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Aref Afsharfard & Anooshiravan Farshidianfar

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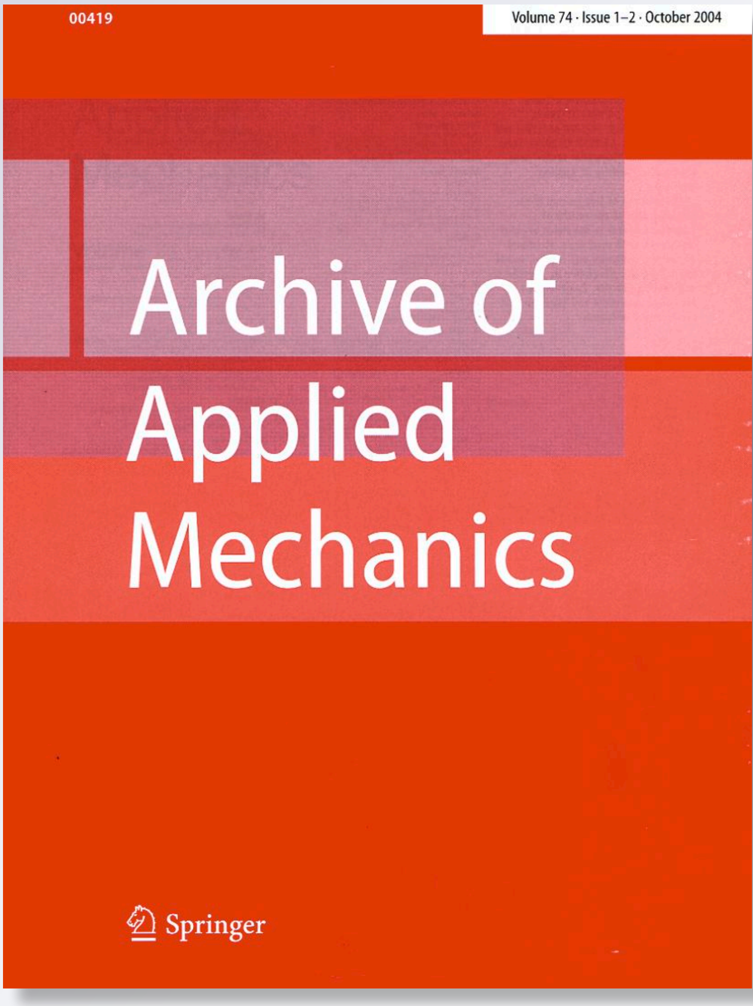

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Abstract In this paper, ongoing studies to solve nonlinear differential equations are extended by combining the Newmark-beta integration method and the piecewise linearization approach. The discussed method is illustrated with a practical example. In doing so, the coupled nonlinear differential equations of an impact oscillator, which incorporates the Hertzian contact, are derived. To investigate this problem, an object-oriented computer code, based on the presented method, is written in MATLAB. Furthermore, the discussed problem is solved numerically using the Runge–Kutta commercial code. To verify the calculated results, the contact durations, which are obtained using the discussed methods, are compared with the previous analytical results. In this study, accuracy of solution and the process time (cost) are selected as two main parameters of the solution method. The so-called adequacy factor is presented to combine the two main parameters of solution. Finally, it is shown that in the case of Hertzian contact, the presented method can be more adequate than the Runge–Kutta method.

Keywords Newmark-beta integration · Impact damper · Hertzian contact · Nonlinear ordinary differential equation

1 Introduction

In reality, every physical process is a nonlinear system and should be described by nonlinear equations. Kerschen et al. [1] studied the sources of nonlinearity and classified them to the following parts:

- Geometry nonlinearity that results in large displacements of structures.
- Inertia nonlinearity that is derived from nonlinear terms containing velocities and/or accelerations in the equation of motion.
- Nonlinear behavior of material that is observed when the stress and strain laws of material are nonlinear.
- Damping sources can introduce nonlinear effects including hysteresis, drag and coulomb frictions.
- Boundary conditions or external nonlinear body forces may also result in nonlinearity.

Nowadays, there is a large tendency toward numerical simulation of nonlinear systems. The reason for this interest lies in the growth of powerful computers. Step by step methods provide the only completely general approach to analysis of nonlinear response; however, the methods are equally valuable in the analysis of linear response [2]. The primary factor to be considered in selecting a step-by-step method is efficiency. This factor concerns the computational effort required to achieve the desired level of accuracy. Note that, accuracy cannot alone be a criterion for method selection because, in general, any desired degree of accuracy can be obtained by any method if the time step is made short enough. But it should be considered, decreasing the time step leads to increase the costs.

A. Afsharfard (✉) · A. Farshidianfar
Department of Mechanical Engineering, Ferdowsi University of Mashhad, Mashhad, Iran
E-mail: aref.afsharfard@gmail.com

One of the most complicated problems in mechanical engineering is contact problem. For many years, researchers have extensively investigated mechanical systems whose elements collide with each other during operation because impacts occur very often in many modern technical devices [3–6]. An impact damper is a small loose mass within a main mass. In these systems, the secondary mass (loose mass) absorbs kinetic energy of the main vibratory system when it collides with the enclosure and converts it into heat. These systems can be extensively applied to attenuate the undesirable vibration of robot arms, turbine blades and so on [7,8]. It is shown that the impact dampers would operate more efficiently than classical dynamic vibration dampers [9]. In the past few years, behavior of impact dampers has been investigated experimentally, analytically and numerically [10–13]. Son et al. [14] proposed active momentum exchange impact dampers to suppress the first large peak value of the acceleration response due to a shock load. Bapat and Sankar [15] showed that the coefficient of restitution has a great effect on the performance of impact dampers. They demonstrated that in the case of single-unit impact dampers, optimized parameters at resonance are not necessarily optimal at other frequencies. Cheng and Xu [16] obtained a relation between coefficient of restitution and impact damping ratio. They showed that optimal initial displacement is a monotonically increasing function of damping.

Although vibro-impact systems often show strongly nonlinear behavior [11], there is a tendency to describe them by linear equations [10]. The reason for this interest may be lies in the fact that it is difficult to solve coupled nonlinear differential equations. In this paper, an efficient method is presented to solve even strongly nonlinear coupled ODEs. This method can be easily applied to the impact damper problem.

The most general method to solve the dynamic response of structural systems is the direct numerical integration. Newmark presented a family of single-step integration methods for the solution of structural dynamic problems for both blast and seismic loading. During the past years, Newmark's method has been applied to dynamic analysis of many practical problems. In the present paper, a nonlinear model of an impact damper that incorporates the Hertzian contact theory is analyzed. The governing differential equations of this system are solved using the combination of the Newmark-beta integration and the piecewise linearization methods.

2 Modeling the elastic stop

The elastic stop can be described using the Hertzian contact model. Based on the Hertzian model, the following relationship holds between the contact force (F_c) and the relative displacement of the impact mass and barrier:

$$F_c = K_{Hz}\,(y - w)^{3/2} \tag{1}$$

where K_{Hz} is the Hertzian contact stiffness and y and w are the colliding masses displacements. Since the collision phenomenon is generally quite complex, the following assumptions are made to validate the above equation:

- The contact area is small compared to the geometry of the colliding bodies.
- The contact areas are perfectly smooth, so there is no friction between the colliding bodies.
- The material is isotropic and linearly elastic, so no plastic deformation occurs.
- The contact time is long enough to establish a quasi-static state.

If contact occurs between a spherical mass and a slab (barrier), the Hertzian contact stiffness will be as follows:

$$K_{Hz} = \frac{4}{3\pi}\frac{\sqrt{r}}{(1-\nu_1^2)/\pi E_1 + (1-\nu_2^2)/\pi E_2} \tag{2}$$

In the above relation, υ is the Poisson's ratio, E is the Young's modulus, r is the radius of the impact mass, and the subscripts 1 and 2 refer to the sphere and the slab, respectively [4,17]. The contact model can be refined by adding a hysteretic damping term accounting for energy loss during collision. The inclusion of hysteretic damping changes Eq. (1) to:

$$F_{hc} = K_{Hz}\,(y - w)^{3/2}\left(1 + \frac{3}{4}\frac{\dot{y} - \dot{w}}{(\dot{y} - \dot{w})_-}\left(1 - e^2\right)\right) \tag{3}$$

where subscript $(-)$ represents the behavior of the two colliding bodies at the beginning of the collision and e is the coefficient of restitution. In this study, for convenience, F_{hc} is simply named "Hyster-Hertz" contact force. Note that, in the case of elastic contact ($e = 1$), the Hyster-Hertz and the Hertzian contact forces are equal ($F_{hc} = F_c$). Effects of varying the contact stiffness and the coefficients of restitution on the Hyster-Hertz contact force are illustrated in Fig. 1.

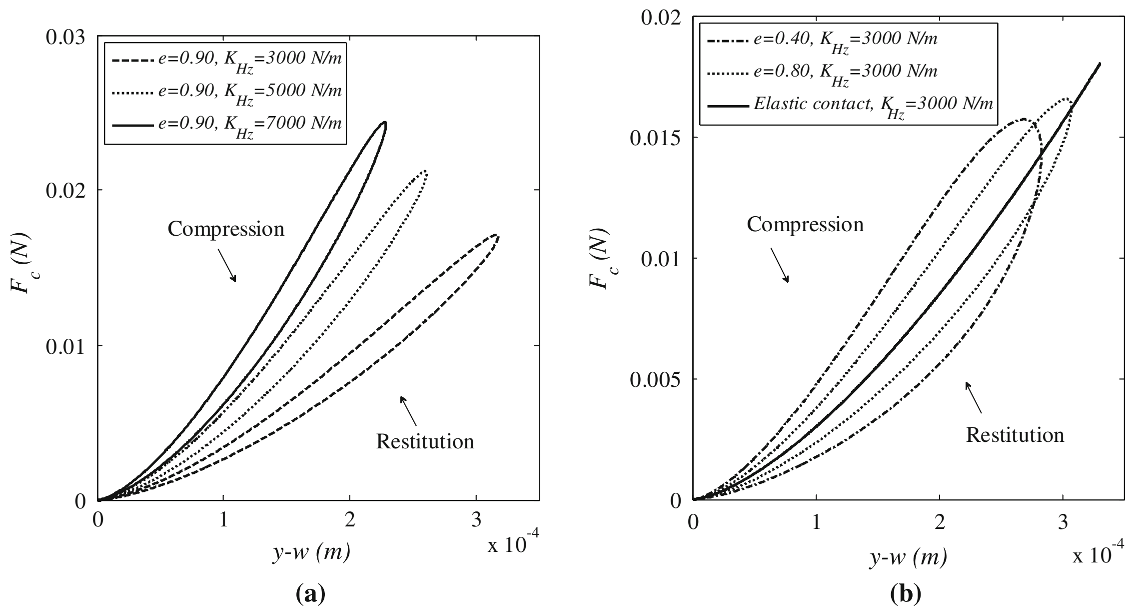

Fig. 1 Dependence of contact force on indentation of the impact mass and barrier of the impact damper system for different contact stiffness (**a**); and for different coefficients of restitutions (**b**)

3 Modeling elastic beam equipped with an impact damper

3.1 Modeling the elastic beam

An impact damper is a small loose mass within a main mass that freely moves through an enclosure. Figure 2a shows a cantilever beam equipped an impact damper. The elastic beam is a continuum with an infinite number of degrees of freedom [18,19]. The transverse vibratory behavior of the beam can be modeled using a spring. The vibratory model of the beam equipped with the impact damper is illustrated in Fig. 2b. As shown in this figure, an impact damper with impact mass m, clearance d, and oscillator with linear stiffness K, main mass M and viscous damping C are considered. To consider other energy losses, damper c_1 is added to the presented model.

Parameters K and M can be derived using the Ritz method. In doing so, the Euler–Lagrange partial differential equation of the beam motion, for constant EI, is as follows [20,21]:

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = f(x, t) \tag{4}$$

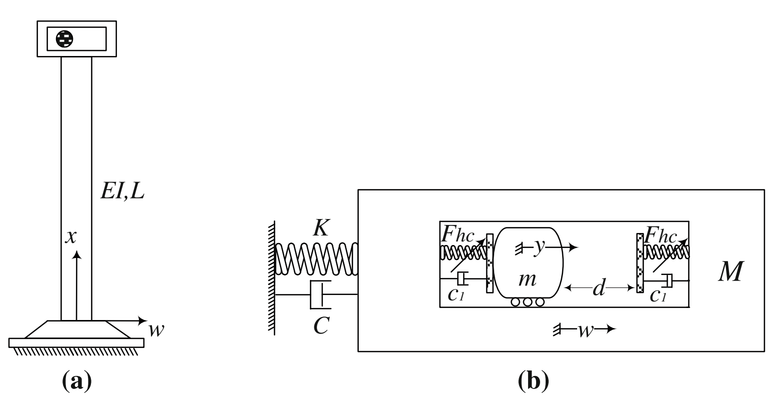

Fig. 2 Schematic diagram of the beam with an impact damper (**a**); and Schematic diagram of the impact damper (**b**)

where w is the transverse displacement of the beam, E is the Young's modulus, ρ is the mass density, A is cross-sectional area of the beam and I is the beam moment of inertia. The above equation can be accurately solved using the method of Ritz in conjunction with the principle of virtual displacements. Truncated series involving a complete set of basis function, which can be used to show the displacements, is given by:

$$w(x,t)=\sum_{i=1}^{N} q_i(t)\phi_i(x) \tag{5}$$

where q_i is the time-dependent response and ϕ_i are the uniform cantilever beam free-vibration mode shapes. The uniform cantilever beam mode shapes can be written as follows [18,19,22]:

$$\phi_i(x)=\sin(\beta_i x)-\sinh(\beta_i x)-\left(\frac{\sin(\beta_i L)+\sinh(\beta_i L)}{\cos(\beta_i L)+\cosh(\beta_i L)}\right)(\cos(\beta_i x)-\cosh(\beta_i x)) \tag{6}$$

where L is the beam length. Note that in case of cantilever beams $\cos(\beta_i L).\cosh(\beta_i L)+1=0$, so that $\beta_1 L=1.875104$. The mode shapes can be normalized such that for all i and j,

$$\int_0^L \phi_i\phi_j \mathrm{d}x=\begin{cases}\eta, \text{ if } & i=j,\\ 0, \text{ if } & i\neq j.\end{cases} \tag{7}$$

Substituting the truncated series into the Euler–Lagrange partial differential equation results in:

$$\rho A\frac{\partial^2 q_1}{\partial t^2}\int_0^L \phi_1^2 \mathrm{d}x+EI\cdot q_1\int_0^L \frac{\partial^4\phi_1}{\partial x^4}\phi_1 \mathrm{d}x=\int_0^L f(x,t)\,\phi_1(L)\,\mathrm{d}x \tag{8}$$

Therefore, a system of linear second-order ordinary differential equation (ODE) will be obtained. This equation, which describes the transverse dynamic behavior of the beam, can be given as follows:

$$\rho AH\,\frac{\partial^2 q_1}{\partial t^2}+\frac{12.362361EI}{L^3}\,q_1=F(x,t) \tag{9}$$

Finally, it can be concluded that the dynamic properties of the system are as $K=12.362361EI/L^3$ and $M=\rho AL$.

3.2 Modeling the dynamic behavior of the vibratory system with impact damper

As illustrated in Fig. 2b, it is clear that when $|y-w|<d$, the impact mass moves freely at a constant speed without causing any collision (free flight of the impact mass). Therefore, differential equations of motion between impacts can be given by:

$$\begin{cases}M\ddot{w}+C\dot{w}+Kw=0\\ m\ddot{y}=0\end{cases} \tag{10}$$

In this study, when contact occurs ($|y-w\geq d$), the impact interfaces between the impact mass and the main mass are described using the so-called Hyster-Hertz contact model. Therefore, the dynamic behavior of the presented vibratory system, when the impact mass collides with the main mass, can be formulated as follows:

$$\begin{cases}M\ddot{w}+C\dot{w}+Kw-c_1(\dot{y}-\dot{w})-K_{\mathrm{Hz}}(y-w)^{3/2}\left(1+\frac{3}{4}\frac{\dot{y}-\dot{w}}{(\dot{y}-\dot{w})_-}\left(1-e^2\right)\right)=0\\ m\ddot{y}+c_1(\dot{y}-\dot{w})+K_{\mathrm{Hz}}(y-w)^{3/2}\left(1+\frac{3}{4}\frac{\dot{y}-\dot{w}}{(\dot{y}-\dot{w})_-}\left(1-e^2\right)\right)=0\end{cases} \tag{11}$$

The homogenous coupled nonlinear ODEs in the above relation can be converted to the following inhomogeneous coupled linear ODEs.

$$\begin{bmatrix}M & 0\\ 0 & m\end{bmatrix}\begin{Bmatrix}\ddot{w}\\ \ddot{y}\end{Bmatrix}_{(t)}+\begin{bmatrix}C+c_1 & -c_1\\ -c_1 & +c_1\end{bmatrix}\begin{Bmatrix}\dot{w}\\ \dot{y}\end{Bmatrix}_{(t)}+\begin{bmatrix}K & 0\\ 0 & 0\end{bmatrix}\begin{Bmatrix}w\\ y\end{Bmatrix}_{(t)}=\begin{Bmatrix}+F_{\mathrm{hc}}\\ -F_{\mathrm{hc}}\end{Bmatrix}_{(t-\Delta t)} \tag{12}$$

The above relation is a kind of piecewise equivalent linearization approach for the presented problem. Therefore, the Newmark-beta integration method can be easily used to solve the piecewise linearized problem. This approach can precisely describe the problem because relative displacement of the colliding bodies $(y - w)$ is small. Furthermore, note that the amount of $(y - w)^{3/2}$ does not vary rapidly when the relative displacement is small. Therefore, the contact force can be estimated using the previous step information. In doing so, the time step (Δt) should be selected small enough.

4 Contact duration

When the impact mass collides with the main mass, it stays in contact with the main mass for half the resonance period, which can be simply named "contact duration." The contact duration depends on the stiffness and damping mechanism of the collided materials. The contact duration (T_c) based on the Hertzian elastic contact is obtained as follows [4]:

$$T_c = \frac{2.9432}{(\dot{y} - \dot{w})_-^{0.2}} \left(\frac{5}{4K^* K_{\mathrm{Hz}}}\right)^{2/5} \tag{13}$$

where $K^* = (m + M)/(m.M)$. Furthermore, the contact duration is obtained based on the linear model of the impact damper [10]. The linear contact duration is given by:

$$T_c = \frac{2mM\pi}{\sqrt{m + M}\sqrt{4mMk_1 - c_1^2\,(m + M)}} \tag{14}$$

In the above relation, the parameter k_1 is the equivalent linear contact stiffness. For example, as shown in Fig. 3, if $K_{\mathrm{Hz}} = 3,000$ N/m, the equivalent linear contact stiffness (k_1) will be equal to 49.05 N/m.

5 Result and discussion

A theoretical model to investigate the behavior of the vibratory system that undergoes an elastic contact has been developed using the Hertzian theory. An impact damper of vibration is selected as a case study. The values of the model parameters are listed in Table 1.

The contact duration is equal to half of relative vibration period of the colliding masses. Figure 4 shows the relative oscillation of the impact mass and the main mass when contact occurs. The so-called Hyster-Hertz contact durations for $e = 0.9$ are calculated as $T_c = 0.1103$s and $T_c = 0.1094$s using the Runge–Kutta and the presented Newmark-beta methods, respectively $(\Delta t = 10^{-4} s)$. In the presented study, the fourth-order Runge–Kutta technique in MATLAB-R2010b software is used to investigate the problem.

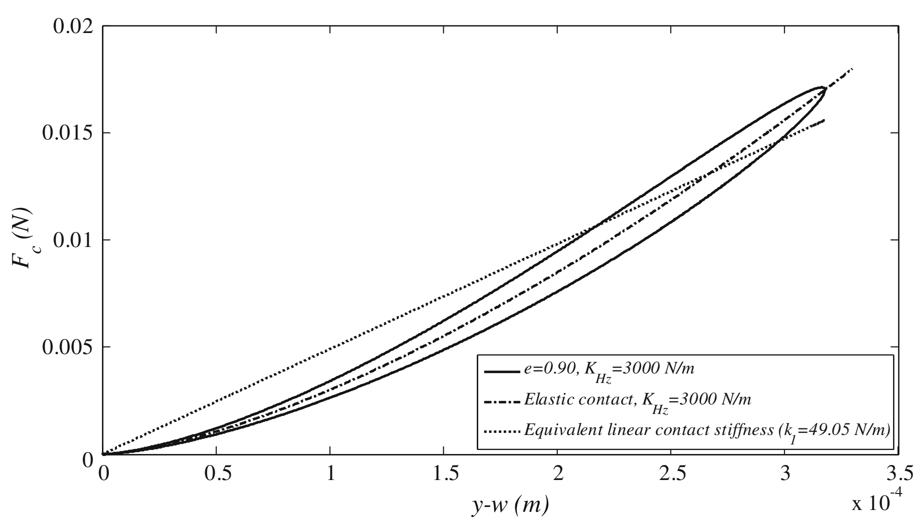

Fig. 3 Comparison of the Hertzian contact force with the linear approximation

Table 1 Parameters for the vibratory system with an impact damper

$M = 1\,\text{Kg}$	$C = 0.01$ N.s/m	$K = 1$ N/m
$m = 0.25\,\text{Kg}$	$c_1 = 0.01$ N.s/m	$K_{\text{Hz}} = 3,000$ N/m

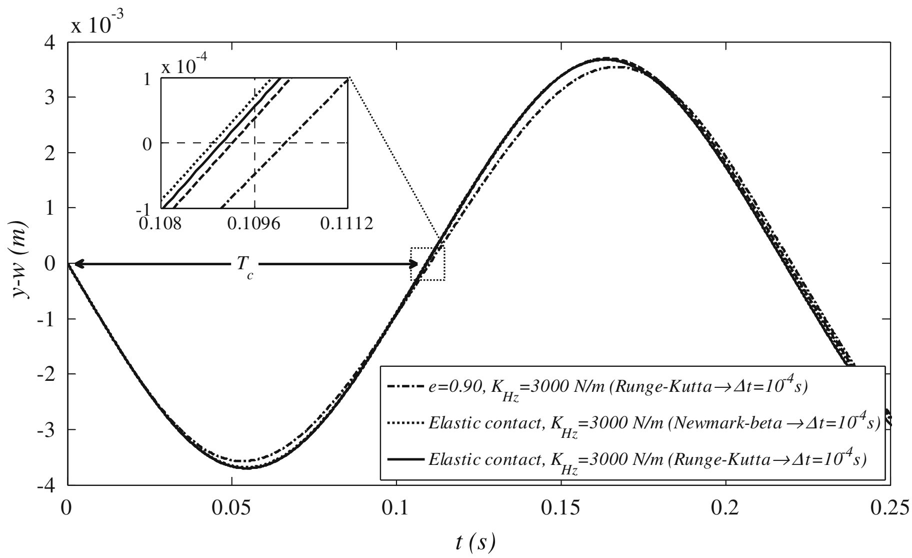

Fig. 4 Relative oscillation of colliding masses

Table 2 Comparison between the calculated elastic contact duration

Method of solution	Δt(s)	T_c (s)	T_{CPU} (s)	Error (%)
Theoretical solution	–	0.1089	–	–
Linear solution [10]	–	0.2006	–	84.21
Newmark-beta	1×10^{-3}	0.1100	0.0624	1.01
	1×10^{-4}	0.1090	0.2028	0.09
	2×10^{-5}	0.1089	2.5584	0.00
Runge–Kutta	1×10^{-3}	0.1110	0.0936	1.93
	1×10^{-4}	0.1092	0.1404	0.27
	2×10^{-5}	0.1091	0.3900	0.18

As shown in the Fig. 4, considering the contact loss leads to increase the contact duration. For example, for $e = 0.9$, the Hyster-Hertz contact duration, which is obtained using the Newmark-beta method, is relatively 0.4% greater than the elastic Hertzian contact duration.

Comparison between the calculated elastic contact duration with the previous theoretical (analytical) result [4] is given in Table 2. Furthermore, inability of linear approximation to describe the Hertzian contact problem is shown in this table. As shown in this table, the presented Newmark-beta method provides more accurate results than the Runge–Kutta method with the same time step. The process time (CPU time or T_{CPU}) in the MATLAB software program is given in Table 2.

As shown in the above table, decreasing the time step leads to increase the accuracy of solution. It should be considered, increasing the accuracy follows with obvious cost increase (increase the CPU time). Adequacy of the presented methods can be studied by weighting the importance of the accuracy and the cost. The so-called adequacy of solution method is defined as follows:

$$\text{Adequacy (\%)} = \left(\left(1 - \frac{\text{Error}}{(\text{Error})_{\max}}\right) \times WF_1 + \left(1 - \frac{T_{\text{CPU}}}{(T_{\text{CPU}})_{\max}}\right) \times WF_2\right) \times 100 \tag{15}$$

where WF_1 and WF_2 are the weight factors of the accuracy of solution and the cost, respectively. Note that $WF_1 + WF_2 = 1$. Effect of varying the weight factors on the adequacy is given in Table 3. Furthermore, in this table, the maximum value of the adequacy for each weight factor is shown in italics.

Table 3 Variation of the adequacy factor with varying the weight factor and the time step

Method of solution	Δt(s)	$WF_1 = 0.0$	$WF_1 = 0.2$	$WF_1 = 0.4$	$WF_1 = 0.6$	$WF_1 = 0.8$	$WF_1 = 1.0$
Newmark-beta	1×10^{-3}	*97.56*	87.58	77.60	67.63	57.65	47.67
	1×10^{-4}	92.07	92.73	*93.38*	*94.03*	*94.68*	95.34
	2×10^{-5}	0.00	20.00	40.00	60.00	80.00	*100.00*
Runge–Kutta	1×10^{-3}	96.34	77.07	57.80	38.54	19.27	0.00
	1×10^{-4}	94.51	*92.81*	91.11	89.41	87.71	86.01
	2×10^{-5}	84.76	85.94	87.12	88.31	89.49	90.67

As shown in the above table, for the Hertzian contact case, the presented method is generally more adequate than the Runge–Kutta commercial code (especially when $\Delta t = 1 \times 10^{-4}$s).

6 Conclusion

This paper presents an efficient method to solve coupled nonlinear ordinary differential equations. This method is used to investigate the vibratory behavior of a cantilever beam equipped with an impact damper. In doing so, the homogenous nonlinear coupled equations of the impact damper system are replaced by the equivalent piecewise inhomogeneous linear equations. Newmark-beta method is used to solve the linearized problem. It is shown that the contact loss in solutions can increase the contact duration up to 0.4%.

To investigate accuracy of solution, the contact durations, which are achieved in this study, are compared with the previous analytical results. It is shown that the linear model of Hertzian contact cannot accurately present the strongly nonlinear behavior of impact dampers. Furthermore, it is shown that the result accuracy of the presented method is better than the Runge–Kutta method up to 65%.

In the presented study, the accuracy and the cost are selected as two main parameters of solution methods. These parameters are combined into a single objective called "adequacy." It is shown that the presented method can be more adequate than the Runge–Kutta method.

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