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A highly computational efficient method to solve nonlinear optimal control problems

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Abstract In this paper, a new analytical technique, called the Optimal Homotopy Perturbation Method (OHPM), is suggested to solve a class of nonlinear Optimal Control Problems (OCP's). Applying the OHPM to a nonlinear OCP, the nonlinear Two-Point Boundary Value Problem (TPBVP), derived from the Pontryagin's maximum principle, is transformed into a sequence of linear time-invariant TPBVP's. Solving the latter problems in a recursive manner provides the optimal trajectory and the optimal control law, in the form of rapid convergent series. Furthermore, the convergence of obtained series is controlled through a number of auxiliary functions involving a number of constants, which are optimally determined. In this study, an efficient algorithm is also presented, which has low computational complexity and fast convergence rate. Just a few iterations are required to find a suboptimal trajectory-control pair for the nonlinear OCP. The results not only demonstrate the efficiency, simplicity and high accuracy of the suggested approach, but also indicate its effectiveness in practical use.

1. Introduction

One of the most active research areas in the control theory is optimal control, which has a wide range of applications in different fields such as physics, economy, aerospace, chemical engineering, robotic, etc. [1–4]. For linear time-invariant systems, theory and application of optimal control have been developed perfectly [5,6]. Although the optimal control of nonlinear systems has been studied extensively, it is still challenging.

In order to solve the nonlinear Optimal Control Problems (OCP's), many computational methods have been developed. One familiar scheme is the State-Dependent Riccati Equation (SDRE) technique [7]. Although this method has been widely used in various applications, its major limitation is that it needs solving a sequence of matrix Riccati algebraic equations. This property may take long computing time and large memory space. Another scheme is called the Approximating Sequence of Riccati Equations (ASRE) [8]. From a practical point of view the ASRE is attractive; however, this scheme suffers from computational complexity, since it needs solving a sequence of linear quadratic time-varying matrix Riccati differential equations.

To determine the optimal control law, there is another approach using dynamic programming [9]. This approach leads to the Hamilton–Jacobi–Bellman (HJB) equation that is hard to solve in most cases. An excellent literature review on the methods for solving the HJB equation is provided in [10], where a Successive Galerkin Approximation (SGA) approach is also considered. In the SGA, a sequence of generalized HJB equations is solved iteratively to obtain a sequence of approximations reaching eventually to the solution of HJB equation. However, the above-mentioned sequence may converge very slowly or even diverge.

The optimal control law can also be derived using the Pontryagin's maximum principle [11]. For the nonlinear OCP's, this approach leads to a nonlinear Two-Point Boundary Value Problem (TPBVP) that unfortunately in general cannot be solved analytically. Therefore, many researchers have tried to find an approximate solution for the nonlinear TPBVP's [12]. In the recent years, some better results have been obtained. For instance,

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a new Successive Approximation Approach (SAA) has been proposed in [13], where instead of directly solving the nonlinear TPBVP, derived from the maximum principle, a sequence of nonhomogeneous linear time-varying TPBVP's is solved iteratively. It should be noted that solving time-varying equations is much more difficult than solving time-invariant ones.

Recently, a growing interest has been appeared toward the application of homotopy techniques in the nonlinear problems, and many new methods have been introduced into the literature. In 1992, Liao [14] utilized the basic ideas of homotopy in topology to propose a general analytical technique, namely the Homotopy Analysis Method (HAM), for solving nonlinear problems. The HAM approximates efficiently the solution of nonlinear problems by means of base functions, and provides a great freedom for using different base functions. This technique has been successfully applied to solve many types of nonlinear problems [15–18]. In 1998, He [19] proposed the Homotopy Perturbation Method (HPM) for solving a large class of nonlinear problems. The HPM is a coupling of the traditional perturbation method and the homotopy concept as used in topology. This strategy has also been utilized to solve many types of nonlinear problems, including fourth-order parabolic equations [20], nonlinear boundary value problems [21], nonlinear partial differential equations of fractional order [22], nonlinear coupled systems of reaction–diffusion equations [23], integro-differential equations [24], delay differential equations [25], etc. In 2010, Marinca and Herişanu [26] proposed a new analytical technique, called the Optimal Homotopy Perturbation Method (OHPM), for solving strongly nonlinear differential equations. This technique starts from the basis of He's HPM, but its homotopy structure is different. In the OHPM, the nonlinear operator is expanded in a series with respect to the parameter p, and a number of auxiliary functions are introduced within the coefficients of this truncated power series. These auxiliary functions depend on a number of unknown constants, which ensure a rapid convergence of the obtained solution when they are optimally determined. In application, the OHPM has been used to study the nonlinear behaviour of an electrical machine rotor-bearing system [27].

The aim of this paper is to employ the OHPM for solving a class of nonlinear OCP's. To reach this goal, the optimal trajectory and the optimal control law are determined in the form of rapid convergent series. Moreover, the convergence of obtained series is controlled through a number of auxiliary functions involving a number of constants, which are optimally determined. The main strength of the proposed technique is its fast convergence. In fact, after only a few iterations it converges to the exact solution of OCP, which proves that the suggested approach is very efficient in practice.

The paper is organized as follows. Section 2 describes the problem statement. The basic idea of OHPM is explained in Section 3. In the following section, the OHPM is employed to propose a new optimal control design strategy. Section 5 explains how to use the results of Section 4 in practice. In Section 6, effectiveness of the proposed approach is verified by solving a numerical example. Finally, conclusions and future works are given in the last section.

2. Statement of the problem

Consider a nonlinear control system described by:

$$\begin{cases} \dot{x}(t) = F(x(t)) + Bu(t), & t \in [t_0, t_f] \\ x(t_0) = x_0, & x(t_f) = x_f \end{cases} \tag{1}$$

where $x \in R^n$ and $u \in R^m$ are respectively the state and control vectors, $F : R^n \to R^n$ is a nonlinear vector field, B is a constant matrix of appropriate dimension, $x_0 \in R^n$ and $x_f \in R^n$ are the initial and final state vectors, respectively. The objective is to find the optimal control law $u^*(t)$, which minimizes the following quadratic performance index subject to the system in Eq. (1):

$$J = \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) \mathrm{d}t, \tag{2}$$

where $Q \in R^{n\times n}$ and $R \in R^{m\times m}$ are positive semi-definite and positive definite matrices, respectively.

According to the Pontryagin's maximum principle, the optimality conditions are obtained as the following nonlinear TPBVP:

$$\begin{cases} \dot{x}(t) = -BR^{-1}B^T\lambda(t) + F(x(t)) \\ \dot{\lambda}(t) = -Qx(t) - \left(\dfrac{\partial F(x(t))}{\partial x(t)}\right)^T \lambda(t) \\ x(t_0) = x_0, \qquad x(t_f) = x_f \end{cases} \tag{3}$$

where $\lambda \in R^n$ is the co-state vector. Also, the optimal control law is given by:

$$u^*(t) = -R^{-1}B^T\lambda(t) \quad t \in [t_0, t_f]. \tag{4}$$

Unfortunately, Eq. (3) contains a nonlinear TPBVP that in general cannot be solved analytically except in a few simple cases. In order to overcome this difficulty, we will introduce the OHPM in the next section.

3. Basic idea of the OHPM

In order to explain the basic idea of OHPM, first we briefly review the main points of He's HPM. To this end, consider the following nonlinear differential equation:

$$L(v(r)) + N(v(r)) = 0 \quad r \in \Omega, \tag{5}$$

with the boundary condition:

$$B\left(v, \frac{\partial v}{\partial n}\right) = 0 \quad r \in \Gamma, \tag{6}$$

where L is a linear operator, N is a nonlinear operator, Γ is the boundary of domain Ω, B is a boundary operator, and $\frac{\partial}{\partial n}$ denotes differential along the normal drawn outwards from Ω.

By means of He's HPM, a homotopy is constructed for Eq. (5) as follows:

$$H(\tilde{v}, p) = L(\tilde{v}) - L(v_{\text{ini}}) + p\left(L(v_{\text{ini}}) + N(\tilde{v})\right) = 0 \qquad p \in [0, 1] \qquad r \in \Omega, \tag{7}$$

where $p \in [0, 1]$ is an embedding parameter called homotopy parameter, and v_{ini} is an initial approximation for the solution of Eq. (5), which satisfies the boundary condition in Eq. (6). Obviously, when $p = 0$ and $p = 1$ it holds:

$$H(\tilde{v}, 0) = L(\tilde{v}) - L(v_{\text{ini}}) = 0, \tag{8a}$$

$$H(\tilde{v}, 1) = L(\tilde{v}) + N(\tilde{v}) = 0. \tag{8b}$$

Thus, when p increases from zero to one, the trivial problem in Eq. (8a) is continuously deformed to the problem in Eq. (8b). Therefore, the changing process of p from zero to unity is just that of $\tilde{v}$ from v_{ini} to v. In topology, this is called deformation, and $L(\tilde{v}) - L(v_{\text{ini}})$ and $L(\tilde{v}) + N(\tilde{v})$ are called homotopic.

According to the He's HPM, the embedding parameter p can be used as a 'small parameter'. Expanding $\tilde{v}$ in a power series with respect to the parameter p, we obtain:

$$\tilde{v} = \tilde{v}^{(0)} + p\tilde{v}^{(1)} + p^2\tilde{v}^{(2)} + \cdots. \tag{9}$$

Setting $p = 1$ in the above series results in the solution of Eq. (5) as:

$$v = \lim_{p\to 1} \tilde{v} = \tilde{v}^{(0)} + \tilde{v}^{(1)} + \tilde{v}^{(2)} + \cdots \tag{10}$$

which is the essence of He's HPM.

We now explain the main idea of OHPM. Substituting $\tilde{v}$ from Eq. (9) into $N(\tilde{v})$ and then expanding N in a power series with respect to the parameter p, we obtain:

$$\begin{aligned} N(\tilde{v}) &= N(\tilde{v})|_{p=0} + \left.\frac{\partial N(\tilde{v})}{\partial p}\right|_{p=0} p + \cdots \\ &= N\left(\tilde{v}^{(0)}\right) + \left.\left(\frac{\partial N(\tilde{v})}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial p}\right)\right|_{p=0} p + \cdots \\ &= N\left(\tilde{v}^{(0)}\right) + \left.\frac{\partial N(\tilde{v})}{\partial \tilde{v}}\right|_{\tilde{v}=\tilde{v}^{(0)}} \tilde{v}^{(1)} p + \cdots. \end{aligned} \tag{11}$$

Then, we construct a new homotopy for Eq. (5) as follows:

$$\begin{aligned} H(\tilde{v}, p) &= L(\tilde{v}) - L(v_{\text{ini}}) + p\left(L(v_{\text{ini}}) + K_0(r, C_0)N(\tilde{v}^{(0)})\right) \\ &\quad + p^2\left(K_1(r, C_1)\left.\frac{\partial N(\tilde{v})}{\partial \tilde{v}}\right|_{\tilde{v}=\tilde{v}^{(0)}} \tilde{v}^{(1)}\right) + \cdots = 0, \end{aligned} \tag{12}$$

where $K_i(r, C_i)$ for $i = 0, 1, \ldots$ is an auxiliary function, and C_i is a vector of unknown constants. By equating the coefficients of the same powers of p in Eq. (12), we obtain:

$$p^0 : L(\tilde{v}^{(0)}) - L(v_{\text{ini}}) = 0, \tag{13a}$$

$$p^1 : L(\tilde{v}^{(1)}) + L(v_{\text{ini}}) + K_0(r, C_0)N(\tilde{v}^{(0)}) = 0, \tag{13b}$$

$$p^2 : L\left(\tilde{v}^{(2)}\right) + K_1(r, C_1)\left.\frac{\partial N(\tilde{v})}{\partial \tilde{v}}\right|_{\tilde{v}=\tilde{v}^{(0)}} \tilde{v}^{(1)} = 0,$$

$$\vdots \tag{13c}$$

and so on.

The functions $K_0, K_1, \ldots$ are not unique and can be chosen as the same form of nonlinear operator N [26]. Also, the constant C_i, that appears in the function $K_i(r, C_i)$, can be optimally determined by minimizing the following residual functional:

$$I = \int_a^b \left(L(v^{(M)}) + N(v^{(M)})\right)^2 \mathrm{d}r, \tag{14}$$

where a and b are two values depending on the given problem, and $v^{(M)}$ is the Mth order approximate solution as:

$$v^{(M)} = \tilde{v}^{(0)} + \tilde{v}^{(1)} + \cdots + \tilde{v}^{(M)}. \tag{15}$$

Once the parameter C_i is known, the solution of nonlinear differential equation in Eq. (5) subject to the boundary condition in Eq. (6) can be immediately determined.

In short, the main idea of OHPM is to construct the new homotopy as Eq. (12), which contains a number of auxiliary functions $K_i(r, C_i)$. These auxiliary functions depend on several unknown constants C_i which ensure a rapid convergence of the obtained solution when they are optimally determined.

4. Optimal control design strategy via OHPM

In this section, we apply the OHPM for solving the nonlinear TPBVP in Eq. (3). In order to perform this methodology, let us define two operators $F_1(x(t), \lambda(t))$ and $F_2(x(t), \lambda(t))$ as follows:

$$F_1(x(t), \lambda(t)) \triangleq \dot{x}(t) + BR^{-1}B^T\lambda(t) - F(x(t)), \tag{16}$$

$$F_2(x(t), \lambda(t)) \triangleq \dot{\lambda}(t) + Qx(t) + \left(\frac{\partial F(x(t))}{\partial x(t)}\right)^T \lambda(t). \tag{17}$$

From the nonlinear TPBVP in Eq. (3) it is obvious that:

$$F_i(x(t), \lambda(t)) = 0 \quad i = 1, 2. \tag{18}$$

The operator F_i can generally be divided into a linear part L_i and a nonlinear part N_i, i.e. we can write:

$$F_i(x(t), \lambda(t)) = L_i(x(t), \lambda(t)) + N_i(x(t), \lambda(t)) \quad i = 1, 2. \tag{19}$$

In accordance with Eqs. (16) and (17), L_i and N_i for $i = 1, 2$ can be defined as:

$$\begin{cases} L_1(x(t), \lambda(t)) \triangleq \dot{x}(t) + BR^{-1}B^T\lambda(t) \\ L_2(x(t), \lambda(t)) \triangleq \dot{\lambda}(t) + Qx(t) \end{cases} \tag{20a}$$

$$\begin{cases} N_1(x(t), \lambda(t)) \triangleq -F(x(t)) \\ N_2(x(t), \lambda(t)) \triangleq \left(\dfrac{\partial F(x(t))}{\partial x(t)}\right)^T \lambda(t). \end{cases} \tag{20b}$$

Also, initial approximations for the solution of nonlinear TPBVP in Eq. (3), i.e. $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$, are chosen as the solution of following linear time-invariant TPBVP:

$$\begin{cases} L_1(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) = 0 \\ L_2(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) = 0 \\ x_{\text{ini}}(t_0) = x_0, \qquad x_{\text{ini}}(t_f) = x_f. \end{cases} \tag{21}$$

Based on the OHPM, the solution of nonlinear TPBVP in Eq. (3) can be expressed as:

$$\begin{cases} x(t) = \tilde{x}^{(0)}(t) + \tilde{x}^{(1)}(t) + \tilde{x}^{(2)}(t) + \cdots = \displaystyle\sum_{i=0}^{\infty} \tilde{x}^{(i)}(t) \\ \lambda(t) = \tilde{\lambda}^{(0)}(t) + \tilde{\lambda}^{(1)}(t) + \tilde{\lambda}^{(2)}(t) + \cdots = \displaystyle\sum_{i=0}^{\infty} \tilde{\lambda}^{(i)}(t) \end{cases} \tag{22}$$

in which $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i \geq 0$ are obtained by solving the following sequence of linear time-invariant TPBVP's in a recursive manner:

$$p^0 : \begin{cases} L_1\left(\tilde{x}^{(0)}(t), \tilde{\lambda}^{(0)}(t)\right) - L_1(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) = 0 \\ L_2\left(\tilde{x}^{(0)}(t), \tilde{\lambda}^{(0)}(t)\right) - L_2(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) = 0 \\ \tilde{x}^{(0)}(t_0) = x_0, \qquad \tilde{x}^{(0)}(t_f) = x_f \end{cases} \tag{23a}$$

$$p^1 : \begin{cases} \begin{bmatrix} L_1(\tilde{x}^{(1)}(t), \tilde{\lambda}^{(1)}(t)) \\ L_2(\tilde{x}^{(1)}(t), \tilde{\lambda}^{(1)}(t)) \end{bmatrix} + \begin{bmatrix} L_1(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) \\ L_2(x_{\text{ini}}(t), \lambda_{\text{ini}}(t)) \end{bmatrix} \\ \qquad + K_0(t, C_0)\begin{bmatrix} N_1(\tilde{x}^{(0)}(t), \tilde{\lambda}^{(0)}(t)) \\ N_2(\tilde{x}^{(0)}(t), \tilde{\lambda}^{(0)}(t)) \end{bmatrix} = 0 \\ \tilde{x}^{(1)}(t_0) = 0, \qquad \tilde{x}^{(1)}(t_f) = 0 \end{cases} \tag{23b}$$

$$p^2: \begin{cases} \begin{bmatrix} L_1(\tilde{x}^{(2)}(t), \tilde{\lambda}^{(2)}(t)) \\ L_2(\tilde{x}^{(2)}(t), \tilde{\lambda}^{(2)}(t)) \end{bmatrix} + K_1(t, C_1) \\ \times \begin{bmatrix} \left.\dfrac{\partial N_1(x,\lambda)}{\partial x}\right|_{\substack{x=\tilde{x}^{(0)}(t)\\ \lambda=\tilde{\lambda}^{(0)}(t)}} \tilde{x}^{(1)}(t) + \left.\dfrac{\partial N_1(x,\lambda)}{\partial \lambda}\right|_{\substack{x=\tilde{x}^{(0)}(t)\\ \lambda=\tilde{\lambda}^{(0)}(t)}} \tilde{\lambda}^{(1)}(t) \\ \left.\dfrac{\partial N_2(x,\lambda)}{\partial x}\right|_{\substack{x=\tilde{x}^{(0)}(t)\\ \lambda=\tilde{\lambda}^{(0)}(t)}} \tilde{x}^{(1)}(t) + \left.\frac{\partial N_2(x,\lambda)}{\partial \lambda}\right|_{\substack{x=\tilde{x}^{(0)}(t)\\ \lambda=\tilde{\lambda}^{(0)}(t)}} \tilde{\lambda}^{(1)}(t) \end{bmatrix} = 0 \\ \tilde{x}^{(2)}(t_0) = 0, \qquad \tilde{x}^{(2)}(t_f) = 0, \end{cases}$$

$$\vdots \tag{23c}$$

and so on, where $K_i(r, C_i)$ for $i = 0, 1, \ldots$ is an auxiliary function, and C_i is a vector of unknown constants.

The parameter C_i can be optimally determined by minimizing the following residual functional:

$$I = \int_{t_0}^{t_f} \sum_{i=1}^{2} \left\| L_i(x^{(M)}(t), \lambda^{(M)}(t)) + N_i(x^{(M)}(t), \lambda^{(M)}(t)) \right\|_2^2 \mathrm{d}t, \tag{24}$$

where $x^{(M)}(t)$ and $\lambda^{(M)}(t)$ are the Mth order approximate solutions as:

$$\begin{cases} x^{(M)}(t) = \sum_{i=0}^{M} \tilde{x}^{(i)}(t) \\ \lambda^{(M)}(t) = \sum_{i=0}^{M} \tilde{\lambda}^{(i)}(t). \end{cases} \tag{25}$$

Finally, according to the previous discussions, the following theorem can be stated:

Theorem 4.1. *Consider the OCP of nonlinear system in Eq. (1) with quadratic performance index in Eq. (2). Using the OHPM, the optimal trajectory and the optimal control law can be determined as follows:*

$$\begin{cases} x^*(t) = \sum_{i=0}^{\infty} \tilde{x}^{(i)}(t), \quad t \in [t_0, t_f] \\ u^*(t) = -R^{-1}B^T \sum_{i=0}^{\infty} \tilde{\lambda}^{(i)}(t), \quad t \in [t_0, t_f]. \end{cases} \tag{26}$$

5. Practical implementation and suboptimal control design strategy

In fact, it is almost impossible to obtain the optimal trajectory and the optimal control law as in Eq. (26), since it contains infinite series. In practice, the Mth order suboptimal trajectory-control pair is obtained by replacing ∞ with a finite positive integer M in Eq. (26) as follows:

$$\begin{cases} x^{(M)}(t) = \sum_{i=0}^{M} \tilde{x}^{(i)}(t) \\ u^{(M)}(t) = -R^{-1}B^T \sum_{i=0}^{M} \tilde{\lambda}^{(i)}(t). \end{cases} \tag{27}$$

The integer M is generally determined according to a concrete control precision. For example, the Mth order suboptimal trajectory-control pair in Eq. (27) has the desired accuracy if for a given positive constant $\varepsilon > 0$, the following condition holds:

$$\left| \frac{J^{(M)} - J^{(M-1)}}{J^{(M)}} \right| < \varepsilon, \tag{28}$$

where:

$$J^{(M)} = \frac{1}{2} \int_{t_0}^{t_f} \left(\left(x^{(M)}(t)\right)^T Q x^{(M)}(t) + \left(u^{(M)}(t)\right)^T R u^{(M)}(t) \right) \mathrm{d}t. \tag{29}$$

In order to obtain an accurate enough suboptimal trajectory-control pair, we present an iterative algorithm with low computational complexity. This algorithm has also a relatively fast convergence rate. Therefore, only a few iterations are required to reach the desired accuracy. This fact reduces the size of computations, effectively.

Algorithm.

Step 1. Obtain $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$ from the linear time-invariant TPBVP in Eq. (21). Set $\tilde{x}^{(0)}(t) = x_{\text{ini}}(t)$, $\tilde{\lambda}^{(0)}(t) = \lambda_{\text{ini}}(t)$, and $i = 1$.

Step 2. Calculate the ith order terms $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ from the sequence of linear time-invariant TPBVP's in Eqs. (23a)–(23c). Set $M = i$ and calculate $x^{(M)}(t)$ and $\lambda^{(M)}(t)$ from Eq. (25).

Step 3. Determine the unknown constant $C_j, j = 0, \ldots, M-1$ by minimizing the residual functional in Eq. (24).

Step 4. Obtain $x^{(M)}(t)$ and $u^{(M)}(t)$ from Eq. (27), and then calculate $J^{(M)}$ according to Eq. (29).

Step 5. If the inequality in Eq. (28) holds for the given small enough constant $\varepsilon > 0$, go to step 6; else replace i by $i+1$ and go to Step 2.

Step 6. Stop the algorithm; $x^{(M)}(t)$ and $u^{(M)}(t)$ are accurate enough.

6. Numerical example

In this section, we consider the optimal manoeuvres of a rigid asymmetric spacecraft [28]. The Euler's equations for the angular velocities of spacecraft are given by:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \underbrace{\begin{bmatrix} -\dfrac{(I_3 - I_2)}{I_1} x_2(t) x_3(t) \\ -\dfrac{(I_1 - I_3)}{I_2} x_1(t) x_3(t) \\ -\dfrac{(I_2 - I_1)}{I_3} x_1(t) x_2(t) \end{bmatrix}}_{F(x(t))} + \underbrace{\begin{bmatrix} \dfrac{1}{I_1} & 0 & 0 \\ 0 & \dfrac{1}{I_2} & 0 \\ 0 & 0 & \dfrac{1}{I_3} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix}}_{u(t)}, \tag{30}$$

where x_1, x_2 and x_3 are the angular velocities of spacecraft, u_1, u_2, and u_3 are control torques, $I_1 = 86.24$ kg m^2, $I_2 = 85.07$ kg m^2 and $I_3 = 113.59$ kg m^2 are the spacecraft principle inertia.

The quadratic performance index to be minimized is given by:

$$J = \frac{1}{2}\int_0^{100} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt, \tag{31}$$

where:

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In addition, the following boundary conditions should be satisfied:

$$\begin{cases} x_1(0) = 0.01 \text{ r/s}, \quad x_2(0) = 0.005 \text{ r/s} \\ \qquad\qquad\qquad\quad x_3(0) = 0.001 \text{ r/s} \\ x_1(100) = x_2(100) = x_3(100) = 0 \text{ r/s}. \end{cases} \tag{32}$$

According to the Pontryagin's maximum principle, the following nonlinear TPBVP is obtained:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \underbrace{-\begin{bmatrix} \frac{\lambda_1(t)}{I_1^2} \\ \frac{\lambda_2(t)}{I_2^2} \\ \frac{\lambda_3(t)}{I_3^2} \end{bmatrix}}_{-BR^{-1}B^T\lambda(t)} + \underbrace{\begin{bmatrix} -\frac{(I_3 - I_2)}{I_1}x_2(t)x_3(t) \\ -\frac{(I_1 - I_3)}{I_2}x_1(t)x_3(t) \\ -\frac{(I_2 - I_1)}{I_3}x_1(t)x_2(t) \end{bmatrix}}_{F(x(t))}, \tag{33a}$$

$$\dot{\lambda}(t) = \begin{bmatrix} \dot{\lambda}_1(t) \\ \dot{\lambda}_2(t) \\ \dot{\lambda}_3(t) \end{bmatrix} = \underbrace{-\begin{bmatrix} -\frac{(I_1 - I_3)}{I_2}x_3(t)\lambda_2(t) - \frac{(I_2 - I_1)}{I_3}x_2(t)\lambda_3(t) \\ -\frac{(I_3 - I_2)}{I_1}x_3(t)\lambda_1(t) - \frac{(I_2 - I_1)}{I_3}x_1(t)\lambda_3(t) \\ -\frac{(I_3 - I_2)}{I_1}x_2(t)\lambda_1(t) - \frac{(I_1 - I_3)}{I_2}x_1(t)\lambda_2(t) \end{bmatrix}}_{-\left(\frac{\partial F(x(t))}{\partial x(t)}\right)^T\lambda(t)}, \tag{33b}$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.005 \\ 0.001 \end{bmatrix} \text{ r/s},$$

$$x(100) = \begin{bmatrix} x_1(100) \\ x_2(100) \\ x_3(100) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ r/s}, \tag{33c}$$

and the optimal control law is given by:

$$u^*(t) = \begin{bmatrix} u_1^*(t) \\ u_2^*(t) \\ u_3^*(t) \end{bmatrix} = \underbrace{-\begin{bmatrix} \frac{\lambda_1(t)}{I_1} \\ \frac{\lambda_2(t)}{I_2} \\ \frac{\lambda_3(t)}{I_3} \end{bmatrix}}_{-R^{-1}B^T\lambda(t)} \quad t \in [0, 100]. \tag{34}$$

For the nonlinear TPBVP in Eqs. (33a)–(33c), linear and nonlinear operators L_i and N_i are defined in accordance with Eqs. (20a) and (20b). Then, the initial approximations, i.e. $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$, are obtained by solving the following linear time-invariant TPBVP:

$$\dot{x}_{\text{ini}}(t) = \begin{bmatrix} \dot{x}_{\text{ini},1}(t) \\ \dot{x}_{\text{ini},2}(t) \\ \dot{x}_{\text{ini},3}(t) \end{bmatrix} = -\begin{bmatrix} \frac{\lambda_{\text{ini},1}(t)}{I_1^2} \\ \frac{\lambda_{\text{ini},2}(t)}{I_2^2} \\ \frac{\lambda_{\text{ini},3}(t)}{I_3^2} \end{bmatrix}, \tag{35a}$$

$$\dot{\lambda}_{\text{ini}}(t) = \begin{bmatrix} \dot{\lambda}_{\text{ini},1}(t) \\ \dot{\lambda}_{\text{ini},2}(t) \\ \dot{\lambda}_{\text{ini},3}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{35b}$$

$$x_{\text{ini}}(0) = \begin{bmatrix} x_{\text{ini},1}(0) \\ x_{\text{ini},2}(0) \\ x_{\text{ini},3}(0) \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.005 \\ 0.001 \end{bmatrix} \text{ r/s},$$

$$x_{\text{ini}}(100) = \begin{bmatrix} x_{\text{ini},1}(100) \\ x_{\text{ini},2}(100) \\ x_{\text{ini},3}(100) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ r/s}, \tag{35c}$$

where $x_{\text{ini},j}(t)$ and $\lambda_{\text{ini},j}(t)$ are the jth elements of vectors $x_{\text{ini}}(t)$ and $\lambda_{\text{ini}}(t)$, respectively. By solving the linear TPBVP in Eqs. (35a)–(35c), we obtain:

$$\begin{cases} x_{\text{ini},1}(t) = -0.0001t + 0.01 \\ x_{\text{ini},2}(t) = -0.00005t + 0.005 \\ x_{\text{ini},3}(t) = -0.00001t + 0.001 \\ \lambda_{\text{ini},1}(t) = 0.7437337601 \\ \lambda_{\text{ini},2}(t) = 0.3618452452 \\ \lambda_{\text{ini},3}(t) = 0.1290268810. \end{cases} \tag{36}$$

Then, based on the proposed method in Section 4, the sequence of linear time-invariant TPBVP's in Eqs. (23a)–(23c) is solved in a recursive manner. Solving the linear TPBVP in Eq. (23a), $\tilde{x}^{(0)}(t)$ and $\tilde{\lambda}^{(0)}(t)$ are obtained as:

$$\begin{cases} \tilde{x}_1^{(0)}(t) = -0.0001t + 0.01 \\ \tilde{x}_2^{(0)}(t) = -0.00005t + 0.005 \\ \tilde{x}_3^{(0)}(t) = -0.00001t + 0.001 \\ \tilde{\lambda}_1^{(0)}(t) = 0.7437337601 \\ \tilde{\lambda}_2^{(0)}(t) = 0.3618452452 \\ \tilde{\lambda}_3^{(0)}(t) = 0.1290268810 \end{cases} \tag{37}$$

where $\tilde{x}_j^{(0)}(t)$ and $\tilde{\lambda}_j^{(0)}(t)$ are the jth elements of vectors $\tilde{x}^{(0)}(t)$ and $\tilde{\lambda}^{(0)}(t)$, respectively.

Substituting $\tilde{x}^{(0)}(t)$ and $\tilde{\lambda}^{(0)}(t)$ from Eq. (37) into Eq. (23b) and choosing $K_0(t, C_0) = c_{00} + c_{01}t + c_{02}t^2$ where c_{0j} for $j = 0, 1, 2$ is unknown constant, Eq. (23b) becomes a nonhomogeneous linear time-invariant TPBVP. Solving the linear TPBVP in Eq. (23b), $\tilde{x}^{(1)}(t)$ and $\tilde{\lambda}^{(1)}(t)$ are obtained as:

$$\begin{aligned} \tilde{x}_1^{(1)}(t) = {} & (-1.653525047 \times 10^{-6}c_{00} \\ & - 3.015834924 \times 10^{-12}c_{02} \\ & - 4.622920645 \times 10^{-14}c_{01})t \\ & + (-8.267625230 \times 10^{-7}c_{01} \\ & + 2.480287570 \times 10^{-8}c_{00})t^2 \\ & + (-5.511750153 \times 10^{-7}c_{02} \\ & - 8.267625233 \times 10^{-11}c_{00} \\ & + 1.377937539 \times 10^{-8}c_{01})t^3 \\ & + (-5.511750155 \times 10^{-11}c_{01} \\ & + 9.645562772 \times 10^{-9}c_{02})t^4 \\ & + (-4.133812616 \times 10^{-11}c_{02})t^5, \end{aligned} \tag{38a}$$

$$\begin{aligned}\tilde{x}_2^{(1)}(t) = {} & (3.214999411 \times 10^{-6} c_{00} \\ & - 5.066935000 \times 10^{-13} c_{02} \\ & - 1.266733750 \times 10^{-14} c_{01}) t \\ & + (1.607499706 \times 10^{-6} c_{01} \\ & - 4.822499117 \times 10^{-8} c_{00}) t^2 \\ & + (1.071666471 \times 10^{-6} c_{02} \\ & + 1.607499706 \times 10^{-10} c_{00} \\ & - 2.679166176 \times 10^{-8} c_{01}) t^3 \\ & + (1.071666470 \times 10^{-10} c_{01} \\ & - 1.875416324 \times 10^{-8} c_{02}) t^4 \\ & + (8.037498529 \times 10^{-11} c_{02}) t^5,\end{aligned} \tag{38b}$$

$$\begin{aligned}\tilde{x}_3^{(1)}(t) = {} & (5.150101215 \times 10^{-7} c_{00} \\ & - 2.548675675 \times 10^{-12} c_{02} \\ & - 6.371689186 \times 10^{-14} c_{01}) t \\ & + (2.575050620 \times 10^{-7} c_{01} \\ & - 7.725151822 \times 10^{-9} c_{00}) t^2 \\ & + (1.716700413 \times 10^{-7} c_{02} \\ & + 2.575050607 \times 10^{-11} c_{00} \\ & - 4.291751021 \times 10^{-9} c_{01}) t^3 \\ & + (1.716700407 \times 10^{-11} c_{01} \\ & - 3.004225717 \times 10^{-9} c_{02}) t^4 \\ & + (1.287525306 \times 10^{-11} c_{02}) t^5,\end{aligned} \tag{38c}$$

$$\begin{aligned}\tilde{\lambda}_1^{(1)}(t) = {} & (2.242978248 \times 10^{-8} c_{02} \\ & + 3.438222154 \times 10^{-10} c_{01} \\ & + 7.555107281 \times 10^{-12} c_{00}) \\ & + (-1.229782401 \times 10^{-4} c_{00}) t \\ & + (6.148912005 \times 10^{-7} c_{00} \\ & - 6.148912005 \times 10^{-5} c_{01}) t^2 \\ & + (4.099274670 \times 10^{-7} c_{01} \\ & - 4.099274670 \times 10^{-5} c_{02}) t^3 \\ & + (3.074456002 \times 10^{-7} c_{02}) t^4,\end{aligned} \tag{38d}$$

$$\begin{aligned}\tilde{\lambda}_2^{(1)}(t) = {} & (3.666892675 \times 10^{-9} c_{02} \\ & + 9.167231687 \times 10^{-11} c_{01} \\ & + 3.666892675 \times 10^{-3} c_{00}) \\ & + (2.326664500 \times 10^{-4} c_{00}) t \\ & + (-1.163332250 \times 10^{-6} c_{00} \\ & + 1.163332250 \times 10^{-4} c_{01}) t^2 \\ & + (-7.755548333 \times 10^{-7} c_{01} \\ & + 7.755548333 \times 10^{-5} c_{02}) t^3 \\ & + (-5.816661250 \times 10^{-7} c_{02}) t^4,\end{aligned} \tag{38e}$$

$$\begin{aligned}\tilde{\lambda}_3^{(1)}(t) = {} & (3.288476730 \times 10^{-8} c_{02} \\ & + 8.221191824 \times 10^{-10} c_{01} \\ & + 3.288476730 \times 10^{-11} c_{00}) \\ & + (6.645014900 \times 10^{-5} c_{00}) t \\ & + (-3.322507450 \times 10^{-7} c_{00} \\ & + 3.322507450 \times 10^{-5} c_{01}) t^2 \\ & + (-2.215004967 \times 10^{-7} c_{01} \\ & + 2.215004967 \times 10^{-5} c_{02}) t^3 \\ & + (-1.661253725 \times 10^{-7} c_{02}) t^4,\end{aligned} \tag{38f}$$

where $\tilde{x}_j^{(1)}(t)$ and $\tilde{\lambda}_j^{(1)}(t)$ are the jth elements of vectors $\tilde{x}^{(1)}(t)$ and $\tilde{\lambda}^{(1)}(t)$, respectively.

Table 1: Simulation results of the proposed method at different iteration times.

i (iteration time)	Performance index value $J^{(i)}$	$\left\lvert\frac{J^{(i)}-J^{(i-1)}}{J^{(i)}}\right\rvert$
0	0.004687795354	–
1	0.004688009428	$4.566415731 \times 10^{-5}$

Continuing as above, $\tilde{x}^{(i)}(t)$ and $\tilde{\lambda}^{(i)}(t)$ for $i \geq 2$ are obtained only by solving a nonhomogeneous linear time-invariant TPBVP.

In order to obtain a suboptimal trajectory-control pair with remarkable accuracy, we applied the proposed algorithm in Section 5 with the tolerance error bound $\varepsilon = 5 \times 10^{-5}$. In this case, convergence was achieved after only one iteration, i.e. $\left\lvert\frac{J^{(1)}-J^{(0)}}{J^{(1)}}\right\rvert = 4.566415731 \times 10^{-5} < 5 \times 10^{-5}$, and a minimum of $J^{(1)} = 0.004688009428$ was obtained. Also, following the proposed procedure, the optimal values of constants c_{0j}, $j = 0, 1, 2$ were obtained as:

$$\begin{aligned} & c_{00} = 0.953752782143730493, \\ & c_{01} = -0.0126091120724424674, \\ & c_{02} = -4.44834663666561910 \times 10^{-5}. \end{aligned} \tag{39}$$

Simulation results are listed in Table 1. From Table 1, it is observed that very accurate results are obtained after only one iteration, which shows that the proposed method is very efficient in practice.

Substituting the optimal values of constants from Eq. (39) into Eqs. (38a)–(38f), and then substituting $\tilde{x}^{(1)}(t)$ and $\tilde{\lambda}^{(1)}(t)$ from Eqs. (38a)–(38f) and $\tilde{x}^{(0)}(t)$ and $\tilde{\lambda}^{(0)}(t)$ from Eq. (37) into Eq. (27) with $M = 1$, the first order suboptimal trajectory and the first order suboptimal control law are obtained as follows:

$$\begin{cases} x_1^{(1)}(t) = \tilde{x}_1^{(0)}(t) + \tilde{x}_1^{(1)}(t) = 0.01 - 1.015770541 \times 10^{-4} t \\ \quad + 3.408055302 \times 10^{-8} t^2 - 2.280802189 \times 10^{-10} t^3 \\ \quad + 2.659146868 \times 10^{-13} t^4 + 1.838863145 \times 10^{-15} t^5 \\ x_2^{(1)}(t) = \tilde{x}_2^{(0)}(t) + \tilde{x}_2^{(1)}(t) = 0.005 - 4.693368537 \times 10^{-5} t \\ \quad - 6.626386345 \times 10^{-8} t^2 + 4.434633580 \times 10^{-10} t^3 \\ \quad - 5.170260734 \times 10^{-13} t^4 - 3.575357955 \times 10^{-15} t^5 \\ x_3^{(1)}(t) = \tilde{x}_3^{(0)}(t) + \tilde{x}_3^{(1)}(t) = 0.001 - 9.508807663 \times 10^{-6} t \\ \quad - 1.061479523 \times 10^{-8} t^2 + 7.103830790 \times 10^{-11} t^3 \\ \quad - 8.28223045 \times 10^{-14} t^4 - 5.727358866 \times 10^{-16} t^5 \end{cases} \tag{40a}$$

$$\begin{cases} u_1^{(1)}(t) = -R^{-1} B^T (\tilde{\lambda}_1^{(0)}(t) + \tilde{\lambda}_1^{(1)}(t)) \\ \quad = -8.624000001 \times 10^{-3} + 1.360051468 \times 10^{-6} t \\ \quad - 1.579055426 \times 10^{-8} t^2 + 3.879083839 \times 10^{-11} t^3 \\ \quad + 1.585835577 \times 10^{-13} t^4 \\ u_2^{(1)}(t) = -R^{-1} B^T (\tilde{\lambda}_2^{(0)}(t) + \tilde{\lambda}_2^{(1)}(t)) \\ \quad = -4.253500001 \times 10^{-3} - 2.608513858 \times 10^{-6} t \\ \quad + 3.028553004 \times 10^{-8} t^2 - 7.439897817 \times 10^{-11} t^3 \\ \quad - 3.041557012 \times 10^{-13} t^4 \\ u_3^{(1)}(t) = -R^{-1} B^T (\tilde{\lambda}_3^{(0)}(t) + \tilde{\lambda}_3^{(1)}(t)) \\ \quad = -1.135900000 \times 10^{-3} - 5.579453691 \times 10^{-7} t \\ \quad + 6.477892070 \times 10^{-9} t^2 - 1.591349235 \times 10^{-11} t^3 \\ \quad - 6.505706859 \times 10^{-14} t^4 \end{cases} \tag{40b}$$

where $x_j^{(1)}(t)$ and $u_j^{(1)}(t)$, are the jth elements of vectors $x^{(1)}(t)$ and $u^{(1)}(t)$, respectively. Simulation curves of the state trajectories and control laws, computed by the suggested technique, have been shown in Figures 1–6. Besides, simulation curves have been obtained by directly solving the nonlinear TPBVP in Eqs. (33a)–(33c), using the collocation method [12]. Figures 1–6 show that the obtained solutions by the proposed

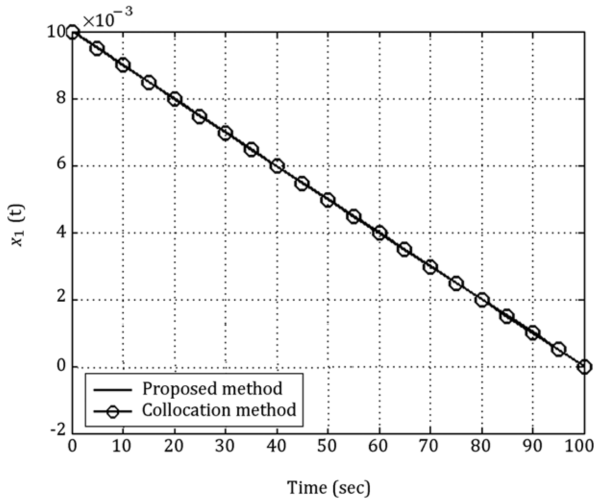

Figure 1: Simulation curves of $x_1(t)$ computed by the proposed method and collocation method.

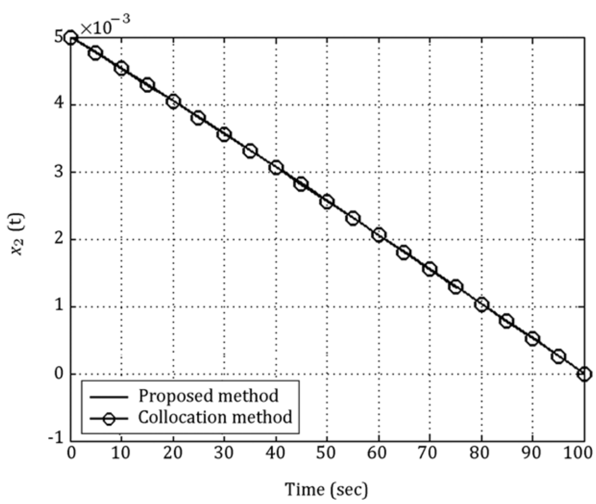

Figure 2: Simulation curves of $x_2(t)$ computed by the proposed method and collocation method.

Table 2: Simulation results of the He's HPM at different iteration times.

i (iteration time)	Performance index value $J^{(i)}$	$\left\lvert \frac{J^{(i)}-J^{(i-1)}}{J^{(i)}} \right\rvert$
0	0.004687795354	–
1	0.004688452416	$1.401447518 \times 10^{-4}$
2	0.004687810140	$1.370098150 \times 10^{-4}$
3	0.004687795533	$3.115963548 \times 10^{-6}$

approach are nearly identical with those of the collocation method. Moreover, in comparison with the collocation method, our computing procedure is very straightforward, which can be done by pencil and paper only.

We have also solved the aforementioned OCP by solving the nonlinear TPBVP in Eqs. (33a)–(33c) via He's HPM [19]. Simulation results are listed in Table 2.

Comparing Tables 1 and 2 verifies that the OHPM is superior to the He's HPM; it converges after only one iteration while the HPM converges after 3 iterations.

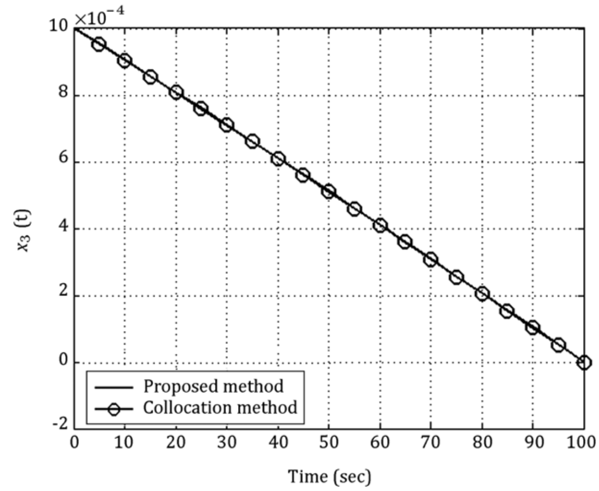

Figure 3: Simulation curves of $x_3(t)$ computed by the proposed method and collocation method.

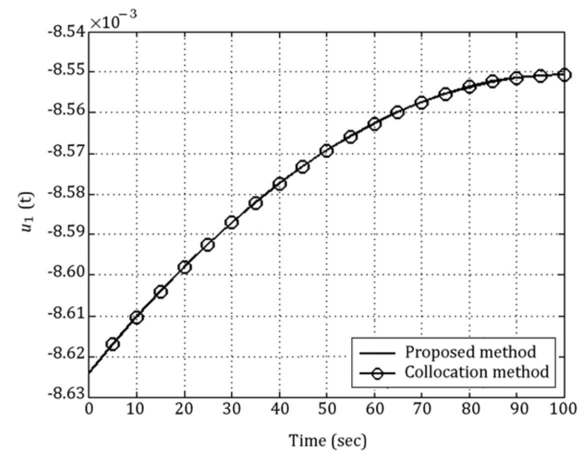

Figure 4: Simulation curves of $u_1(t)$ computed by the proposed method and collocation method.

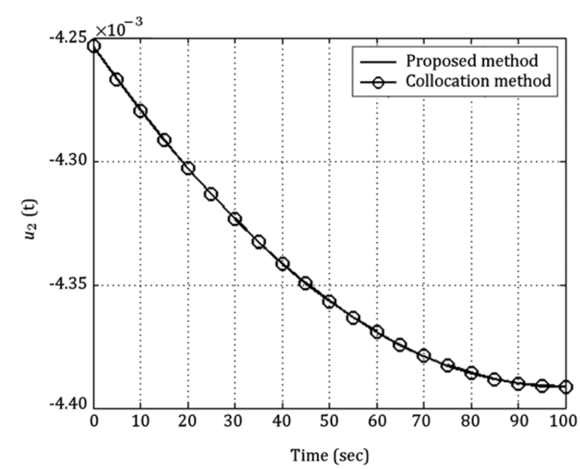

Figure 5: Simulation curves of $u_2(t)$ computed by the proposed method and collocation method.

7. Conclusions

This paper presented a new analytical technique, called the OHPM, for solving a class of nonlinear OCP's. The proposed

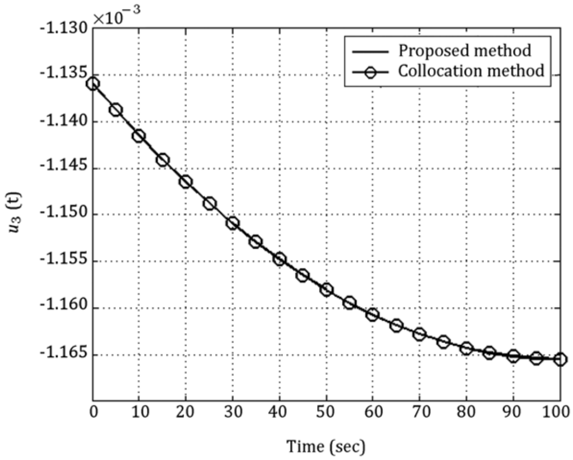

Figure 6: Simulation curves of $u_3(t)$ computed by the proposed method and collocation method.

method avoids directly solving the nonlinear TPBVP or the HJB equation. Furthermore, despite the other approximate approaches such as SAA [13], ASRE [8], SDRE [7] and SGA [10], the suggested technique keeps away from solving a sequence of linear time-varying TPBVP's or a sequence of matrix Riccati differential (or algebraic) equations or a sequence of generalized HJB equations. It only requires solving a sequence of linear time-invariant TPBVP's, and it needs only a few iterations to obtain a remarkable accuracy due to its fast convergence. Therefore, in view of computational complexity, the proposed method is more practical than the other approximate approaches. Future works can be focused on extending this method for solving more general form of nonlinear OCP's than one, which was considered in this paper.

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References

[1] Garrard, W.L. and Jordan, J.M. "Design of nonlinear automatic flight control systems", *Automatica*, 13(5), pp. 497–505 (1997).
[2] Manousiouthakis, V. and Chmielewski, D.J. "On constrained infinite-time nonlinear optimal control", *Chem. Eng. Sci.*, 57(1), pp. 105–114 (2002).
[3] Notsu, T., Konishi, M. and Imai, J. "Optimal water cooling control for plate rolling", *Int. J. Innov. Comput. Inform. Control*, 4(12), pp. 3169–3181 (2008).
[4] Tang, L., Zhao, L.D. and Guo, J. "Research on pricing policies for seasonal goods based on optimal control theory", *ICIC Expr. Lett.*, 3(4B), pp. 1333–1338 (2009).
[5] Bryson, A.E., *Applied Linear Optimal Control: Examples and Algorithms*, Cambridge University Press, UK (2002).
[6] Yousefi, S.A., Dehghan, M. and Lotfi, A. "Finding the optimal control of linear systems via He's variational iteration method", *Int. J. Comput. Math.*, 87(5), pp. 1042–1050 (2010).
[7] Cimen, T. "State dependent Riccati equation (SDRE) control: a survey", *17th IFAC World Congress*, Seoul, Korea (2008).
[8] Banks, S.P. and Dinesh, K. "Approximate optimal control and stability of nonlinear finite- and infinite-dimensional systems", *Ann. Oper. Res.*, 98(7), pp. 19–44 (2000).
[9] Bellman, R. "On the theory of dynamic programming", *Proc. Natl. Acad. Sci. USA*, 38(8), pp. 716–719 (1952).
[10] Beard, R.W., Saridis, G.N. and Wen, J.T. "Galerkin approximations of the generalized Hamilton–Jacobi–Bellman equation", *Automatica*, 33(12), pp. 2159–2177 (1997).
[11] Pontryagin, L.S. "Optimal control processes", *Uspekhi Mat. Nauk*, 14, pp. 3–20 (1959).
[12] Ascher, U.M., Mattheij, R.M.M. and Russel, R.D., *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, Philadelphia (1995).
[13] Tang, G.Y. "Suboptimal control for nonlinear systems: a successive approximation approach", *Systems Control Lett.*, 54(5), pp. 429–434 (2005).
[14] Liao, S.J. "On the proposed homotopy analysis technique for nonlinear problems and its applications", Ph.D. Dissertation, Shanghai Jio Tong University (1992).
[15] Liao, S.J. "An approximate solution technique not depending on small parameters: a special example", *Internat. J. Non-Linear Mech.*, 30(3), pp. 371–380 (1995).
[16] Liao, S.J. and Chwang, A.T. "Application of homotopy analysis method in nonlinear oscillations", *ASME J. Appl. Mech.*, 65(91), pp. 914–922 (1998).
[17] Abbasbandy, S. "The application of homotopy analysis method to solve a generalized Hirota–Satsuma coupled KdV equation", *Phys. Lett. A*, 361(6), pp. 478–481 (2007).
[18] Bataineh, A.S., Noorani, M.S.M. and Hashim, I. "The homotopy analysis method for Cauchy reaction–diffusion problems", *Phys. Lett. A*, 372(5), pp. 613–618 (2008).
[19] He, J.H. "An approximate solution technique depending upon an artificial parameter", *Commun. Nonlinear Sci.*, 3(2), pp. 92–97 (1998).
[20] Agirseven, D. and Ozis, T. "He's homotopy perturbation method for fourth-order parabolic equations", *Int. J. Comput. Math.*, 87(7), pp. 1555–1568 (2010).
[21] Saadatmandi, A., Dehghan, M. and Eftekhari, A. "Application of He's homotopy perturbation method for non-linear system of second-order boundary value problems", *Nonlinear Anal. RWA*, 10(3), pp. 1912–1922 (2009).
[22] Momani, S. and Odibat, Z. "Homotopy perturbation method for nonlinear partial differential equations of fractional order", *Phys. Lett. A*, 365(5–6), pp. 345–350 (2007).
[23] Ganji, D. and Sadighi, A. "Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction–diffusion equations", *Int. J. Nonlinear Sci. Numer. Simul.*, 7(4), pp. 411–418 (2006).
[24] Alawneh, A., Al-Khaled, K. and Al-Towaiq, M. "Reliable algorithms for solving integro-differential equations with applications", *Int. J. Comput. Math.*, 87(7), pp. 1538–1554 (2010).
[25] Shakeri, F. and Dehghan, M. "Solution of delay differential equations via a homotopy perturbation method", *Math. Comput. Modelling*, 48(3–4), pp. 486–498 (2008).
[26] Marinca, V. and Herişanu, N. "Optimal homotopy perturbation method for strongly nonlinear differential equations", *Nonlinear Sci. Lett. A*, 1(3), pp. 273–280 (2010).
[27] Marinca, V. and Herişanu, N. "Nonlinear dynamic analysis of an electrical machine rotor-bearing system by the optimal homotopy perturbation method", *Comput. Math. Appl.*, 87(7), pp. 1555–1568 (2010).
[28] Junkins, J.L. and Turner, J.D., *Optimal Spacecraft Rotational Maneuvers*, Elsevier, Amsterdam (1986).

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