^*)^*(B)\leq (V^*)^*(A\times B).$$

This note leads us to our main result.

Theorem 2.3. *Let $\mathcal{V}$ be a variety, A and B be two groups with $\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)$, then $(V^*)^*(A\times B)=(V^*)^*(A)\times(V^*)^*(B)$. Consequently $A\times B$ is $\mathcal{V}$-capable if and only if A and B are both $\mathcal{V}$-capable.*

Remark 2.4. In some famous varieties as the variety of abelian groups and the variety of nilpotent groups, the isomorphism $\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)$ holds, where $(|A^{ab}|,|B^{ab}|)=1$ ([3, 9]). Thus, using Theorem 2.3, for a family of groups $\{A_i \mid 1\leq i\leq n\}$ whose abelianizations have mutually coprime orders, $\prod_{i=1}^{n}A_i$ is capable ($\mathcal{N}_c$-capable) if and only if every A_i is capable ($\mathcal{N}_c$-capable). Note that in these varieties, for finitely generated groups A and B, $\mathcal{V}M(A\times B)\cong \mathcal{V}M(A)\times\mathcal{V}M(B)$ if and only if $|A^{ab}|$ and $|B^{ab}|$ are finite with $(|A^{ab}|,|B^{ab}|)=1$ ([3, 9]).

Corollary 2.5. *Let $\{A_i \mid 1 \leq i \leq n\}$ be a family of groups whose abelianizations have mutually coprime orders. If $\prod_{i=1}^{n} A_i$ is nilpotent of class at most c_1, then it is $\mathcal{N}_{c_1,\cdots,c_s}$-capable if and only if every A_i is $\mathcal{N}_{c_1,\cdots,c_s}$-capable.*

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